Topic 2: Difference Equations

1. Introduction

In this chapter we shall consider systems of equations where each variable has a time index $t = 0, 1, 2, \ldots$ and variables of different time–periods are connected in a non–trivial way. Such systems are called *systems of difference equations* and are useful to describe *dynamical systems with discrete time*. The study of dynamics in economics is important because it allows to drop out the (static) assumption that the process of economic adjustment inevitable leads to an equilibrium. In a dynamic context, this stability property has to be checked, rather than assumed away.
Let time be a discrete denoted $t = 0, 1, \ldots$. A function $X : \mathbb{N} \longrightarrow \mathbb{R}^n$ that depends on this variable is simply a sequence of vectors of n dimensions

$$X_0, X_1, X_2, \ldots$$

If each vector is connected with the previous vector by means of a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as

$$X_{t+1} = f(X_t), \qquad t = 0, 1, \ldots,$$

then we have a *system of first–order difference equations*. In the following definition, we generalize the concept to systems with longer time lags and that can include t explicitly.

Definition 1.1. A kth order discrete system of difference equations is an expression of the form

$$X_{t+k} = f(X_{t+k-1}, \ldots, X_t, t), \qquad t = 0, 1, \ldots, \tag{1.1}$$

where every $X_t \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \longrightarrow \mathbb{R}^n$. The system is

- *autonomous*, if f does not depend on t;
- *linear*, if the mapping f is linear in the variables $(X_{t+k-1}, \ldots, X_t)$;
- *of first order*, if $k = 1$.

Definition 1.2. A sequence $\{X_0, X_1, X_2, \ldots\}$ obtained from the recursion (1.1) with initial value X_0 is called a trajectory, orbit or path of the dynamical system from X_0.

In what follows we will write x_t instead of X_t if the variable X_t is a scalar.

Example 1.3. [Geometrical sequence] Let $\{x_t\}$ be a scalar sequence, $x_{t+1} = qx_t$, $t = 0, 1, \ldots$, with $q \in \mathbb{R}$. This a first–order, autonomous and linear difference equation. Obviously $x_t = q^t x_0$. Similarly, for arithmetic sequence, $x_{t+1} = x_t + d$, with $d \in \mathbb{R}$, $x_t = x_0 + td$.

Example 1.4.

- $x_{t+1} = x_t + t$ is linear, non–autonomous and of first order;
- $x_{t+2} = -x_t$ is linear, autonomous and of second order;
- $x_{t+1} = x_t^2 + 1$ is non–linear, autonomous and of first order;

Example 1.5. [Fibonacci numbers (1202)] "How many pairs of rabbits will be produced in a year, beginning with a single pair, if every month each pair bears a new pair which becomes productive from the second month on?". With x_t denoting the pairs of rabbits in month t, the problem leads to the following recursion

$$x_{t+2} = x_{t+1} + x_t, \qquad t = 0, 1, 2, \ldots, \text{ with } x_0 = 1 \text{ and } x_1 = 1.$$

This is an autonomous and linear second–order difference equation.

2. Systems of first order difference equations

Systems of order $k > 1$ can be reduced to first order systems by augmenting the number of variables. This is the reason we study mainly first order systems. Instead of giving a general formula for the reduction, we present a simple example.

Example 2.1. Consider the second–order difference equation $y_{t+2} = g(y_{t+1}, y_t)$. Let $x_{1,t} = y_{t+1}$, $x_{2,t} = y_t$, then $x_{2,t+1} = y_{t+1} = x_{1,t}$ and the resulting first order system is

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} g(x_{1,t}, x_{2,t}) \\ x_{1,t} \end{pmatrix}.$$

If we denote $X_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$, $f(X_t) = \begin{pmatrix} g(X_t) \\ x_{1,t} \end{pmatrix}$, then the system can be written $X_{t+1} = f(X_t)$. For example, $y_{t+2} = 4y_{t+1} + y_t^2 + 1$ can be reduced to the first order system

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 4x_{1,t} + x_{2,t}^2 + 1 \\ x_{1,t} \end{pmatrix},$$

and the Fibonacci equation of Example 1.5 is reduced to

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} x_{1,t} + x_{2,t} \\ x_{1,t} \end{pmatrix},$$

For a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we shall use the following notation: f^t denotes the t–fold composition of f, i.e. $f^1 = f$, $f^2 = f \circ f$ and, in general, $f^t = f \circ f^{t-1}$ for $t = 1, 2, \ldots$. We also define f^0 as the identity function, $f^0(X) = X$.

