

# Chapter 1

## Introduction to Game Theory. Normal Form Games

September 2, 2024

### 1 Normal Form Games

A normal form Game  $G$  consists of three elements  $G = (I, S, u)$ ; where

1.  $I = \{1, 2, \dots, n\}$  is the set of decision makers, the **players**.
2.  $S = S_1 \times S_2 \times \dots \times S_n$  describes the feasible **actions** of the players. The players choose a strategy  $s = (s_1, s_2, \dots, s_n) \in S$  **simultaneously**. An outcome is obtained.
3.  $u = (u_1, u_2, \dots, u_n)$  describes the satisfaction (the **payoffs**) of the players with the outcomes. For each  $i = 1, 2, \dots, n$ ,  $u_i : S \rightarrow \mathbb{R}$ .

*Observation 1.1.* In the course, almost all the time  $I = 2$ .

**Example 1.2** (Prisoners' Dilemma). Two suspects are arrested by the police and kept in separated cells. They cannot communicate with each other. The police have enough evidence to convince both of them of crime  $A$ . However, the police 'know' that they have committed a much more serious crime, crime  $B$ . Unfortunately, the police lack the evidence to convict them of crime  $B$ . The police interrogate each of them separately and offers the possibility of confessing ( $C$ ) crime  $B$  and implicating the other partner. Thus, each of the suspects has to decide whether to confess crime  $B$  and implicate his partner or remain silent ( $N$ ). If one of them confesses and the other does not, then the one that has confessed goes free and his partner is sentenced to 6 years in prison. If both of them confess, they are sentenced to 5 years in prison each (the judge takes into account that they have collaborated). If both of them remain silent, the police use the evidence they have to convict them for crime  $A$  and they are both sentenced to 1 year in prison each.

Why does this fit into the framework of a normal form game? Because,

1. The **players** are the suspects.  $I = \{1, 2\}$ .
2. Their feasible actions are to confess ( $C$ ) or remain silent ( $N$ ). Thus,  $S_1 = S_2 = \{C, N\}$  describe their **strategies**.
3. The **payoffs** may be represented by

$$\begin{array}{ll}
 u_1(C, C) & = -5 & u_2(C, C) & = -5 \\
 u_1(C, N) & = 0 & u_2(C, N) & = -6 \\
 u_1(N, C) & = -6 & u_2(N, C) & = 0 \\
 u_1(N, N) & = -1 & u_2(N, N) & = -1
 \end{array}$$

The above is a bit cumbersome. From now we will use the following notation.

		Suspect 2	
		$N$	$C$
Suspect 1	$N$	-1, -1	-6, 0
	$C$	0, -6	-5, -5

The prisoner's dilemma

What do you think it will happen?

**Example 1.3** (Advertising). Two firms operate in a market. Presently, each one of them earns \$50 million from its customers. Now both have to decide whether to advertise ( $A$ ) their product or not ( $N$ ). Advertising costs each firm \$20 million. If one firm advertises and the other does not, the former captures \$30 million from the latter. If both firms advertise, they both earn \$50 million.

The above situation can be described as a normal form game with

1. The **players** are the firms.  $I = \{1, 2\}$ .
2. Their feasible actions are to advertise ( $A$ ) or not to advertise ( $N$ ). Thus  $S_1 = S_2 = \{A, N\}$  describe their **strategies**.
3. The **payoffs** (profits) may be represented by

		Firm 2	
		$N$	$A$
Firm 1	$N$	50, 50	20, 60
	$A$	60, 20	30, 30

Advertising

Shi Qi: 'The impact of Advertising Regulation on Industry. The cigarette Advertising Ban of 1971', Rand Journal (2013).

**Example 1.4** (Tourists and Natives). Two bars compete for customers. They can charge prices \$2, \$4 or \$5 for coffee. 6000 tourists choose a bar randomly. 4000 natives choose the bar with the lowest price.

We write this as a normal form game.

1. The **players** are bar 1 and bar 2.  $I = \{1, 2\}$ .
2. Their feasible actions are the prices. Thus the **strategies** are  $S_1 = S_2 = \{2, 4, 5\}$ .
3. The **payoffs** (profits) may be represented by

		Bar 2		
		2	4	5
Bar 1	2	10, 10	14, 12	14, 15
	4	12, 14	20, 20	28, 15
	5	15, 14	15, 28	25, 25

Tourists and Natives

## 2 Equilibrium

What does Game Theory predict in each of the above situations? (Standard) Game Theory makes the following assumptions.

- Rationality.
  1. Players maximize their payoffs.
  2. Players can make all the necessary inferences and computations.
- Common knowledge.
  1. Every player knows the rules of the game.
  2. Every player knows that every other player knows (1).

