Consider an economy in which there are two goods x and y, one firm whose production set is

$$Y = \{(-x, y) \in \mathbb{R}_- \times \mathbb{R}_+ \mid y \leq 2\sqrt{x}\},$$

and two consumers with identical preferences, represented by $u(x, y) = xy$, and endowments, $(\bar{x}_i, \bar{y}_i) = (24, 0)$, and with fractions $\theta \in [0, 1]$ and $1 - \theta$ of the firm's property.

Calculate the set of Pareto optimal and competitive equilibrium allocations for $\theta = 1$ and $\theta = 1/2$.

THE CE OF THE ECONOMY

W: price of good x (leisure)
p: " " " y (consumption)

$w = \frac{W}{p}$: real wage

Firm

$$\max_{x \geq 0} p(2\sqrt{x}) - Wx \iff \max_{x \geq 0} 2\sqrt{x} - wx$$

Solution: $\frac{1}{\sqrt{x}} = w \Rightarrow \boxed{x_f(w) = \frac{1}{w^2}}$. Firm's Labor Demand

Hence: $y^s(w) = 2\sqrt{x_f(w)} = \frac{2}{w}$. $\pi(w) = \frac{2}{w} - w\left(\frac{1}{w^2}\right) = \frac{1}{w}$

Consumers

$$\max_{(x,y)\in\mathbb{R}^2_+} xy$$
$$\text{s.t.} \quad wx + y \le 24w + \pi(w)\theta_i$$

$$\begin{cases} MRS = \dfrac{y}{x} = w \\ wx + y = 24w + \pi(w)\theta_i \end{cases}$$

HENCE,

$$\boxed{x_i(w) = 12 + \frac{\pi(w)\theta_i}{2w} = 12 + \frac{\theta_i}{2w^2}}$$ WORKER'S LABOR SUPPLY

$$\boxed{y_i(w) = 12w + \frac{\theta_i}{2w}}.$$ WORKER'S DEMAND of CONSUMPTION

MARKET CLEARING: $(24 - x_1(w)) + (24 - x_2(w)) = x_f(w)$

i.e.,

$$24 - \frac{\theta_1 + \theta_2}{2w^2} = \frac{1}{w^2} \iff 24 = \frac{3}{2}\left(\frac{1}{w^2}\right) \iff \boxed{w^* = \frac{1}{4}}.$$ CE REAL WAGE.

THUS, IN THE CE FIRM'S

LABOR is $x_f(w^*) = \left(\frac{1}{w^{*2}}\right) = \underline{16}$,

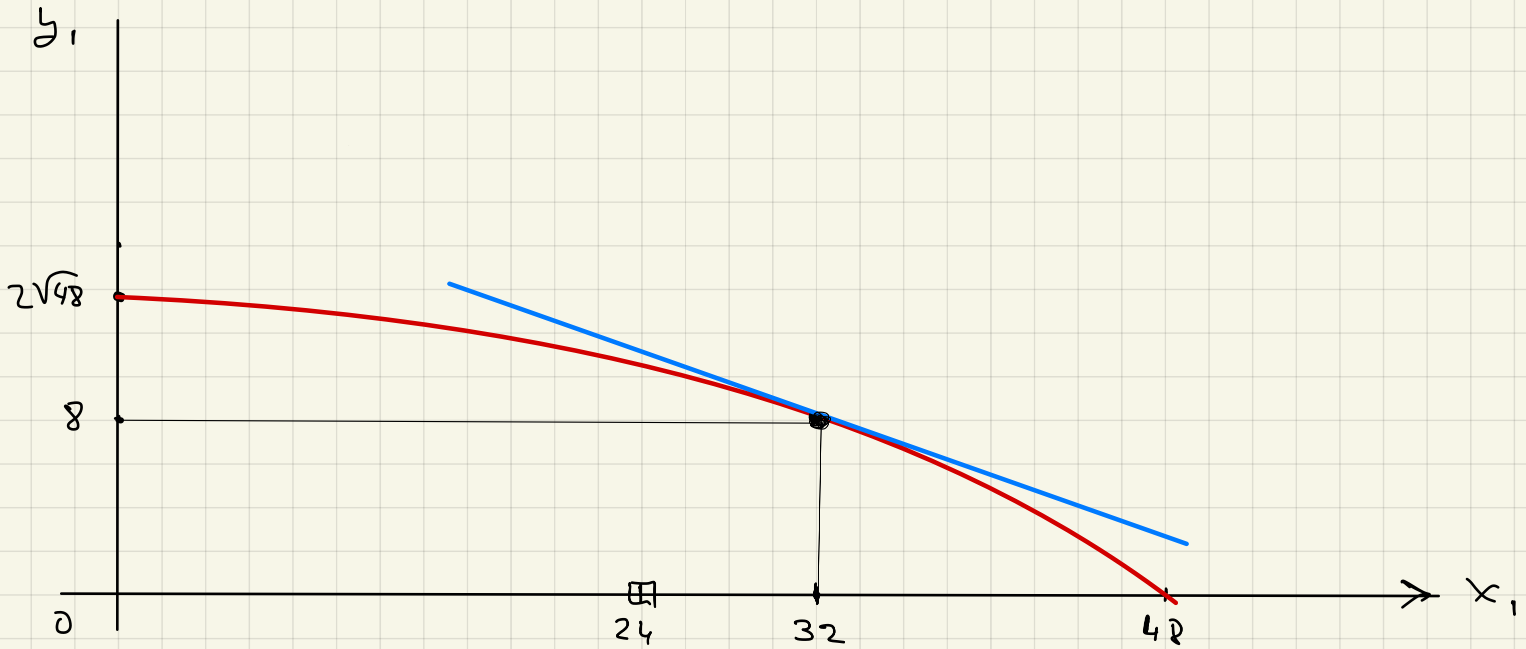
OUTPUT is $2\sqrt{16} = \underline{8}$.

