UC3M
Mathematics for Economics II (second chance exam)
June 28, 2023

1

Given the parameter $m \in \mathbb{R}$, consider the matrix

$$A = \begin{pmatrix} 1 & m & m \\ m & 1 & m \\ m & m & 1 \end{pmatrix}$$

(a) (10 points) Study the rank of A, according to the values of m. Hint: if you need to find the roots of the determinant of A, use Ruffini's Rule. Also, note that $m = 1$ makes $|A| = 0$.

(b) (10 points) Since A is symmetric, it is the matrix of a quadratic form Q. Classify the quadratic form according to the values of m.

Solution:

(a) The determinant of A is (after making zeroes in the first column)

$$|A| = \begin{vmatrix} 1 & m & m \\ 0 & (1-m^2) & (m-m^2) \\ 0 & (m-m^2) & (1-m^2) \end{vmatrix} = (1-m^2)^2 - (m-m^2)^2 = 2m^3 - 3m^2 + 1.$$

By using Ruffini's rule twice, knowing that $m = 1$ is a root of $|A|$, we obtain $|A| = 2(m-1)^2(m+\frac{1}{2})$.

Case 1. $m \neq 1$ and $m \neq -\frac{1}{2}$: the rank is 3, since $|A| \neq 0$.

Case 2. $m = -\frac{1}{2}$: the rank is 2, since $|A| = 0$, but the minor of order 2 formed by lines 1 and 2 is not null, $\begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4} \neq 0$.

Case 3. $m = 1$: all rows (and columns) are equal, hence the rank is 1.

(b) Denote the principal minors by D_1, D_2 and D_3. Then $D_1 = 1 > 0$, $D_2 = 1 - m^2$ and D_3 was computed above, $D_3 = |A| = 2(m-1)^2(m+\frac{1}{2})$.

The only possibility for Q is to be positive, thus we impose $D_2 > 0$, that is $|m| < 1$.

The sign of D_3 is positive if and only if $(m+\frac{1}{2}) > 0$ and $m \neq 1$, that is $m > -\frac{1}{2}$ and $m \neq 1$.

Thus: Q is positive definite iff $-\frac{1}{2} < m < 1$. Clearly, it is positive semidefinite for $m = -\frac{1}{2}$ and for $m = 1$.

For the rest of values of m, Q is indefinite.

2

Given the parameter a, consider the matrix

$$A = \begin{pmatrix} a & 0 & 2a \\ -1 & -a & -1 \\ 2a & 0 & a \end{pmatrix}.$$

(a) (10 points) Study whether the matrix A is diagonalizable. For the values of the parameter a for which the matriz is diagonalizable, calculate its eigenvalues and eigenvectors.

(b) (10 points) In the cases in which the matrix A is diagonalizable, find a diagonal matrix D and a matrix P such that $P^{-1}AP = D$. Find P^{-1} explicitly

Solution:

(a) Characteristic polynomial: $-(\lambda+a)\left((\lambda-a)^2-4a^2\right)$.

Roots: $\lambda=-a$ and $\lambda=3a$. Note that $(\lambda-a)^2-4a^2=0$ iff $\lambda-a=\pm 2a$.

Case $a=0$: it appears the eigenvalue $\lambda=0$ with multiplicity 3. The matrix is not diagonalizable.

Case $a\neq 0$: $\lambda=-a$ is double and $\lambda=3a$ is simple.

Rank of matrix

$$A+aI_3=\begin{pmatrix} 2a & 0 & 2a \\ -1 & 0 & -1 \\ 2a & 0 & 2a \end{pmatrix}$$

is 1 for all $a\neq 0$, thus the matrix is diagonalizable for all $a\neq 0$.

The proper subspaces are: $S(-a)=\langle(0,1,0),(1,0,-1)\rangle$ and $S(3a)=\langle(1,-\frac{2}{a},1)\rangle$. This can be easily found from the systems below.

$$S(-a)\quad\left\{\begin{array}{rrl} 2ax & +2az & =0 \\ -x & -z & =0 \\ 2ax & +2az & =0 \end{array}\right. ;\qquad S(3a)\left\{\begin{array}{rrrl} -2ax & & +2az & =0 \\ -x & -4ay & -z & =0 \\ 2ax & & -2az & =0 \end{array}\right.$$

The solutions of the first system have $x=-z$ and y free (two parameters). The solutions of the second one have $x=z$ and then from the second equation, $y=-\frac{2}{a}z$ (one parameter).