3. Every player knows that every other player knows (2).
4. Etc.

**Example 2.1. The power of common knowledge.**

1.  $n = 3$  people sit in a circle. Each of them wears a red (R) hat.
2. They are numbered: person 1, person 2 and person 3.
3. All agents are rational.
4. Each person sees the hat of the other  $n - 1 = 2$  people. But not his own.
5. They all know that the hats can be either red (R) or white (W). This is common knowledge.
6. Person 1 is asked if he knows the color of his hat. He answers sincerely. What is his answer?
7. Person 2 is asked if he knows the color of his hat. He answers sincerely. What is his answer?
8. ...
9. All the above is common knowledge.
10. Person  $n = 3$  is asked if he knows the color of his hat. He answers sincerely. What is his answer?

Suppose now that before they start asking the teacher says:

At least one of you is wearing a red hat.

Repeat the above, does anything change?

**Some notation:** Fix a player  $i = 1, 2, \dots, n$ . We write the strategies by

$$s = (s_1, s_2, \dots, s_n) = (s_i, s_{-i})$$

where

$$s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$$

## 2.1 Dominant strategies and Dominance Solvability.

**Definition 2.2.** Let a game  $G = (I, S, u)$ . Fix a player  $i = 1, 2, \dots, n$  and consider two actions  $a_i, b_i \in S_i$  for that player. We say that

- action  $a_i$  **weakly dominates** action  $b_i$  (or that action  $b_i$  is **weakly dominated** by action  $a_i$ ) for player  $i$  if

$$u_i(a_i, s_{-i}) \geq u_i(b_i, s_{-i}), \quad \text{for every } s_{-i} \in S_{-i}$$

and

$$u_i(a_i, s_{-i}) > u_i(b_i, s_{-i}), \quad \text{for some } s_{-i} \in S_{-i}.$$

- item action  $a_i$  **strictly dominates** action  $b_i$  (or that action  $b_i$  is **strictly** by action  $a_i$ ) for player  $i$  if

$$u_i(a_i, s_{-i}) > u_i(b_i, s_{-i}), \quad \text{for every } s_{-i} \in S_{-i}.$$

**Definition 2.3.** Let a normal form game  $G = (I, S, u)$ . Fix a player  $i = 1, 2, \dots, n$ . An action  $s_i \in S_i$  is

- **weakly dominant** if weakly dominates every other action in  $S_i$ .
- **strictly dominant** if strictly dominates every other action in  $S_i$ .

Examples.

In this course ‘dominant’ means ‘strictly dominant’.

### Iterated Elimination of Strictly Dominated Strategies (IESD):

Start with a normal form game  $G_0$ .

- Choose a player and remove all the strictly dominated strategies for that player. We obtain a new game  $G_1$ .
- Consider now the game  $G_1$ . Choose a player and remove all the strictly dominated strategies for that player. We obtain a new game  $G_2$ .
- Repeat the process until there are no strictly dominated strategies in the remaining game, say  $G_n$ .

The remaining strategies in the game  $G_n$  are called the **rationalizable** strategies.

**Definition 2.4.** A normal form game is **dominance solvable** if the IESD procedure yields a unique outcome.

**Example 2.5** (Stug Hunt Game). Two firms operate in a market. Presently, each one of them earns \$45 million from its customers. Now both have to decide whether to invest ( $I$ ) all the money in R&D or not ( $N$ ). However, R&D is successful only if both firms invest. If R&D is successful the net profit for each firm is \$50 million.

The above situation can be described as a normal form game with

1. The **players** are the firms,  $I = \{1, 2\}$ .
2. Their feasible actions are to invest ( $I$ ) or not to invest ( $N$ ). Thus the **strategies** are  $S_1 = S_2 = \{I, N\}$ .
3. The **payoffs** (profits) may be represented by the following table.

		Firm 2	
		$I$	$N$
Firm 1	$I$	50, 50	0, 45
	$N$	45, 0	45, 45

Stug Hunt Game

Every strategy is rationalizable.

## 2.2 Nash Equilibrium in pure strategies.

**Definition 2.6** (Best response for pure strategies). Let a game  $G = (I, S, u)$ . Fix a player  $i = 1, 2, \dots, n$ . Suppose he knows that all the other players are playing the strategy  $s_{-i}$ . We say an action  $s_i \in S_i$  is a best response for player  $i$  to  $s_{-i}$  (and we write  $s_i \in \text{BR}_i(s_{-i})$  if

$$u_i(s_i, s_{-i}) \geq u_i(a_i, s_{-i}), \quad \text{for every other } a_i \in S_i$$

**Definition 2.7** (pure strategy Nash Equilibrium). Let a game  $G = (I, S, u)$ . A strategy profile  $s^* = (s_1^*, s_2^*, \dots, s_n^*)$  is a (pure strategy) Nash Equilibrium of  $G$  if for every player  $i = 1, 2, \dots, n$  we have that

$$s_i^* \in \text{BR}_i(s_{-i}^*).$$

**Theorem 2.8** (Nash’s Existence Theorem. Version 1). Let  $G = (I, S, u)$  be a strategic form game such that for each  $i \in I$ ,  $S_i$  is a finite set. Then, the set of NE is non-empty.