PROFIT is $\frac{1}{w^*} = \underline{4}$

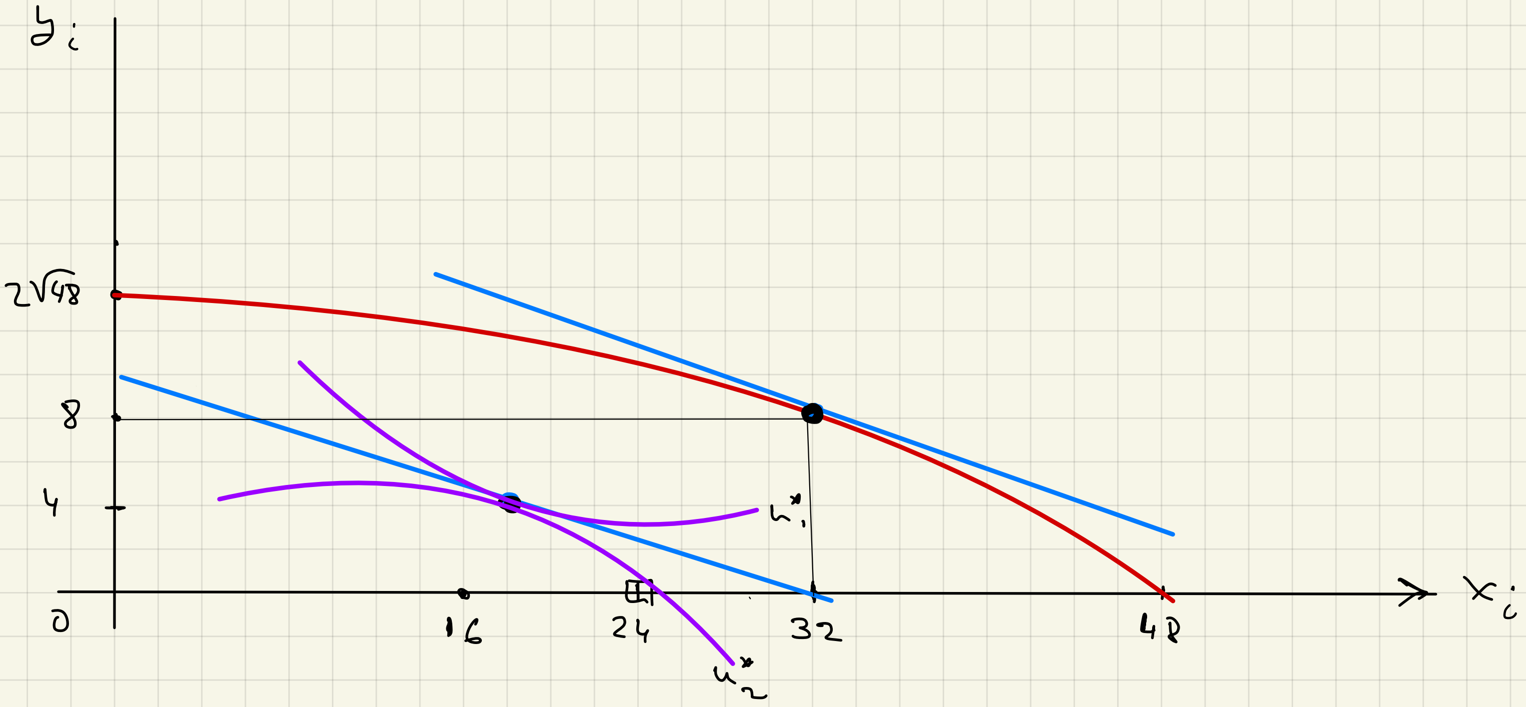
CE ALLOCATION :