Theorem 2.2. *Consider the autonomous first order system $X_{t+1} = f(X_t)$ and suppose that there exists some subset D such that for any $X \in D$, $f(X, t) \subseteq D$. Then, given any initial condition $X_0 \in D$, the sequence $\{X_t\}$ is given by*

$$X_t = f^t(X_0).$$

Proof. Notice that

$$\begin{aligned} X_1 &= f(X_0), \\ X_2 &= f(X_1) = f(f(X_0)) = f^2(X_0), \\ &\vdots \\ X_t &= f(f \cdots f(X_0) \cdots)) = f^t(X_0). \end{aligned}$$

□

The theorem provides the current value of X, X_t, in terms of the initial value, X_0. We are interested what is the behavior of X_t in the future, that is, in the limit

$$\lim_{t \to \infty} f^t(X_0).$$

Generally, we are more interested in this limit that in the analytical expression of X_t. Nevertheless, there are some cases where the solution can be found explicitly, so we can study the above limit behavior quite well. Observe that if the limit exists, $\lim_{t \to \infty} f^t(X_0) = X^0$, say, and f is continuous

$$f(X^0) = f(\lim_{t \to \infty} f^t(X_0)) = \lim_{t \to \infty} f^{t+1}(X_0) = X^0,$$

hence the limit X^0 is a *fixed point* of map f. This is the reason fixed points play a distinguished role in dynamical systems.

Definition 2.3. A point $X^0 \in D$ is called a fixed point of the autonomous system f if, starting the system from X^0, it stays there:

$$\text{If } X_0 = X^0, \text{ then } X_t = X^0, \qquad t = 1, 2, \dots.$$

Obviously, X^0 is also a fixed point of map f. A fixed point is also called *equilibrium*, *stationary point*, or *steady state.*

Example 2.4. In Example 1.3 ($x_{t+1} = qx_t$), if $q = 1$, then every point is a fixed point; if $q \neq 1$, then there exists a unique fixed point: $x^0 = 0$. Notice that the solution $x_t = q^t x_0$ has the following limit ($x_0 \neq 0$) depending the value of q.

$$\begin{aligned} -1 < q < 1 &\Rightarrow \lim_{t\to\infty} q^t x_0 = 0, \\ q = 1 &\Rightarrow \lim_{t\to\infty} q^t x_0 = x_0, \\ q \leq -1 &\Rightarrow \text{ the sequence oscillates between } + \text{ and } - \text{ and the limit does not exist} \end{aligned}$$

In Example 1.5, $x^0 = 0$ is the unique fixed point. Consider now the difference equation $x_{t+1} = x_t^2 - 6$. Then, the fixed points are the solutions of $x = x^2 - 6$, that is, $x^0 = -2$ and $x^0 = 3$.

In the following definitions, $\|X - Y\|$ stands for the Euclidean distance between X and Y. For example, if $X = (1, 2, 3)$ and $Y = (3, 6, 7)$, then

$$\|X - Y\| = \sqrt{(3-1)^2 + (6-2)^2 + (7-3)^2} = \sqrt{36} = 6.$$

Definition 2.5.

- A fixed point X^0 is called stable if for any close enough initial state X_0, the resulting trajectory $\{X_t\}$ exists and stays close forever to X^0, that is, for any positive real ε, there exists a positive real $\delta(\varepsilon)$ such that if $\|X_0 - X^0\| < \delta(\epsilon)$, then $\|X_t - X^0\| < \varepsilon$ for every t.
- A stable fixed point X^0 is called locally asymptotically stable (l.a.s.) if the trajectory $\{X_t\}$ starting from any initial point X_0 close to enough to X^0, converges to the fixed point.
- A stable fixed point is called globally asymptotically stable (g.a.s.) if any trajectory generated by any initial point X_0 converges to it.
- A fixed point is unstable if it is not stable or asymptotically stable.

Remark 2.6.

- If X^0 is stable, but not l.a.s., $\{X_t\}$ need not approach X^0.
- A g.a.s. fixed point is necessarily unique.
- If X^0 is l.a.s., then small perturbations around X^0 decay and the trajectory generated by the system returns to the fixed point as the time grows.

Definition 2.7. Let P be an integer larger than 1. A series of vectors $X_0, X_1, \dots, X_{P-1}$ is called a P–period cycle of system f if a trajectory starting from X_0 goes through $X_1, \dots, X_{P-1}$ and returns to X_0, that is

$$X_{t+1} = f(X_t), \qquad t = 0, 1, \dots, P-1, \qquad X_P = X_0.$$

Observe that the series of vectors $X_0, X_1, \dots, X_P$ repeats indefinitely in the trajectory,

$$\{X_t\} = \{X_0, X_1, \dots, X_{P-1}, X_0, X_1, \dots, X_{P-1}, \dots\}.$$

For this reason, the trajectory itself is called a P–cycle.