(b) By the item above,

$$P=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -\frac{2}{a} \\ 0 & -1 & 1 \end{pmatrix},\qquad D=\begin{pmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 3a \end{pmatrix}$$

The inverse of P is

$$P^{-1}=-\frac{1}{2}\begin{pmatrix} -\frac{2}{a} & -2 & -\frac{2}{a} \\ -1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

3

Consider the triangular region

$$T=\{(x,y)\in\mathbb{R}^2\,:\,0\leq x\leq\sqrt{\pi},\ x\leq y\leq 2x\}.$$

(a) (10 points) Draw T and calculate its area by the method you wish.

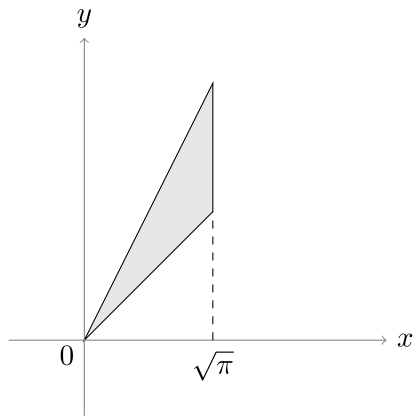
(b) (10 points) Calculate

$$\iint_T \sin\left(x^2\right)dxdy,$$

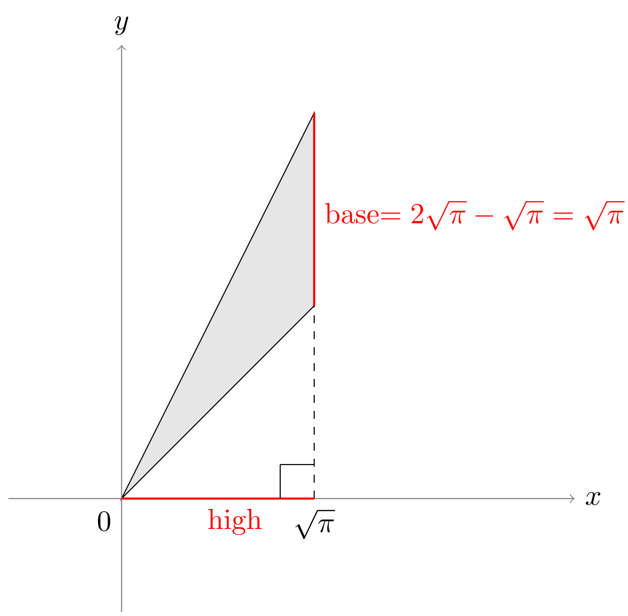
where T is the region considered above. Hint: here are some popular trigonometric values: $\sin 0=0$, $\sin\pi/4=\sqrt{2}/2$, $\sin\pi/2=1$, $\sin\pi=0$; $\cos 0=1$, $\cos\pi/4=\sqrt{2}/2$, $\cos\pi/2=1$, $\cos\pi=-1$.

Solution:

(a) The region T is represented below.



By elementary geometry, we know that the area of a triangle is one half base times high. In this case, the area of T is $\frac{\pi}{2}$. See the figure below.



Also, the area of T can be calculated as the double integral

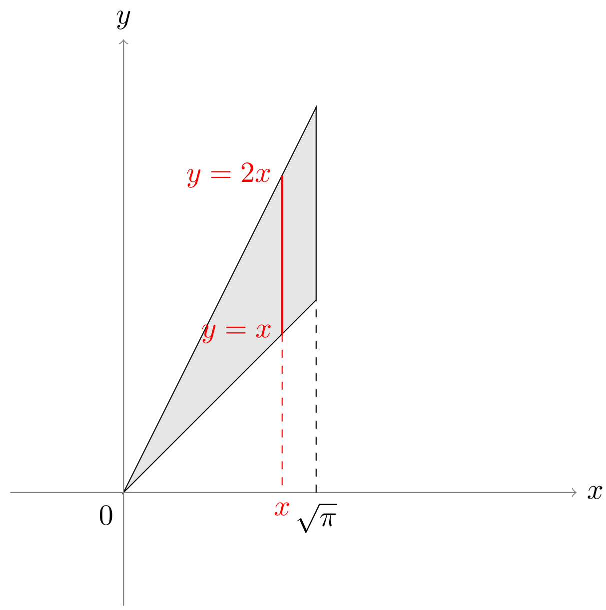
$$\iint_T 1\,dxdy = \int_0^{\sqrt{\pi}} \left(\int_x^{2x} 1\,dy \right) dx = \int_0^{\sqrt{\pi}} (2x - x)\,dx = \frac{x^2}{2}\bigg|_{x=0}^{x=\sqrt{\pi}} = \frac{1}{2}(\pi - 0) = \frac{\pi}{2}.$$