**Theorem 2.9** (Nash's Existence Theorem. Version 2). *Let  $G = (I, S, u)$  be a strategic form game such that for each  $i \in I$ ,*

- $S_i$  is a nonempty, convex and compact subset of  $\mathbb{R}^{m_i}$ ;
- $u_i$  is continuous on  $S$ ; and
- $u_i$  is quasi-concave on  $S_i$ .

*Then, the set of NE is non-empty.*

**Example 2.10** (Chicken Game). Two drivers drive towards each other in a one lane road. Each driver has to decide whether to swerve ( $S$ ) or not ( $N$ ). If none of them changes the direction, they both collide.

		Driver 2	
		$S$	$N$
Driver 1	$S$	0, 0	-1, 1
	$N$	1, -1	-100, -100

Chicken Game

**Example 2.11** (Matching pennies). Two individuals have a coin each. They simultaneously choose heads tails and show their coin. If both coins are matched player 1 pays \$1 to player 2. Otherwise, player 2 pays \$1 to player 1.

		Player 2	
		$H$	$T$
Player 1	$H$	-1, 1	1, -1
	$T$	1, -1	-1, 1

Matching pennies

No NE in pure strategies.

### Some Remarks about NE.

1. A NE is part of the rationalizable strategies.
2. If the IESD procedure yields a unique strategy for each player, then these strategies constitute the unique NE of the game.
3. However a NE may include weakly dominated strategies.

		Player 2	
		$H$	$T$
Player 1	$H$	0, 0	0, 0
	$T$	0, 0	1, 1

4. The unique NE of a game may not be Pareto Optimal. (Prisoner's Dilemma).
5. A game may have more than one NE in pure strategies. (Stug Hunt Game).
6. A game may not have any NE in pure strategies. (Matching pennies).

### 2.3 Nash Equilibrium in mixed strategies.

**Definition 2.12** (Mixed Strategies). Let a game  $G = (I, S, u)$ . Fix a player  $i = 1, 2, \dots, n$ . Suppose the set of strategies  $S_i = \{s_1^i, \dots, s_{k^i}^i\}$  of agent  $i$  has  $k^i$  elements. A mixed strategy for player  $i$  is a probability distribution  $\sigma^i = (\alpha_1^i, \dots, \alpha_{k^i}^i) \in \mathbb{R}^{k^i}$  on  $S_i$  such that

- $\alpha_1^i, \dots, \alpha_{k^i}^i \geq 0$ .
- $\alpha_1^i + \dots + \alpha_{k^i}^i = 1$ .

Instead of  $\sigma^i = (\alpha_1^i, \dots, \alpha_{k^i}^i)$  we will write

$$\sigma^i = \alpha_1^i s_1^i + \dots + \alpha_{k^i}^i s_{k^i}^i$$

We interpret that player  $i$  plays the strategy  $s_l^i$  with probability  $\alpha_l^i$ . We denote by  $\Delta_i$  the set of mixed strategies for player  $i$ .

- Examples.
- Notation.
- The payoffs extend to mixed strategies in a linear way. Payoffs are expected payoffs.

That is, Let  $(\sigma^1, \dots, \sigma^n)$  a mixed strategy profile. Then

$$u_i(\sigma) = \sum_{l_1=1}^{k_1} \sum_{l_2=1}^{k_2} \dots \sum_{l_n=1}^{k_n} \alpha_{l_1}^1 \alpha_{l_2}^2 \dots \alpha_{l_n}^n u_i(s_{l_1}^1, s_{l_2}^2, \dots, s_{l_n}^n)$$

**Definition 2.13** (Best response for mixed strategies). Let a game  $G = (I, S, u)$ . Fix a player  $i = 1, 2, \dots, n$ . Suppose he knows that all the other players are playing the mixed strategy  $\sigma_{-i}$ . We say a mixed strategy  $\sigma_i \in \Delta_i$  is a best response for player  $i$  to  $\sigma_{-i}$  (and we write  $\sigma_i \in \text{BR}_i(\sigma_{-i})$  if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\bar{\sigma}_i, \sigma_{-i}), \quad \text{for every other mixed (or pure) strategy } \bar{\sigma}_i \in \Delta_i$$

**Definition 2.14** (Mixed strategy Nash Equilibrium). Let a game  $G = (I, S, u)$ . A strategy profile  $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ , consisting of mixed strategies, is a (mixed strategy) Nash Equilibrium of  $G$  if for every player  $i = 1, 2, \dots, n$  we have that

$$\sigma_i^* \in \text{BR}_i(\sigma_{-i}^*).$$

**Definition 2.15** (Nash Equilibrium (NE)). Let a normal game  $G = (I, S, u)$ . A Nash Equilibrium of  $G$  is either a pure or a mixed strategy Nash Equilibrium of  $G$ .