Example 2.8. In Example 1.3 ($x_{t+1} = qx_t$) with $q = -1$ all the trajectories contains 2–cycles, because a typical path is

$$\{x_0, -x_0, x_0, -x_0, \dots\}.$$

Example 2.9. In Example 1.4 where $y_{t+2} = -y_t$, to find the possible cycles of the equation, first we write it as first order system using Example 2.1, to obtain

$$X_{t+1} = \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} -x_{2,t} \\ x_{1,t} \end{pmatrix} \equiv f(X_t).$$

Let $X_0 = (2,4)$. Then

$$\begin{aligned} X_1 &= f(X_0) = (-4,2), \\ X_2 &= f(X_1) = (-2,-4), \\ X_3 &= f(X_2) = (4,-2), \\ X_4 &= f(X_3) = (2,4) = X_0. \end{aligned}$$

Thus, a 4–cycle appears starting at X_0. In fact, any trajectory is a 4–cycle.

3. First order linear difference equations

The linear equation is of the form

$$x_{t+1} = ax_t + b, \qquad x_t \in \mathbb{R}, \qquad a,b \in \mathbb{R}. \tag{3.1}$$

Consider first the case $b = 0$ (homogeneous case). Then, by Theorem 2.2 the solution is $x_t = a^t x_0$, $t = 0,1,\dots$. Consider now the non–homogeneous case, $b \neq 0$. Let us find the fixed points of the equation. They solve (see Definition 2.3)

$$x^0 = ax^0 + b,$$

hence there is no fixed point if $a = 1$. However, if $a \neq 1$, the unique fixed point is

$$x^0 = \frac{b}{1-a}.$$

Define now $y_t = x_t - x^0$ and replace $x_t = y_t + x^0$ into (3.1) to get

$$y_{t+1} = ay_t,$$

hence $y_t = a^t y_0$. Returning to the variable x_t we find that the solution of the linear equation is

$$\begin{aligned} x_t &= x^0 + a^t(x_0 - x^0) \\ &= \frac{b}{1-a} + a^t\left(x_0 - \frac{b}{1-a}\right). \end{aligned}$$

Theorem 3.1. *In (3.1), the fixed point $x^0 = \frac{b}{1-a}$ is g.a.s. if and only if $|a| < 1$.*

Proof. Notice that $\lim_{t\to\infty} a^t = 0$ iff $|a| < 1$ and hence $\lim_{t\to\infty} x_t = \lim_{t\to\infty} x^0 + a^t(x_0 - x^0) = x^0$ iff $|a| < 1$, independently of the initial x_0. □

The convergence is monotonous if $0 < a < 1$ and oscillating if $-1 < a < 0$.

Example 3.2 (A Multiplier–Accelerator Model of Growth)**.** Let Y_t denote national income, I_t total investment, and S_t total saving—all in period t. Suppose that savings are proportional to national income, and that investment is proportional to the change in income from period t to $t+1$. Then, for $t = 0,1,2,\dots$,

$$\begin{aligned} S_t &= \alpha Y_t, \\ I_{t+1} &= \beta(Y_{t+1} - Y_t), \\ S_t &= I_t. \end{aligned}$$

The last equation is the equilibrium condition that saving equals investment in each period. Here $\beta > \alpha > 0$. We can deduce a difference equation for Y_t and solve it as follows. From the first and third equation, $I_t = \alpha Y_t$, and so $I_{t+1} = \alpha Y_{t+1}$. Inserting these into the second equation yields $\alpha Y_{t+1} = \beta(Y_{t+1} - Y_t)$, or $(\alpha - \beta)Y_{t+1} = -\beta Y_t$. Thus,

$$Y_{t+1} = \frac{\beta}{\beta - \alpha} Y_t = \left(1 + \frac{\alpha}{\beta - \alpha}\right) Y_t, \qquad t = 0, 1, 2, \ldots.$$

The solution is

$$Y_t = \left(1 + \frac{\alpha}{\beta - \alpha}\right)^t Y_0, \qquad t = 0, 1, 2, \ldots.$$

Thus, Y grows at the constant proportional rate $g = \alpha/(\beta - \alpha)$ each period. Note that $g = (Y_{t+1} - Y_t)/Y_t$.

