ADVANCED MATHEMATICS FOR ECONOMICS - 2012/2013

Sheet 3: Differential equations

$\boxed{1}$ Check that $x(t) = \pm\sqrt{\ln(C(t^2+1))}$, where C is a positive constant, is solution of the ODE

$$x'(t) = \frac{t}{x(t)(t^2+1)}.$$

$\boxed{2}$ Solve the following differential equations:

(a) $x' = 3(t-4)(x^2+1)$, con $x(0) = 0$.

(b) $x' = \frac{3t^{\frac{-2}{3}}+1}{4x^3-1}$, con $x(0) = 1$.

(c) $dx = (x^2 - y)dt$.

(d) $\sqrt{x'} - x = 0$, con $x(1) = 1$.

(e) $x' = \frac{\sqrt{x}}{1+t^2}$.

(f) $x' = \frac{1}{2}(x^2-1)$.

(g) $x' = x^{\frac{1}{3}}$.

$\boxed{3}$ Solve the following differential equations:

(a) $x^2 + 2txx' = 0$

(b) $(1+2tx^2)dt + 2(t^2+1)xdx = 0$

(c) $(\sin x - 1 - x\sin t)dt + (\cos t + t\cos x)dx = 0$

(d) $(2x + te^{tx})x' = -(1 + xe^{tx})$

(e) $(3t^2 + 4tx)dt + (2t^2 + 2x)dx = 0$

$\boxed{4}$ Solve the following differential equations:

(a) $t(1-x) + (x+t^2)x' = 0$

(b) $tdx - xdt = t^2e^tdt$

(c) $x^2 + 2txx' = 0$

(d) $2tx + (x^2 - 3t^2)x' = 0$

$\boxed{5}$ Solve the following differential equations:

(a) $x' + 2x = e^t$ con $x(0) = 1$.

(b) $3x' - 2tx = t$ con $x(0) = -1$.

(c) $t(x' - x) = (1+t^2)e^t$.

(d) $x' - x = \frac{e^t}{t}$.

(e) $e^{-t^2}(2tx - x') = t$.

(f) $tx' - \frac{t^2x}{t+2} = te^t$ con $x(1) = 0$.

$\boxed{6}$ Solve the following differential equations:

(a) $x' = \frac{t^3}{x^3}$.

(b) $x' = \frac{x^3}{t^3}$.

(c) $x' = \frac{\sqrt{t+1}}{x^2}$, with $y(0) = \frac{5}{3}$.

(d) $(t + 3x)\,dt + (t - 20)\,dx = 0$.

(e) $(2xy - \cos x)\,dx + (x^2 - 1)\,dy = 0$, with $y(0) = 0$.

(f) $x' + 2tx = \cos t e^{-t^2}$, with $x(0) = 0$.

(g) $x' + \frac{x}{t} = e^{-t^2}$.

$\boxed{7}$ The equation

$$x' + a(t)x = b(t)x^n$$

is a *Bernoulli equation*. It is a linear equation for $n = 0$ or $n = 1$, but it is not linear for $n \neq 0, 1$. Suppose that this is the case.

(a) Prove that the change of variable $y = x^{1-n}$ transforms the equation into a linear equation for $y(t)$.

(b) Solve $x' + 2x = x^3$, $x(0) = 2$.

$\boxed{8}$ Draw the phase diagrams of each of the following equations, find the equilibrium points and study their stability.

(a) $x' = g_1(x) = (x + 1)(x - 1)^2(x - 2)$.

(b) $x' = g_2(x) = (x + 1)(x - 1)(x - 2)$.

$\boxed{9}$ The Verhuslt model of population dynamics in continuous time is analogous to the model in discrete time. It is supposed that the rate growth is proportional to the product of the population level times the population remaining until reach a saturation level K. Thus, the population evolves as

$$P'(t) = kP(t)\,(K - P(t))\,, \qquad k, K > 0.$$

Here, K is the carrying capacity. Find P and draw some solution trajectories.

$\boxed{10}$ Suppose that the population y of a certain species of fish in a given area of the ocean is described by the logistic equation

$$y' = r\left(1 - \frac{y}{K}\right)y.$$

The resource is used for food. Suppose that the rate at which fish are caught, $E(y)$, is proportional to the population y. Thus, we assume that $E(y) = Ey$, with E a positive constant. Then the logistic equation is replaced by

$$y' = r\left(1 - \frac{y}{K}\right)y - Ey.$$

This equation is known as *Schaefer model.*

(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$, $y_2 > 0$;

(b) Show that y_1 is unstable and y_2 is asymptotically stable.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is defined as Ey_2. Find Y as a function of the effort E and graph the function (it is known as the yield–effort curve).