*Observation 2.16* (A property of mixed strategy Nash Equilibria). Suppose  $\sigma^*$  is a mixed strategy NE. Then for each player  $i \in N$ ,

1. given  $\sigma_{-i}^*$ , player  $i$  obtains the same payoff, say  $v_i$ , with any action to which  $\sigma_i^*$  assigns **strictly positive probability**. That is player  $i$  is indifferent between any of the strategies that he uses with **strictly positive probability** in the NE  $\sigma^*$ .
2. given  $\sigma_{-i}^*$ , player  $i$  obtains at most the payoff  $v_i$  with any action to which  $\sigma_i^*$  assigns **zero probability**.

*Observation 2.17* (Existence Nash Equilibria). Every finite normal game has at least one NE.

### 3 Examples

**Example 3.1** (Employee monitoring). An employee can work ( $W$ ) or shirk ( $S$ ). His salary is \$100 unless he is caught shirking. In the latter case the salary is \$0. The cost of working for the employee is \$50.

The manager may monitor ( $M$ ) the employee or not ( $N$ ). The cost of monitoring is \$10. The value of the employee's work for the manager is \$200 if the employee works and \$0 if he shirks.

		Manager	
		$M$	$N$
Employee	$S$	0, -10	100, -100
	$W$	50, 90	50, 100

Employee monitoring

Note that there are no NE in pure strategies. Let us look for a NE in mixed strategies of the form

$$\begin{aligned}\sigma_1 &= xS + (1-x)W, & 0 \leq x \leq 1 \\ \sigma_2 &= yM + (1-y)N & 0 \leq y \leq 1\end{aligned}$$

The expected utility of the players in this equilibrium are

$$\begin{aligned}u_1(\sigma_1, \sigma_2) &= 0 \times xy + 100x(1-y) + 50y(1-x) + 50(1-x)(1-y) = 50 + x(50 - 100y) \\ u_2(\sigma_1, \sigma_2) &= -10xy - 100x(1-y) + 90y(1-x) + 100(1-x)(1-y) = 100 - 200x + y(100x - 10)\end{aligned}$$

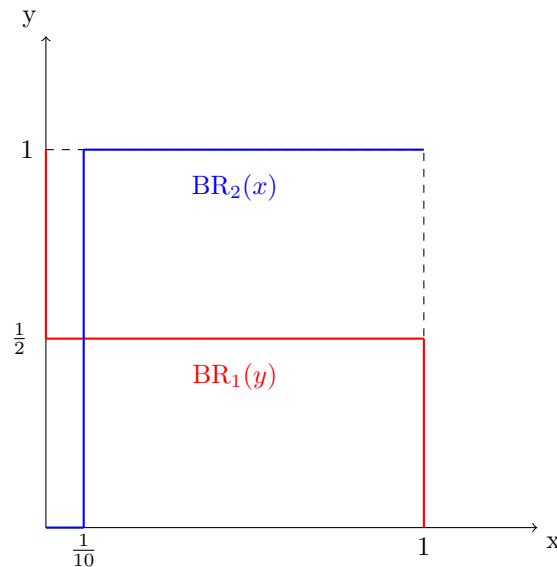
From this, we see that

$$BR_1(y) = BR_1(\sigma_2) = \begin{cases} x = 0 & \text{if } y > 1/2 \\ x \in [0, 1] & \text{if } y = 1/2 \\ x = 1 & \text{if } y < 1/2 \end{cases}$$

and

$$BR_2(x) = BR_2(\sigma_1) = \begin{cases} y = 0 & \text{if } x < 1/10 \\ y \in [0, 1] & \text{if } x = 1/10 \\ y = 1 & \text{if } x > 1/10 \end{cases}$$

Graphically,



The intersection of the graphs yields the Nash equilibria of the game. The unique NE is  $x^* = \frac{1}{10}$ ,  $y = \frac{1}{2}$ . That is,

$$\begin{aligned}\sigma_1^* &= \frac{1}{10}S + \frac{9}{10}W, \\ \sigma_2^* &= \frac{1}{2}M + \frac{1}{2}N\end{aligned}$$

The expected payoffs are  $u_1 = \$50$  for the employee and  $u_2 = \$80$  for the manager.