Example 3.3 (A Cobweb Model)**.** Consider a market model with a single commodity where producer's output decision must be made one period in advance of the actual sale—such as in agricultural production, where planting must precede by an appreciable length of time the harvesting and sale of the output. Let us assume that the output decision in period t is based in the prevailing price P_t, but since this output will no be available until period $t+1$, the supply function is lagged one period,

$$Q_{s,t+1} = S(P_t).$$

Suppose that demand at time t is determined by a function that depends on P_t,

$$Q_{d,t+1} = D(P_t).$$

Supposing that functions S and D are linear and that in each time period the market clears, we have the following three equations

$$\begin{aligned} Q_{d,t} &= Q_{s,t}, \\ Q_{d,t+1} &= \alpha - \beta P_{t+1}, \qquad \alpha, \beta > 0, \\ Q_{s,t+1} &= -\gamma + \delta P_t, \qquad \gamma, \delta > 0. \end{aligned}$$

By substituting the last two equations into the first the model is reduced to the difference equation for prices

$$P_{t+1} = -\frac{\delta}{\beta} P_t + \frac{\alpha + \gamma}{\beta}.$$

The fixed point is $P^0 = (\alpha + \gamma)/(\beta + \delta)$, which is also the equilibrium price of the market, that is, $S(P^0) = D(P^0)$. The solution is

$$P_{t+1} = P^0 + \left(-\frac{\delta}{\beta}\right)^t (P_0 - P^0).$$

Since $-\delta/\beta$ is negative, the solution path is oscillating. It is this fact which gives rise to the cobweb phenomenon. There are three oscillations patterns: it is *explosive* if $\delta > \beta$ (S steeper than D), *uniform* if $\delta = \beta$, and *damped* if $\delta < \beta$ (S flatter than D). The three possibilities are illustrated in the graphics below. The demand is the downward–slopping line, with slope $-\beta$. The supply is the upward–slopping line, with slope δ. When $\delta > \beta$, as in Figure 3, the interaction of demand and supply will produce an explosive oscillation as follows: Given an initial price P_0, the quantity supplied in the next period will be $Q_1 = S(P_0)$. In order to clear the market, the quantity demanded in period 1 must be also Q_1, which is possible if and only if price is set at the level of P_1 given by the equation $Q_1 = D(P_1)$. Now, via the S curve, the price P_1 will lead to $Q_2 = S(P_1)$ as the quantity supplied in period 2, and to clear the market, price must be set at the level of P_2 according to the demand curve. Repeating this reasoning, we can trace out a "cobweb" around the demand and supply curves.

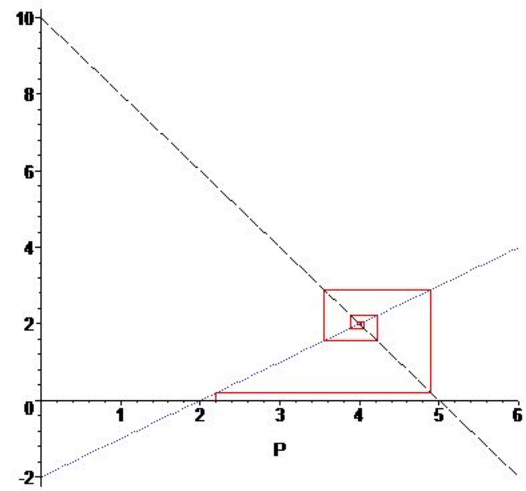


FIGURE 1. Cobweb diagram with damped oscillations

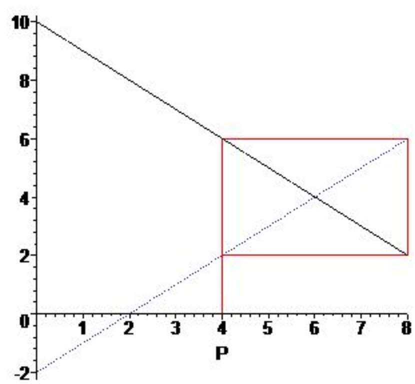


FIGURE 2. Cobweb diagram with uniform oscillations

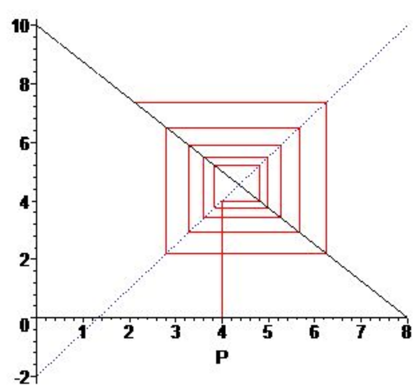


FIGURE 3. Cobweb diagram with explosive oscillations

4. Second order linear difference equations

The second–order linear difference equation is

$$x_{t+2} + a_1 x_{t+1} + a_0 x_t = b_t,$$

where a_0 and $a_1$are constants and b_t is a given function of t. The associated *homogeneous equation* is

$$x_{t+2} + a_1 x_{t+1} + a_0 x_t = 0,$$

and the associated *characteristic equation* is

$$r^2 + a_1 r + a_0 = 0.$$

This quadratic equation has solutions

$$r_1 = -\frac{1}{2}a_1 + \frac{1}{2}\sqrt{a_1^2 - 4a_0}, \qquad r_2 = -\frac{1}{2}a_1 - \frac{1}{2}\sqrt{a_1^2 - 4a_0}.$$