(d) Determine E so as to maximize Y and thereby find the *maximum sustainable yield* Y_m.

$\boxed{11}$ Five college students with the flu return after Christmas Holidays to an isolated campus of 2500 students. If the rate at which this virus spreads is proportional to the number of infected students y and to the number not infected $2500 - y$, solve the initial value problem

$$y' = ky(2500 - y), \qquad y(0) = 5$$

to find the number of infected students after t days if 25 students have the virus after one day. How many students have the flu after five days? Determine the number of days required for half the campus to be infected.

$\boxed{12}$ Answer the following questions:

(a) Form the homogeneous linear ODEs from their characteristic equation.

- $r^2 - 3r + 5 = 0$;
- $r(r+2) = 0$.

(b) Form the homogeneous linear ODEs from the roots of their characteristic equation.

- $r_1 = 1$, $r_2 = 4$;
- $r_1 = 3 - 4i$, $r_2 = 3 + 4i$.

(c) Form the homogeneous linear ODEs from their general solution.

- $C_1 e^t + C_2 e^{-2t}$;
- $C_1 e^{-2t} + C_2 t e^{-2t}$;
- $e^{-t/2}(C_1 \sin 2t + C_2 \cos 2t)$.

$\boxed{13}$ Find the solution of the following equations:

(a) $x''' + 3x'' + 3x' + 2x = 0$

(b) $x''' - 2x'' - x' + 2x = 0$

(c) $x''' - 4x'' + 5x' - 2x = 0$

(d) $x'' - 2x' + 5x = 0$

(e) $x'' - 10x' + 25x = 0$

(f) $y^{iv} - 3y''' + 4y' = 0$

(g) $x''' + 3x'' + 3x' + 2x = 4$

(h) $x'' + 9x = e^t$

(i) $x''' - 3x' - 2x = \cos t$

(j) $x'' + x = sent$

(k) $x'' - 3x' + 2x = (t^2 + t)e^{3t}$

$\boxed{14}$ Find the solution of the following equations, considering the initial conditions:

(a) $y'' - 3y' - 4y = t^3$, such that $y(1) = 1$, $y'(1) = 2$

(b) $y'' + y' - 2y = 2$, such that $y(0) = -1$, $y'(1) = 1$

$\boxed{15}$ An equation of the form

$$t^2 x'' + atx' + bx = 0,$$

where a and b are real constants, is called an *Euler equation.* Show that the substitution of the independent variable $s = \ln t$ transforms an Euler equation into an equation with constant coefficients for the new dependent variable $y(s) = x(e^s)$. As an application, find the solution of the equation $t^2 x'' - 4tx' - 6x = 0$ for $t > 0$.

$\boxed{16}$ Suppose that a risky asset X grows at an average exponential rate of α but it is subjected to random fluctuations of instantaneous volatility σ. Let $V(x)$ be the value of a security that collects $x\,dt$ euros continuously when the price of the stock is $X = x$. Supposing that the risk free interest rate in the economy is $r < \alpha$, it can be shown by arbitrage reasonings that the value of the stock $V(x)$ satisfies the equation the Euler equation

$$\frac{\sigma^2}{2} x^2 V''(x) + axV'(x) - rV(x) = x.$$

Find the general solution and pick up the economically sensible solution among these.

$\boxed{17}$ Let the demand and supply functions for a single commodity be given by

$$D(t) = 42 - 4P(t) - 4P'(t) + P''(t),$$
$$S(t) = -6 + 8P(t).$$

We have assumed that the demand depends not only on current price, P, but also in expectations about the first and second variation of prices, given by P' and P'', respectively. Assuming that market clears at every time t, i.e. $D(t) = S(t)$, determine the path of P. Determine a linear relation between initial conditions $P(0)$ and $P'(0)$ such that the solution is bounded.

$\boxed{18}$ An entomologist is studying two neighboring populations of red and black ants. She has estimated that the number of black ants is approximately $60,000$ and that of red ants is $15,000$. The ants begin fighting and our entomologist observe that at any time, the number of ants killed of one population is proportional to the number of ants alive of the other population. However, red ants are more aggressive than black ants in such a way that their effectiveness in the fight is quadruple that of black ants. The observer receives a call to her mobile phone and must leave the observation, coming back to the camp. She knows that these two species of ants fight until one of them is annihilated. She conjecture that, given that the initial population of ants is 4:1 in favor of blacks, but the effectiveness is 4:1 for reds, both populations will practically extinct at once. However, when she returns next day to the anthill, the situation is quite different. Could you help our hero to understand what happened by answering the following questions?