*Observation 3.2.*

1. No NE in pure strategies.
2. A unique NE in mixed strategies. Employee shirks with probability  $1/10$ . Manager monitors with probability  $1/2$ . The expected payoffs are \$50 for the employee and \$80 for the manager.
3. **Both players are indifferent between their strategies. Each player chooses the mixed strategy that makes his opponent indifferent. This yields a general procedure to compute the NE in mixed strategies.**

**Example 3.3** (Natural Monopoly). Two firms consider entering a market. The market generates total profits of \$300. There is a fixed cost of entry of \$200. If both firms enter the market they share equally the profits.

		Firm 2	
		<i>I</i>	<i>O</i>
Firm 1	<i>I</i>	-50, -50	100, 0
	<i>O</i>	0, 100	0, 0

Natural Monopoly

Note that there are two NE in pure strategies:  $(I, O)$  and  $(O, I)$ . Let us look for a NE in mixed strategies of the form

$$\begin{aligned}\sigma_1 &= xI + (1-x)O, \quad 0 \leq x \leq 1 \\ \sigma_2 &= yI + (1-y)O \quad 0 \leq y \leq 1\end{aligned}$$

The expected utility of the players in this equilibrium are

$$\begin{aligned}u_1(\sigma_1, \sigma_2) &= -50xy + 100x(1-y) + 0 \times y(1-x) + 0 \times (1-x)(1-y) = 100x - 150xy = 10x(10 - 15y) \\ u_2(\sigma_1, \sigma_2) &= -50xy + 0 \times x(1-y) + 100y(1-x) + 0 \times (1-x)(1-y) = 100y - 150xy = 10y(10 - 15x)\end{aligned}$$

From this, we see that

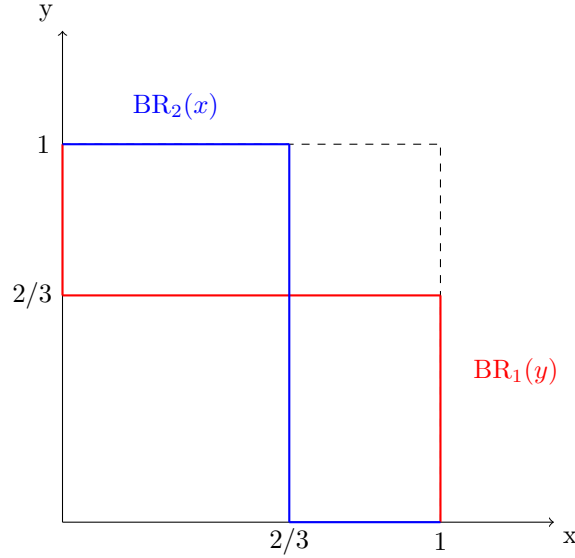
$$BR_1(y) = BR_1(\sigma_2) = \begin{cases} x = 1 & \text{if } y < 2/3 \\ x \in [0, 1] & \text{if } y = 2/3 \\ x = 0 & \text{if } y > 2/3 \end{cases}$$

and

$$BR_2(x) = BR_2(\sigma_1) = \begin{cases} y = 1 & \text{if } x < 2/3 \\ y \in [0, 1] & \text{if } x = 2/3 \\ y = 0 & \text{if } x > 2/3 \end{cases}$$

Graphically,





The intersection of the graphs yields the Nash equilibria of the game. There are three NE,

(a)  $x^* = 0, y = 1$ . That is,

$$\begin{aligned}\sigma_1 &= I \\ \sigma_2 &= O\end{aligned}$$

with payoffs  $u_1 = \$100, u_2 = \$0$ .

(b)  $x^* = 1, y = 0$ . That is,

$$\begin{aligned}\sigma_1 &= O \\ \sigma_2 &= I\end{aligned}$$

with payoffs  $u_1 = \$0, u_2 = \$100$ .

(c)  $x^* = y = \frac{2}{3}$ . That is,

$$\begin{aligned}\sigma_1 &= \frac{2}{3}I + \frac{1}{3}O, \quad 0 \leq x \leq 1 \\ \sigma_2 &= \frac{2}{3}I + \frac{1}{3}O \quad 0 \leq y, \leq 1\end{aligned}$$

with payoffs  $u_1 = u_2 = \$0$ .

*Observation 3.4.* Note that in the mixed NE we have the following two types of (bad) situations

1. Coordination failure: Both firms enter the market. The probability that this happens is

$$\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$$

2. Loss of opportunity: No firm enters the market. The probability that this happens is

$$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

If both firm coordinate on a coin (for example, firm 1 enters when heads and firm 2 enters when heads) the expected profits are

$$\frac{1}{9} \times 0 + \frac{1}{9} \times 100 = 50$$

## 4 Applications

## 5 Credibility and Commitment

- Two firms: Firm 1 and Firm 2;
- Two prices: L (\$4) or H (\$5 );
- 3000 captive consumers per firm;
- 4000 floating consumers who go to the firm with the lowest price.