There are three different cases depending of the sign of the discriminant $a_1^2 - 4a_0$ of the equation. When it is negative, the solutions are (conjugate) complex numbers. Recall that a complex number is $z = a + ib$, where a and b are real numbers and $i = \sqrt{-1}$ is called the imaginary unit, so that $i^2 = -1$. The *real part* of z is a, and the *imaginary part* of z is b. The conjugate of $z = a + ib$ is $\overline{z} = a - ib$. Complex numbers can be added, $z + z' = (a + a') + i(b + b')$, and multiplied,

$$zz' = (a + ib)(a' + ib') = aa' + iab' + ia'b + i^2bb' = (aa' - bb') + i(ab' + a'b).$$

For the following theorem we need the *modulus of* z, $\rho = |z| = \sqrt{a^2 + b^2}$, and the *argument of* z, which is the angle $\theta \in (-\pi/2, \pi/2]$ such that $\tan\theta = b/a$. It is useful to recall the following table of trigonometric values

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

For the negative values of the argument θ, observe that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$, so that $\tan(-\theta) = -\tan\theta$.
For example, the modulus and argument of $1 - i$ is $\rho = \sqrt{2}$ and $\theta = -\pi/2$, respectively, since $\tan\theta = -1/1 = -1$.

Theorem 4.1. *The general solution of*

$$x_{t+2} + a_1 x_{t+1} + a_0 x_t = 0 \quad (a_0 \neq 0) \tag{4.1}$$

is as follows:

(1) If $a_1^2 - 4a_0 > 0$ (the characteristic equation has two distinct real roots),

$$x_t = Ar_1^t + Br_2^t, \qquad r_{1,2} = -\frac{1}{2}a_1 \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0}.$$

(2) If $a_1^2 - 4a_0 = 0$ (the characteristic equation has one real double roots),

$$x_t = (A + Bt)r^t, \qquad r = -\frac{1}{2}a_1.$$

(3) If $a_1^2 - 4a_0 < 0$ (the characteristic equation has no real roots),

$$x_t = \rho^t(A\cos\theta t + B\sin\theta t), \quad \rho = \sqrt{a_0}, \quad \tan\theta = -\frac{\sqrt{4a_0 - a_1^2}}{a_1}, \quad \theta \in [0, \pi].$$

Remark 4.2. When the characteristic equation has complex roots, the solution of (4.1) involves oscillations. Note that when $\rho < 1$, ρ^t tends to 0 as $t \to \infty$ and the oscillations are damped. If $\rho > 1$, the oscillations are explosive, and in the case $\rho = 1$, we have undamped oscillations.

Example 4.3. Find the general solutions of

(a) $x_{t+2} - 7x_{t+1} + 6x_t = 0$, (b) $x_{t+2} - 6x_{t+1} + 9x_t = 0$, (c) $x_{t+2} - 2x_{t+1} + 4x_t = 0$.

SOLUTION: (a) The characteristic equation is $r^2 - 7r + 6 = 0$, whose roots are $r_1 = 6$ and $r_2 = 1$, so the general solution is

$$x_t = A6^t + B, \qquad A, B \in \mathbb{R}.$$

(b) The characteristic equation is $r^2 - 6r + 9 = 0$, which has a double root $r = 3$. The general solution is

$$x_t = 3^t(A + Bt).$$

(c) The characteristic equation is $r^2 - 2r + 4 = 0$, with complex solutions $r_1 = \frac{1}{2}(2 + \sqrt{-12}) = (1 + i\sqrt{3})$, $r_2 = (1 - i\sqrt{3})$. Here $\rho = 2$ and $\tan\theta = -\frac{\sqrt{12}}{-2} = \sqrt{3}$. this means that $\theta = \pi/3$. The general solution is

$$x_t = 2^t \left(A \cos \frac{\pi}{3} t + B \sin \frac{\pi}{3} t \right).$$

4.1. **The nonhomogeneous case.** Now consider the nonhomogeneous equation

$$x_{t+2} + a_1 x_{t+1} + a_0 x_t = b_t, \tag{4.2}$$

and let x_t^* be a *particular solution*. It turns out that solutions of the equation have an interesting structure, due to the linearity of the equation.