(a) Which is the survival species?

(b) How many ants of the survival species remain alive?

(c) Which should be the initial proportion of both populations in order that both species become extinct at once?

Hint: Denoting $x(t)$ = black ants at time t, $y(t)$ = red ants at time t (both in thousands), justify why the interaction between ants can be given by

$$x'(t) = -4ky(t),$$
$$y'(t) = -kx(t),$$

with $k > 0$ a constant which is the fight effectiveness of black ants. This system can be converted into a second order ODE for $x(t)$ alone (or for $y(t)$). Then, solve and find the paths of $x(t)$ and $y(t)$, knowing that $x(0) = 60$ and $y(0) = 15$.

$\boxed{19}$ Obtain the general solution of the following complete systems of equations. In Case (d) compute the particular solution considering the initial conditions.

(a)

$$X' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} X + \begin{pmatrix} \text{sent} \\ \text{cost} \\ \text{t} \end{pmatrix}$$

(b)

$$X' = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} X + \begin{pmatrix} 2 \\ \mathrm{e}^{2t} \\ \mathrm{e}^{-t} \end{pmatrix}$$

(c)

$$\begin{cases} x' = -x + y + z + e^t \\ y' = x - y + z + e^{3t} \\ z' = x + y - z + 4 \end{cases}$$

(d)

$$X' = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} X + \begin{pmatrix} 2\mathrm{e}^{-t} \\ 3\mathrm{t} \end{pmatrix}, \quad X(0) = \begin{pmatrix} 1 \\ \text{-1} \end{pmatrix}$$

$\boxed{20}$ Classify the equilibrium point $(0,0)$ of the following systems, in terms of the parameter α.

(a) $X' = \begin{pmatrix} \alpha & 0 \\ 6 & 2\alpha \end{pmatrix} X, \quad (\alpha \neq 0).$

(b) $X' = \begin{pmatrix} \alpha & -3 \\ 3 & \alpha \end{pmatrix} X.$

$\boxed{21}$ The model of Obst[1] of monetary policy in the presence of an inflation adjustment mechanism is as follows. The quotient M_d/M_s (money demand/money supply), is denoted by μ; $p = P'/P$ is the inflation rate (P is the price level of the economy); $q = Q'/Q$ the constant (exogenous) rate of growth of GDP, Q, and $m = M_s'/M_s$ the monetary expansion rate. The evolution of p follows the Walrasian adjustment mechanism

$$p' = h(1-\mu), \qquad 0 < h < 1 \text{ a parameter.}$$

Hence an excess in the monetary supply $M_s > M_d$, leads to a positive increment in the inflation rate. To stipulate the time evolution of μ we consider the following assumption: monetary demand is proportional to GDP in nominal terms, that is,

$$M_d = aPQ, \qquad a > 0 \text{ constant,}$$

hence

$$\mu = a\frac{PQ}{M_s}.$$

Taking logarithms

$$\ln \mu = \ln a + \ln P + \ln Q - \ln M_s,$$

and taking the derivative with respect to time we get

$$\frac{\mu'}{\mu} = \frac{P'}{P} + \frac{Q'}{Q} - \frac{M_s'}{M_s} = p + q - m.$$

Hence, the system of ODEs in the model of Obst is

$$\begin{aligned} p' &= h(1-\mu), \\ \mu' &= (p+q-m)\mu. \end{aligned}$$

The exercise studies the effect of the monetary policy chosen by the central bank, given by m.

(a) Suppose that $m = \overline{m}$ is constant (*exogenous and constant monetary expansion rate*) and that $\overline{m} > q$. Show that the system has a center.

(b) Suppose that $m = \overline{m} - \alpha p$ with $\alpha > 0$ (*countercyclical conventional monetary policy*) and $\overline{m} > q$. Show by means of the phase portrait that the qualitative behavior of the system is similar to (a) above.

(c) Suppose that $m = \overline{m} - \alpha p'$ (*Obst's Rule*) with $\alpha > 0$ and $\overline{m} > q$. Prove that for some values of α the system has a spiral attractor.

(d) What do you think about the stabilization properties of the countercyclical rule and Obst's Rule?

[1] N. P. Obst (1978) "Stabilization policy with an inflation adjustment mechanism". *Quarterly Journal of Economics*, May, pp. 355–359.