The payoffs are as follows

		Firm 2	
		<i>L</i>	<i>H</i>
Firm 1	<i>L</i>	20, 20	28, 15
	<i>H</i>	15, 28	25, 25

Non Commitment

The situation may be described as a prisoner's dilemma. Both firms prefer  $(H, H)$  but the unique NE is  $(L, L)$ . Firms cannot credibly commit to play  $H$ . If one firm commits to  $H$ , the other firm should respond with  $L$ . How to resolve this?

One possibility is a **low price guarantee**. Each firm announces that it will match the lowest price in the market. Now the situation is as follows

		Firm 2	
		<i>L</i>	<i>H</i>
Firm 1	<i>L</i>	20, 20	20, 20
	<i>H</i>	20, 20	25, 25

With Commitment

Now  $(H, H)$  is a NE with payoff  $u_1 = u_2 = 25$ .

### 5.1 Bertrand Duopoly with Differentiated Products

We now study the case in which firms offer different products to the market; each firm produces a different good. Firms choose prices.

Suppose there are two firms: firm 1 and firm 2. Each firm chooses the price for its product without knowing the price the other firm has chosen. Prices are denoted by  $p_1$  and  $p_2$ , respectively. The cost to firm  $i$  of producing quantity  $q_i$  is  $c_i(q_i) = cq_i$ .

Products are differentiated. That is, if the prevailing prices in the market are  $(p_1, p_2)$  the quantity that consumers demand from firm 1 is

$$q_1(p_1, p_2) = \max\{0, a - p_1 + bp_2\}$$

and the quantity that consumers demand from firm 2:

$$q_2(p_1, p_2) = \max\{0, a - p_2 + bp_1\}$$

with  $0 \leq b < 2$ . Thus, products are different, but they compete in a single market: low prices of the rival's product lowers the demand for my own product.

We may study this situation as a normal game in which players have a continuum of strategies:

1. The set of players is  $I = \{\text{Firm 1}, \text{Firm 2}\}$ .
2. The sets of strategies are  $S_1 = S_2 = [0, +\infty)$ .
3. The payoff functions are

$$\begin{aligned} u_1(p_1, p_2) &= q_1(p_1, p_2)(p_1 - c) \\ u_2(p_1, p_2) &= q_2(p_1, p_2)(p_2 - c) \end{aligned}$$

If firms choose prices such that

$$a - p_1 + bp_2 \geq 0, \quad a - p_2 + bp_1 \geq 0$$

then

$$\begin{aligned} u_1(p_1, p_2) &= (a - p_1 + bp_2)(p_1 - c) \\ u_2(p_1, p_2) &= (a - p_2 + bp_1)(p_2 - c) \end{aligned}$$

We compute now the NE of the above game. This consists of price pair  $(p_1^*, p_2^*)$  such that  $p_1^*$  is Firm 1's best response to Firm 2's price  $p_2^*$ , and  $p_2^*$  is Firm 2's best response to Firm 1's price  $p_1^*$ . That is,  $p_1^*$  is a solution of

$$\max_{p_1 \geq 0} u_1(p_1, p_2^*) = (a - p_1 + bp_2^*)(p_1 - c)$$

and  $p_2^*$  is a solution of

$$\max_{p_2 \geq 0} u_2(p_1^*, p_2) = (a - p_2 + bp_1^*)(p_2 - c)$$

The FOC for firm 1 is

$$\frac{\partial u_1(p_1, p_2^*)}{\partial p_1} = 0$$

That is,  $a + c - 2p_1 + bp_2 = 0$  so, we must have

$$p_1^* = \frac{a + c + bp_2^*}{2}$$

Likewise, the FOC for firm 2 yields

$$p_2^* = \frac{a + c + bp_1^*}{2}$$

Note that the SOC hold for both firms

$$\frac{\partial^2 u_1(p_1, p_2^*)}{\partial p_1^2} = \frac{\partial^2 u_2(p_1, p_2^*)}{\partial p_2^2} = -2 < 0$$

Therefore, if  $(p_1^*, p_2^*)$  is a NE we must have that

$$\begin{aligned} p_1^* &= \frac{a + c + bp_2^*}{2} \\ p_2^* &= \frac{a + c + bp_1^*}{2} \end{aligned}$$

We obtain

$$p_1^* = p_2^* = \frac{a + c}{2 - b}$$

## 5.2 Public goods

Two agent have an initial wealth  $w_1 > 1$ ,  $w_2 > 1$ . They have to contribute part of their wealth towards a public good. If agent  $i = 1, 2$  contributes the amount  $c_i$  towards the public good, the utilities of the agents is

$$u_1(c_1, c_2) = \ln(2c_1 + c_2) - c_1, \quad u_2(c_1, c_2) = \ln(c_1 + 2c_2) - c_2$$

We first describe the situation as a static game. The players are  $N = \{1, 2\}$ . Their strategy sets are  $S_1 = [0, w_1]$ ,  $S_2 = [0, w_2]$ . And their payoffs are represented by the utility functions above.