Theorem 4.4. *The general solution of the nonhomogeneous equation* (4.2) *is the sum of the general solution of the homogeneous equation* (4.1) *and a particular solution* x_t^* *of the nonhomogeneous equation.*

Example 4.5. Find the general solution of $x_{t+2} - 4x_t = 3$.

SOLUTION: Note that $x_t^* = -1$ is a particular solution. To find the general solution of the homogeneous equation, consider the solutions of the characteristic equation, $m^2 - 4 = 0$, $m_{1,2} = \pm 2$. Hence, the general solution of the nonhomogeneous equation is

$$x_t = A(-2)^t + B2^t - 1.$$

Example 4.6. Find the general solution of $x_{t+2} - 4x_t = t$.

SOLUTION: Now it is not obvious how to find a particular solution. We can try with the *method of undetermined coefficients* and try with some expression of the form $x_t^* = Ct + D$. Then, we look for constants a, b such that x_t^* is a solution. This requires

$$C(t+2) + D - 4(Ct + D) = t, \qquad \forall t = 0, 1, 2, \ldots$$

One must have $C - 4C = 1$ and $2C + D - 4D = 0$. It follows that $C = -1/3$ and $D = -2/9$. Thus, the general solution is

$$x_t = A(-2)^t + B2^t - t/3 - 2/9.$$

Example 4.7. Find the solution of $x_{t+2} - 4x_t = t$ satisfying $x_0 = 0$ and $x_1 = 1/3$.

SOLUTION: Using the general solution found above, we have two equations for the two unknown parameters A and B:

$$\left.\begin{array}{ll} A + B + \frac{2}{9} & = 0 \\ -2A + 2B - \frac{1}{3} + \frac{2}{9} & = \frac{1}{3} \end{array}\right\}.$$

The solution is $A = -2/9$ and $B = 0$. Thus, the solution of the nonhomogeneous equation is

$$x_t = -\frac{2}{9}(-2)^t - \frac{t}{2} + \frac{2}{9}.$$

The method of undetermined coefficients for solving equation (4.2) suppose that a particular solution has the form of the nonhomogeneous term, b_t. The method works quite well when this term is of the form

$$a^t, \quad t^m, \quad \cos at, \quad \sin at$$

or linear combinations of them.

Example 4.8. Solve the equation $x_{t+2} - 5x_{t+1} + 6x_t = 4^t + t^2 + 3$.

Solution: The homogeneous equation has characteristic equation $r^2 - 5r + 6 = 0$, with two different real roots $r_{1,2} = 2, 3$. Its general solution is, therefore, $A2^t + B3^t$. To find a particular solution we look for constants C, D, E and F such that a particular solution is

$$x_t^* = C4^t + Dt^2 + Et + F.$$

Plugging this into the equation we find

$$\begin{aligned} C4^{t+2} + D(t+2)^2 + E(t+2) + F - 5(C4^{t+1} + D(t+1)^2 + E(t+1) + F) \\ + 6(C4^t + Dt^2 + Et + F) = 4^t + t^2 + 3. \end{aligned}$$

Expanding and rearranging yields

$$2C4^t + 2Dt^2 + (-6D + 2E)t + (-D - 3E + 2F) = 4^t + t^2 + 3.$$

This must hold for every $t = 0, 1, 2, \ldots$ thus,

$$\begin{aligned} 2C &= 4, \\ 2D &= 1, \\ -6D + 2E &= 0, \\ -D - 3E + 2F &= 3. \end{aligned}$$

It follows that $C = 1/2$, $D = 1/2$, $E = 3/3$ and $F = 4$. The general solution is

$$x_t = A2^t + B3^t + \frac{1}{2}4^t + \frac{1}{2}t^2 + \frac{3}{2}t + 4.$$

Example 4.9 (A Multiplier–Accelerator Growth Model)**.** Let Y_t denote national income, C_t total consumption, and I_t total investment in a country at time t. Assume that for $t = 0, 1, \ldots,$

(i) $Y_t = C_t + I_t$ (income is divided between consumption and investment)

(ii) $C_{t+1} = aY_t + b$ (consumption is a linear function of previous income)

(iii) $I_{t+1} = c(C_{t+1} - C_t)$ (investment is proportional to to the change in consumption),

where $a, b, c > 0$. Find a second order difference equation describing this national economy.

