UC3M
Mathematics for Economics II
Final Exam, 31 May 2024 **Solutions**

$\boxed{1}$

Consider the following system of linear equations in the unknowns (x, y, z, t), where m is a parameter:

$$\left\{\begin{array}{rrrrcr} x & -2y & -z & +2t & = & 2 \\ 2x & +5y & & -t & = & -1 \\ 3x & +3y & -z & -3t & = & 1 \\ 4x & +y & -2z & +t & = & m \end{array}\right.$$

(a) (10 points) Study the system according to the values of m.

(b) (10 points) Solve the system for those values of m for which the system admits solutions

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Solution:

The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & +2 & 2 \\ 2 & 5 & 0 & -1 & -1 \\ 3 & 3 & -1 & -3 & 1 \\ 4 & 1 & -2 & 1 & m \end{array}\right)$$

Subtracting the first row multiplied by 2, 3 and 4 to the second, third and fourth row, respectively we obtain a first column of zeroes under the pivotal element.

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & +2 & 2 \\ 0 & 9 & 2 & -5 & -5 \\ 0 & 9 & 2 & -9 & -5 \\ 0 & 9 & 2 & -7 & m-8 \end{array}\right)$$

Subtracting the second row to the third and fourth, we get

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & +2 & 2 \\ 0 & 9 & 2 & -5 & -5 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -2 & m-3 \end{array}\right)$$

This is not in reduced form yet, as se can make another operation to obtain

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & +2 & 2 \\ 0 & 9 & 2 & -5 & -5 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & m-3 \end{array}\right)$$

1. The ranks of the system matrix and the augmented matrix coincide if and only if $m = 3$. Since this rank is 3, there are infinitely many solutions. There are no solutions if $m \neq 3$.

2. To solve the system, we substitute $m = 3$ into the reduced system and solve from the bottom equation

$$\left\{\begin{array}{rrrrcr} x & -2y & -z & +2t & = & 2 \\ & 9y & +2z & -5t & = & -5 \\ & & & -4t & = & 0 \end{array}\right.$$

   obtaining $t = 0$. Letting z be the parameter, then

$$9y + 2z = -5 \Rightarrow y = -\frac{5}{9} - \frac{2z}{9},$$

$$x - 2y - z = 2 \Rightarrow x = 2 - \frac{10}{9} - \frac{4z}{9} + z = \frac{8}{9} + \frac{5z}{9}.$$

   The solutions are

$$\left(\frac{8}{9}, -\frac{5}{9}, 0, 0\right) + z\left(\frac{5}{9}, -\frac{2}{9}, 1, 0\right),$$

   with z arbitrary.

$\boxed{2}$

Consider the symmetric matrix

$$A = \begin{pmatrix} m+1 & 0 & -1 \\ 0 & m & 0 \\ -1 & 0 & m+1 \end{pmatrix},$$

where m is a parameter.

(a) (10 points) The matrix A is diagonalizable for all m. Why? Find the eigenvalues and eigenvectors of A find matrices P regular and D diagonal such that $P^{-1}AP = D$.

(b) (10 points) Classify the quadratic form Q defined by the matrix A.

Solution:

(a)

$$p_A(\lambda) = \begin{vmatrix} m+1-\lambda & 0 & -1 \\ 0 & m-\lambda & 0 \\ -1 & 0 & m+1-\lambda \end{vmatrix} = (m-\lambda)\begin{vmatrix} m+1-\lambda & -1 \\ -1 & m+1-\lambda \end{vmatrix}$$

es decir

$$p_A(\lambda) = (m-\lambda)((m+1-\lambda)^2 - 1).$$

This polynomial has roots $\lambda_1 = m$, double, and $\lambda_2 = m+2$, simple. The rank of the matrix

$$A - mI_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

1s 1, which is the dimension of matrix A (3) minus the multiplicity of the eigenvalue (2). Thus, the matrix is diagonalizable for all m. (Another way: A is diagonalizable for every m, since A is symmetric for all m.)

Eigenvectors associated to $\lambda_1 = m$: they are solutions of the homogenous system with matrix

$$A - mI_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

which reduces to the equation $x - z = 0$. Two representative vectors are $(1, 0, 1)$ and $(0, 1, 0)$.

igenvectors associated to $\lambda_1 = m + 2$: they are solutions of the homogenous system with matrix

$$A - (m+2)I_3 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

that is, $-x - z = 0$ ad $-2y = 0$. A representative vector is $(1, 0, -1)$.

In summary, we can take

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m+2 \end{pmatrix}.$$

(b) We will study the sign of the eigenvalues found in part (a), which are m and $m + 2$.

$m > 0$: Q is positive definite;

$m = 0$: Q is positive semidefinite;

$m < -2$: Q is negative definite;

$m = -2$: Q is negative semidefinite;

$-2 < m < 2$: Q is indefinite.

$\boxed{3}$

Consider the plane region

$$A = \{(x,y) \in \mathbb{R}^2 \,:\, x \geq 0, x^2 + y^2 \leq 2, y \geq x^2\}.$$

(a) (10 points) Draw A.

