Name: ____

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

Instructions:

- DURATION OF THE EXAM: 120'.
- \bullet Calculators are ${\bf NOT}$ allowed. Turn off your smart phone.
- DO NOT UNSTAPLE the exam.
- Please show a valid ID to the professor if required.
- Read the exam carefully. The exam has 5 questions, for a total of 100 points.
- Justify all your answers. You can use this sheet to write your answers.

Given the parameter $m \in \mathbb{R}$, consider the matrix

$$A = \left(\begin{array}{rrrr} 1 & m & m \\ m & 1 & m \\ m & m & 1 \end{array}\right)$$

- (a) (10 points) Study the rank of A, according to the values of m. Hint: if you need to find the roots of the determinant of A, use Ruffini's Rule. Also, note that m = 1 is a root of the determinant of A.
- (b) (10 points) Since A is symmetric, it is the matrix of a quadratic form Q. Classify the quadratic form according to the values of m.

Given the parameter a, consider the matrix

$$A = \left(\begin{array}{rrr} a & 0 & 2a \\ -1 & -a & -1 \\ 2a & 0 & a \end{array} \right).$$

- (a) (10 points) Study whether the matrix A is diagonalizable. For the values of the parameter a for which the matriz is diagonalizable, calculate its eigenvalues and eigenvectors.
- (b) (10 points) In the cases in which the matrix A is diagonalizable, find a diagonal matrix D and a matrix P such that $P^{-1}AP = D$. Find P^{-1} explicitly

Consider the triangular region

$$T = \{ (x, y) \in \mathbb{R}^2 : 0 \le x \le \sqrt{\pi}, \ x \le y \le 2x \}.$$

- (a) (10 points) Draw T and calculate its area by the method you wish.
- (b) (10 points) Calculate

$$\iint_T \sin\left(x^2\right) dx dy,$$

where T is the region considered above. Hint: here are some popular trigonometric values: $\sin 0 = 0$, $\sin \pi/4 = \sqrt{2}/2$, $\sin \pi/2 = 1$, $\sin \pi = 0$; $\cos 0 = 1$, $\cos \pi/4 = \sqrt{2}/2$, $\cos \pi/2 = 1$, $\cos \pi = -1$.

(a) (10 points) Calculate the indefinite integral

$$I = \int \frac{1 + e^x}{1 - e^x} \, dx.$$

Hint: it may be helpful to change variable $t = e^x$.

(b) (10 points) Study the convergence of the improper integral

$$I = \int_{1}^{3} \frac{1}{\sqrt[3]{(x-1)^2}} \, dx.$$

In the case it is convergent, find its value.

(a) (10 points) Calculate the limit of the sequence $\{x_n\}_{n=1}^{\infty}$ with general term

$$x_n = \left(\frac{n}{1+n}\right)^{2n}.$$

(b) (10 points) Consider the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{\sqrt{2}}\right)^n.$$

Prove that it is convergent. If possible, calculate its sum. Hint: Expand a few terms of the series. Is it alternating, geometric, telescoping, \ldots ? Answering these questions will help you to find the sum.