Solution: We eliminate two of the unknown functions as follows. From (i), we get (iv) $Y_{t+2} = C_{t+2} + I_{t+2}$. Replace now t by $t+1$ in (ii) and (iii) to get (v) $C_{t+2} = aY_{t+1} + b$ and (vi) $I_{t+2} = c(C_{t+2} - C_{t+1})$, respectively. Then, inserting (iii) and (v) into (vi) gives $I_{t+2} = ac(Y_{t+1} - Y_t)$. Inserting this result and (v) into (iv) we get $Y_{t+2} = aY_{t+1} + b + ac(Y_{t+1} - Y_t)$ and rearranging we arrive to

$$Y_{t+2} - a(1+c)Y_{t+1} + acY_t = b, \qquad t = 0, 1, \ldots$$

The form of the solution depends on the coefficients a, b, c.

5. Linear systems of difference equations

Now we suppose that the dynamic variables are vectors, $X_t \in \mathbb{R}^n$. A *first order system of linear difference equations with constant coefficients* is given by

$$\begin{aligned} x_{1,t+1} &= a_{11}x_{1,t} + \cdots + a_{1n}x_{n,t} + b_{1,t} \\ &\vdots \\ x_{n,t+1} &= a_{n1}x_{1,t} + \cdots + a_{nn}x_{n,t} + b_{n,t} \end{aligned}$$

An example is

$$\begin{aligned} x_{1,t+1} &= 2x_{1,t} - x_{2,t} + 1 \\ x_{2,t+1} &= x_{1,t} + x_{2,t} + e^{-t}. \end{aligned}$$

Most often we will rewrite systems omitting subscripts using different letters for different variables, as in

$$\begin{aligned} x_{t+1} &= 2x_t - y_t + 1 \\ y_{t+1} &= x_t + y_t + e^{-t}. \end{aligned}$$

A linear system is equivalent to the matrix equation

$$X_{t+1} = AX_t + B_t,$$

where

$$X_t = \begin{pmatrix} x_{1,t} \\ \vdots \\ x_{n,t} \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{1,t} \\ \vdots \\ b_{n,t} \end{pmatrix}$$

We will center on the case where the independent term $B_t \equiv B$ is a constant vector.

5.1. **Homogeneous systems.** Consider the homogeneous system $X_{t+1} = AX_t$.
Note that $X_1 = AX_0$, $X_2 = AX_1 = AAX_0 = A^2X_0$. Thus, given the initial vector X_0, the solution is

$$X_t = A^tX_0.$$

In the case that A be diagonalizable, $P^{-1}AP = D$ with D diagonal, the expression above simplifies to

$$X_t = PD^tP^{-1}X_0,$$

that is easy to compute since D is diagonal.

Example 5.1. Find the general solution of the system

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

Solution: The matrix $A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 - 5\lambda + 6$, with roots $\lambda_1 = 3$ and $\lambda_2 = 2$. Thus, the matrix is diagonalizable. It is easy to find the eigenspaces

$$S(3) =< (1,1) >, \qquad S(2) =< (1,2) > .$$

Hence, the matrix P is

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

and the solution

$$X_t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & 2^t \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} X_0 = \begin{pmatrix} 2\,3^t - 2^t & -3^t + 2^t \\ 2\,3^t - 2^{t+1} - 1 & -3^t + 2^{t+1} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Supposing that the initial condition is $(x_0, y_0) = (1, 2)$, the solution is given by

$$\begin{aligned} x_t &= 2\,3^t - 2^t + 2(-3^t + 2^t), \\ y_t &= 2\,3^t - 2^{t+1} - 1 + 2(-3^t + 2^{t+1}). \end{aligned}$$

5.2. **Nonhomogeneous systems.** Consider the system $X_{t+1} = AX_t + B$, where B is a non–null, constant vector.
To obtain a closed-form solution of the system, we begin by noting that

$$\begin{aligned} X_1 &= AX_0 + B, \\ X_2 &= AX_1 + B = A(AX_0 + B) + B = A^2 X_0 + (A + I_n)B, \\ &\vdots \\ X_t &= AX_{t-1} + B = \cdots = A^t X_0 + (A^{t-1} + A^{t-2} + \cdots + I_n)B. \end{aligned}$$

Observe that

$$(A^{t-1} + A^{t-2} + \cdots + I_n)(A - I_n) = A^t + A^{t-1} + \cdots + A - A^{t-1} - \cdots - A - I_n = A^t - I_n.$$