We now compute the Nash equilibria of the game. Let us start with the best reply of player 1. We have to compute the solution of the following maximization problem.

$$\max_{c_1} \ln(2c_1 + c_2) - c_1$$

The first order condition is

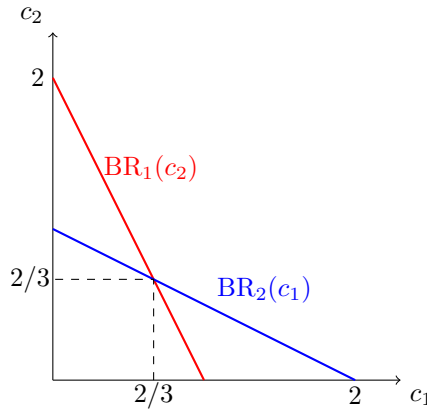
$$\frac{2}{2c_1 + c_2} = 1$$

So,

$$BR_1(c_2) = \frac{2 - c_2}{2}$$

Similarly the best reply of player 2 is

$$BR_2(c_1) = \frac{2 - c_1}{2}$$



And the NE have to satisfy the equations

$$\begin{aligned} c_1 &= \frac{2 - c_2}{2} \\ c_2 &= \frac{2 - c_1}{2} \end{aligned}$$

The solution is

$$c_1^* = c_2^* = \frac{2}{3}$$

The utilities are  $u_1^* = u_2^* = \ln(5/3) - 2/3 \approx 0.026$

The social optimal, given by  $\max_{c_1, c_2} (u_1(c_1, c_2) + u_2(c_1, c_2))$ , is given by the first order conditions

$$\begin{aligned} \frac{2}{2c_1 + c_2} + \frac{1}{c_1 + 2c_2} &= 1 \\ \frac{1}{2c_1 + c_2} + \frac{2}{c_1 + 2c_2} &= 1 \end{aligned}$$

The solution is  $\bar{c}_1 = \bar{c}_2 = 1$ . The utilities are  $\bar{u}_1 = \bar{u}_2 = \ln(3) - 1 \approx 0.098$ . Under the NE the production of the public good is below the optimum.

### 5.3 Hotelling/Downs model of electoral competition

Proposed by Hotelling (1929) and Downs (1957).

**Voters:**

- Each voter  $v$  has a favorite policy on  $v \in [0, 1]$ ; She has single-peaked preference and her utility decreases as the winner's position is further away from her favorite policy. For example,  $u_v(t) = -|v - t|$ .  
Each voter will vote sincerely, choosing the party whose position is closest to her favorite policy.
- Voters are distributed according to a distribution function  $F(v)$  defined on  $[0, 1]$ .
  - $F(0) = 0$ .
  - $F$  is strictly increasing  $[0, 1]$ .
  - $F(1) = 1$ .

**Political parties:**

- There are 2 political parties.
- Parties compete by choosing a policy on the interval  $t_1, t_2 \in [0, 1]$ .
- The party with most votes wins; if there is a draw, each party has a 50% chance of winning.
- Parties only care about winning, and will commit to the platforms they have chosen.

**Median Voter Theorem:** Let  $m$  be the median voter position,  $F(m) = 1/2$ . There is a unique NE:  $(m, m)$ .

## 6 Reporting a crime (From the book of Martin J. Osborne)

A crime is observed by  $n$  people. Each person would like the police to be informed, but prefers that somebody else goes through the trouble of informing the police.

We assume that the utility of each agent  $i = 1, \dots, n$  is

- $v > 0$  if somebody else informs the police.
- $v - c > 0$  if agent  $i$  informs the police.
- 0 if nobody informs the police.

This is a game with  $N = \{1, \dots, n\}$  agents. The set of strategies of agent  $i \in N$  is  $S_i = \{I, N\}$ . This game has  $n$  NE. In each of these NE one of the agents informs the police and no other does. Why?

Let us look for a **symmetric** NE in mixed strategies of the form

$$\sigma = \sigma_i = pI + (1 - p)N$$

Then player  $i$  is indifferent between  $I$  and  $N$ . Since,

$$\begin{aligned} u_i(I, \sigma_{-i}) &= v - c \\ u_i(N, \sigma_{-i}) &= 0 \cdot \Pr\{\text{no other agent calls}\} + v \cdot (1 - \Pr\{\text{no other agent calls}\}) \\ &= v \cdot (1 - \Pr\{\text{no other agent calls}\}) \end{aligned}$$

we must have

$$v - c = v \cdot (1 - \Pr\{\text{no other agent calls}\})$$

That is

$$\frac{c}{v} = \Pr\{\text{no other agent calls}\} = (1 - p)^{n-1}$$

So,

$$p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$$

Note that  $0 < c/v < 1$ , so  $0 < p < 1$ . We conclude that there is a unique symmetric NE in mixed strategies of the form

$$\sigma_i = pI + (1 - p)N, \quad p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}, \quad i = 1, \dots, n$$

Note that,

1. Since,

$$\lim_{n \rightarrow \infty} \left(\frac{c}{v}\right)^{\frac{1}{n-1}} = 1$$

and  $\left(\frac{c}{v}\right)^{\frac{1}{n-1}}$  is increasing we have that, as  $n$  increases, the probability that a given agent  $i$  reports decreases to 0.