(b) (10 points) Calculate the double integral

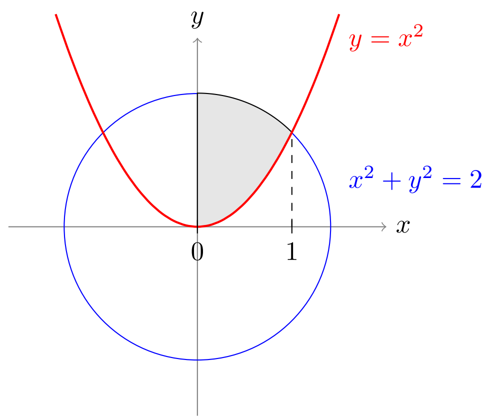
$$\iint_A x\,dx\,dy.$$

Solution:

(a) $x^2 + y^2 = 2$ is the equation of a circumference centered at $(0,0)$ and radius $\sqrt{2}$. The intersection points of the circumference with the parabola solve

$$\begin{cases} x^2 + y^2 &= 2, \\ x^2 - y &= 0, \end{cases}$$

Hence $y + y^2 = 2$, that is, $y = 1$ and hence $x^2 = 1$, or $x = 1$ (reject $x = -1$, since x cannot be negative, as well as reject $y = -2$, since $y = x^2 \geq 0$). The region A is the shadowed region in the figure below.



(b)

$$\iint_A x\,dx\,dy = \int_0^1 x\,dx \int_{x^2}^{\sqrt{2-x^2}} dy = \int_0^1 x(\sqrt{2-x^2} - x^2)\,dx$$
$$= \int_0^1 x\sqrt{2-x^2}\,dx - \int_0^1 x^3\,dx.$$

The first integral is immediate, or make the change of varaible $t = 2 - x^2$, $dt = -2x\,dx$. Thus

$$\int_0^1 x\sqrt{2-x^2}\,dx = \int_2^1 -\frac{1}{2}\sqrt{t}\,dt = \frac{1}{2}\int_1^2 \sqrt{t}\,dt = \frac{1}{3}t^{\frac{3}{2}}\bigg|_1^2 = \frac{1}{3}(2^{\frac{3}{2}} - 1) = \frac{1}{3}(2\sqrt{2} - 1).$$

Also, $\int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$. So the value of the integral is

$$\frac{1}{3}(2\sqrt{2} - 1) - \frac{1}{4} = \frac{2\sqrt{2}}{3} - \frac{7}{12}.$$

$\boxed{4}$

Study whether the following improper integrals are convergent or divergent. When convergent, calculate its value.

(a) (10 points)

$$\int_0^1 \frac{dx}{x^2+x}.$$

(b) (10 points)

$$\int_1^\infty \frac{2x+1}{(x^2+x)^3}\,dx.$$

Solution:

(a) Since

$$\lim_{x\to 0^+} \frac{\frac{1}{x^2+x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{x}{x^2+x} = \lim_{x\to 0^+} \frac{1}{x+1} = 1,$$

the integral has the same character than

$$\int_0^1 \frac{dx}{x},$$

which is divergent.

Another way is to see that $\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$, with $A = 1$ and $B = -1$. Hence

$$\int \frac{dx}{x^2+x} = \int \frac{dx}{x} - \int \frac{dx}{x+1} = \ln\left(\frac{x}{x+1}\right) + C.$$

Thus

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{x^2+x} = \lim_{a\to 0^+} \ln\left(\frac{x}{x+1}\right)\Big|_a^1 = \lim_{a\to 0^+} \left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{a}{a+1}\right)\right) = -\infty.$$

(b) Note that the numerator of the proposed integrand, $2x+1$, is the derivative of x^2+x, thus the integrand admits the primitive

$$-\frac{1}{2}\frac{1}{(x^2+x)^2}.$$

Hence

$$\int_1^\infty \frac{2x+1}{(x^2+x)^3}\,dx = \lim_{b\to\infty} -\frac{1}{2}\frac{1}{(x^2+x)^2}\Big|_1^b = \frac{1}{8} - \lim_{b\to\infty} \frac{1}{2}\frac{1}{(b^2+b)^2} = \frac{1}{8}.$$

$\boxed{5}$

(a) (10 points) Study the following limit

$$\lim_{n\to\infty} \frac{p^n}{1+p^n},$$

where $p > 0$ is a parameter. *Hint:* $p^n \to 0$ *if* $0 < p < 1$ *and* $p^n \to \infty$ *if* $p > 1$*, as* $n \to \infty$*.*

(b) (10 points) Study the character of the series

$$\sum_{n=1}^{\infty} \frac{p^n}{1+p^n},$$

where $p > 0$ is a parameter. *Hint: use part (a).*

Solution:

(a) When $p = 1$, the sequence $\frac{p^n}{1+p^n} = \frac{1}{2}$ is constant for all n, thus the limit is $\frac{1}{2}$. For values $p \neq 1$ we have

$$\lim_{n\to\infty} \frac{p^n}{1+p^n} = \frac{0}{1+0} \quad \text{if } 0 < p < 1.$$

$$\lim_{n\to\infty} \frac{p^n}{1+p^n} = \lim_{n\to\infty} \frac{1}{p^{-n}+1} = \frac{1}{0+1} = 1 \quad \text{if } p > 1.$$

(b) When $p \geq 1$ the general term does not converge to 0 (part (a)), thus the series is not convergent. It a series of positive terms. Applying the ratio test to the case $p < 1$:

$$\lim_{n\to\infty} \frac{\frac{p^{n+1}}{1+p^{n+1}}}{\frac{p^n}{1+p^n}} = \lim_{n\to\infty} \frac{p+p^{n+1}}{1+p^{n+1}} = \frac{p+0}{1+0} = p < 1,$$

thus the series converges if and only if $p < 1$.