$$x_i^* = 12 + \frac{\theta_i}{2\left(\frac{1}{4}\right)^2} = 12 + 8\theta_i$$

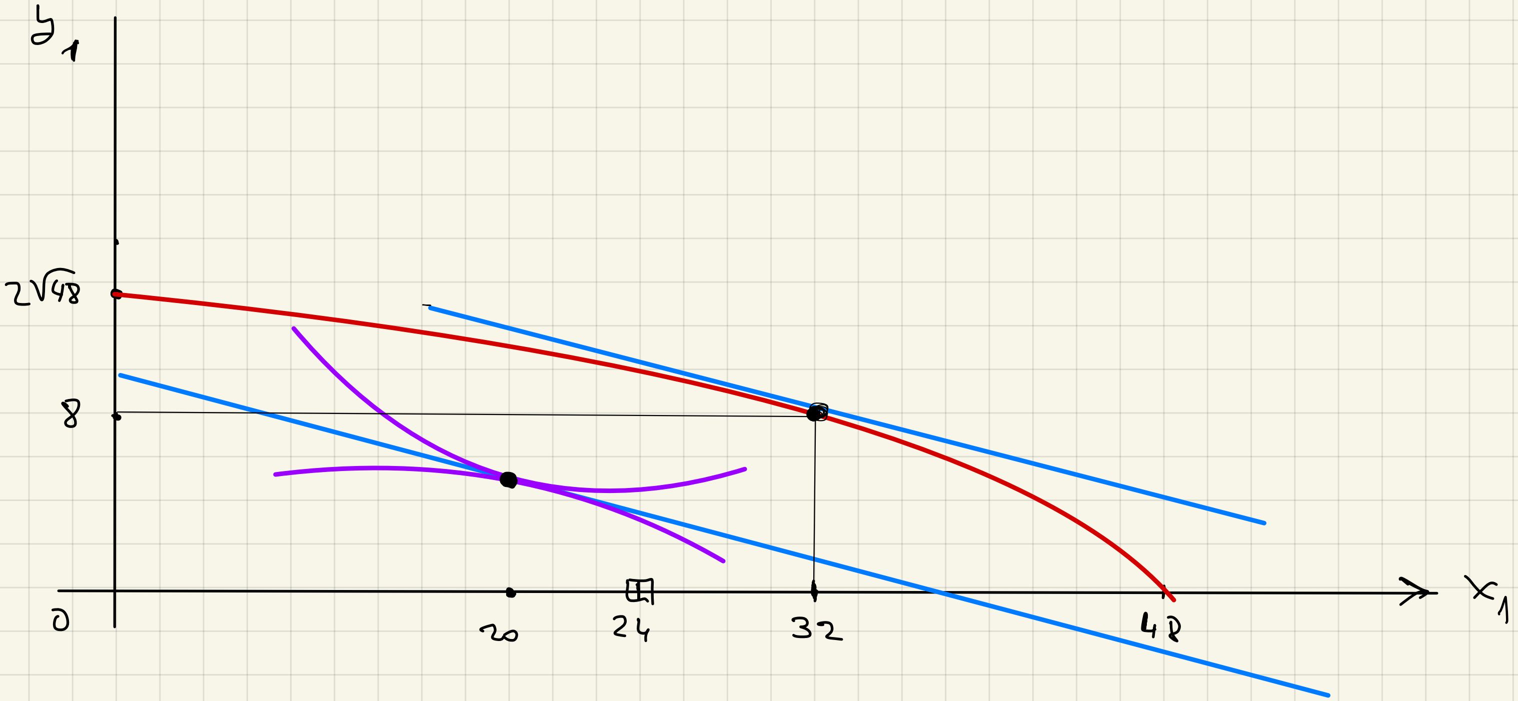
$$y_i^* = 12 \cdot \left(\frac{1}{4}\right) + \frac{\theta_i}{2\left(\frac{1}{4}\right)} = 3 + 2\theta_i$$



$$\theta_1 = \theta_2 = \frac{1}{2} \Rightarrow (x_i^*, y_i^*) = (16, 4) \quad , \quad i \in \{1, 2\}$$



$$\theta_1 = 1, \quad \theta_2 = 0 \quad \Rightarrow \quad (x_1, y_1) = (20, 5), \quad (x_2, y_2) = (12, 3)$$



Two firms (rather than 1), same technology, $\theta_1 = (1,0)$, $\theta_2 = (0,1)$

Market clearing:

$$(24 - x_1(w)) + (24 - x_2(w)) = 2x^d(w) \qquad w^* = \frac{1}{\sqrt{8}}$$

$$2\left(12 - \frac{1}{2w^2}\right) = \frac{2}{w^2} \iff 24 - \frac{1}{w^2} = \frac{2}{w^2} \iff 8 = \frac{1}{w^2}$$

Employment: $\frac{2}{w^2} = 16$

Output $2(2\sqrt{8}) \simeq 11.3$

CE allocation: $[(16, 5.65), (16, 5.65)]$