(b) It is better to integrate first with respect to y and then with respect to x, see the figure below.

$$\iint_S \sin(x^2)\,dxdy = \int_0^{\sqrt{\pi}} \sin(x^2) \int_x^{2x} dy\,dx = \int_0^{\sqrt{\pi}} (2x - x)\sin(x^2)\,dx = \int_0^{\sqrt{\pi}} x\sin(x^2)\,dx.$$

Noticing that a primitive of $x\sin(x^2)$ is $-\frac{1}{2}\cos(x^2)$, we obtain

$$\iint_S \sin(x^2)\,dxdy = -\frac{1}{2}\cos(x^2)\bigg|_{x=0}^{x=\sqrt{\pi}} = -\frac{1}{2}(\cos\pi - \cos 0) = 1.$$



4

(a) (10 points) Calculate the indefinite integral

$$I = \int \frac{1 + e^x}{1 - e^x}\, dx.$$

Hint: it may be helpful to change variable $t = e^x$

(b) (10 points) Study the convergence of the improper integral

$$I = \int_1^3 \frac{1}{\sqrt[3]{(x-1)^2}}\, dx.$$

In the case it is convergent, find its value.

Solution:

(a) $t = e^x$ transforms the integral to the rational integral

$$\int \frac{1+t}{1-t}\frac{1}{t}\, dt.$$

Then, simple fractions

$$\frac{1+t}{t(1-t)} = \frac{A}{t} + \frac{B}{1-t}$$

gives $A = 1$ and $B = 2$. Hence

$$\int \frac{1+t}{1-t}\frac{1}{t}\, dt = \int \frac{1}{t}\, dt + \int \frac{2}{1-t}\, dt = \ln t - 2\ln(1-t) + C.$$

Coming back to the original x

$$I = x - 2\ln(1 - e^x) + C.$$

(b)

$$I = \lim_{a \to 1^+} \int_a^3 \frac{1}{\sqrt[3]{(x-1)^2}}\, dx = \lim_{a \to 1^+} \left. \frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}} \right|_{x=a}^{x=3} = 3\sqrt[3]{2}.$$

5

(a) (10 points) Calculate the limit of the sequence $\{x_n\}_{n=1}^{\infty}$ with general term

$$x_n = \left(\frac{n}{1+n}\right)^{2n}.$$

(b) (10 points) Consider the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{2}}\right)^{n}.$$

Prove that it is convergent. If possible, calculate its sum. Hint: Expand a few terms of the series. Is it alternating, geometric, telescoping, ...? Answering these questions will help you to find the sum.

Solution:

(a) The limit is e^{-2}.

(b) Expanding the few first terms of the series

$$(*) \qquad \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^3 - \left(\frac{1}{\sqrt{2}}\right)^4 + \cdots$$

we see that it is geometric, with negative ratio $-\frac{1}{\sqrt{2}}$. Since the absolute value of the ratio is smaller than 1, the series is convergent. Moreover, the sum is equal to the first term over one minus the ratio, thus

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{2}}\right)^{n} = \frac{\frac{1}{\sqrt{2}}}{1-(-\frac{1}{\sqrt{2}})} = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}+1}{\sqrt{2}}} = \frac{1}{\sqrt{2}+1}\left(=\sqrt{2}-1\right).$$

Note: since the series is alternating, and the terms shown in (*), without the sign, form a decreasing sequence converging to zero, by Leibniz Theorem the series is convergent. However, it may be difficult to find its value if one does not realize that it is a geometric series.