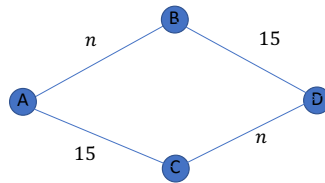
2. The probability that in the NE nobody reports to the police is

$$\begin{aligned} \Pr\{\text{no agent calls}\} &= \Pr\{\text{agent } i \text{ does not call}\} \cdot \Pr\{\text{no other agent calls}\} \\ &= (1 - p) \cdot \left(\frac{c}{v}\right) \end{aligned}$$

Since  $p$  decreases to 0 with  $n$ , we see that  $\Pr\{\text{no agent calls}\}$  increases to  $\frac{c}{v}$  with  $n$ . That is, the larger the group of witnesses the less likely it is that the police are informed about the crime.

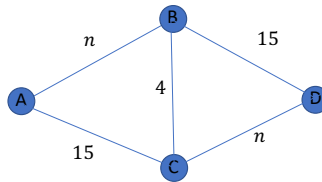
## 7 Braess's Paradox.

Originally formulated by Dietrich Braess, a mathematician at Ruhr University, Germany. The exposition here borrows heavily from the article in Wikipedia.

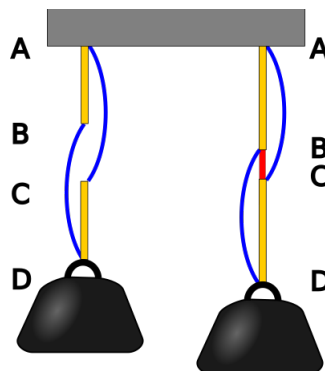


- There is a continuous flow of (10 cars per hour) that travel from  $A \mapsto B$ .
- The roads from  $B \mapsto D$  and from  $A \mapsto C$  are very wide. They are never congested. But They are long. They take each 15 hours of travel time, no matter how much traffic.
- The roads from  $A \mapsto B$  and from  $C \mapsto D$  are very short, but narrow. They get easily congested. If  $n$  cars travel in those roads, it takes  $n$  hours to travel them.
- What is the NE?
- We must have the same number of cars in the route  $A \mapsto B$ , than in the route  $C \mapsto D$ . Otherwise, some drivers would prefer to switch to the faster route.
- Hence, the NE is 5 cars take the route  $A \mapsto B$  and 5 cars take the route  $C \mapsto D$ . Total travel time is 20 hours.

- Suppose now a new fast route is opened between  $B$  and  $C$ . This route takes 4 hours, regardless of the traffic.



- What is the new NE?
- No one would want to use the route  $A \mapsto C$ , because the route  $A \mapsto B \mapsto C$  takes at most 14 hours. Thus, all traffic goes through  $B$ .
- Likewise, no one would want to use the route  $B \mapsto D$ , because the route  $B \mapsto C \mapsto D$  takes at most 14 hours.
- The new NE is everybody travels the route  $A \mapsto B \mapsto C \mapsto D$ . Total travel time is 24 hours.
- Wikipedia documents the following real life examples.
  1. In Stuttgart, Germany, after investments into the road network in 1969, the traffic situation did not improve until a section of newly built road was closed for traffic again.
  2. In Seoul, South Korea, traffic around the city sped up when a motorway was removed.
  3. In 1990 the temporary closing of 42nd Street in Manhattan, New York City, for Earth Day reduced the amount of congestion in the area.
  4. In 2009, New York experimented with closures of Broadway at Times Square and Herald Square, which resulted in improved traffic flow and permanent pedestrian plazas.
  5. In 2012 in Rouen, a bridge was destroyed by fire. Over the next two years, other bridges were used more, but the total number of cars crossing bridges was reduced.
  - 6.
- Wikipedia also shows instances where the same phenomenon may occur in power transmission networks.
- Wikipedia also shows that this might happen with springs:



- When the short rope connecting  $B$  and  $C$  on the left is removed, the weight hangs higher. Check also ‘The Spring Paradox’ by Steve Mould : <https://www.youtube.com/watch?v=Cg73j3QYRJe>, for a fun explanation.