Thus, assuming that $(A - I_n)$ is invertible, we find

$$A^{t-1} + A^{t-2} + \cdots + I_n = (A^t - I_n)(A - I_n)^{-1}.$$

Plugging this equality into the expression for X_t above one gets

$$X_t = A^t X_0 + (A^t - I_n)(A - I_n)^{-1} B.$$

On the other hand, note that the constant solutions of the nonhomogeneous system (or fixed points of the system) satisfy

$$X^0 = AX^0 + B.$$

Assuming again that the matrix $A - I_n$ has inverse, we can solve for X^0

$$(I_n - A)X^0 = B \Rightarrow X^0 = (I_n - A)^{-1} B.$$

Then, collecting all the above observations, we can write the solution of the nonhomogeneous system in a nice form as

$$X_t = A^t X_0 - (A^t - I_n)X^0 = X^0 + A^t(X_0 - X^0). \tag{5.1}$$

Theorem 5.2. *Suppose that $|A - I_n| \neq 0$. Then, the general solution of the nonhomogeneous system is given in Eqn.* (5.1)*. Moreover, when A is diagonalizable, the above expression may be written as*

$$X_t = X^0 + PD^t P^{-1}(X_0 - X^0), \tag{5.2}$$

where $P^{-1}AP = D$ with D diagonal.

Proof. Eqn. (5.2) easily follows from Eqn. (5.1), taking into account that $A^t = PD^tP^{-1}$. □

Example 5.3. Find the general solution of the system

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution: The fixed point X^* is given by

$$(I_3 - A)^{-1}B = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 5/2 \end{pmatrix}$$

By the example above we already know the general solution of the homogeneous system. The general solution of the nonhomogeneous system is then

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 2\,3^t - 2^t & -3^t + 2^t \\ 2\,3^t - 2^{t+1} - 1 & -3^t + 2^{t+1} \end{pmatrix} \begin{pmatrix} x_0 - 1/2 \\ y_0 - 5/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 5/2 \end{pmatrix}.$$

5.3. **Stability of linear systems.** We study here the stability properties of a first order system $X_{t+1} = AX_t + B$ where $|I_n - A| \neq 0$.
For the following theorem, recall that for a complex number $z = \alpha + \beta i$, the modulus is $\rho = \sqrt{\alpha^2 + \beta^2}$. For a real number α, the modulus is $|\alpha|$.

Theorem 5.4. *A necessary and sufficient condition for system $X_{t+1} = AX_t + B$ to be g.a.s. is that all roots of the characteristic polynomial $p_A(\lambda)$ (real or complex) have moduli less than 1. In this case, any trajectory converges to $X^* = (I_n - A)^{-1}B$ as $t \to \infty$.*

We can give an idea of the proof of the above theorem in the case where the matrix A is diagonalizable. As we have shown above, the solution of the nonhomogeneous system in this case is

$$X_t = X^0 + PD^tP^{-1}(X_0 - X^0),$$

where

$$D = \begin{pmatrix} \lambda_1^t & 0 & \dots & 0 \\ 0 & \lambda_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^t \end{pmatrix},$$

and $\lambda_1, \dots, \lambda_n$ are the real roots (possibly repeated) of $p_A(\lambda)$. Since $|\lambda_j| < 1$ for all j, the diagonal elements of D^t tends to 0 as t goes to ∞, since $\lambda_j^t \leq |\lambda_j|^t \to 0$. Hence

$$\lim_{t\to\infty} X_t = X^0.$$

Example 5.5. Study the stability of the system

$$\begin{aligned} x_{t+1} &= x_t - \frac{1}{2}y_t + 1, \\ y_{t+1} &= x_t - 1. \end{aligned}$$

Solution: The matrix of the system is $\begin{pmatrix} 1 & -1/2 \\ 1 & 0 \end{pmatrix}$, with characteristic equation $\lambda^2 - \lambda + 1/2 = 0$. The (complex) roots are $\lambda_{1,2} = 1/2 \pm i/2$. Both have modulus $\rho = \sqrt{1/4 + 1/4} = 1/\sqrt{2} < 1$, hence the system is g.a.s. and the limit of any trajectory is the equilibrium point,

$$X^0 = \begin{pmatrix} 1-1 & 0-(-1/2) \\ 0-1 & 1-0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Example 5.6. Study the stability of the system

$$\begin{aligned} x_{t+1} &= -x_t + y_t, \\ y_{t+1} &= -x_t/2 - y_t/2. \end{aligned}$$

Solution: The matrix of the system is $\begin{pmatrix} 1 & 3 \\ 1/2 & 1/2 \end{pmatrix}$, with characteristic equation $\lambda^2 - (3/2)\lambda - 1 = 0$. The roots are $\lambda_1 = 2$ and $\lambda_2 = -1/2$. The system is not g.a.s. However, there are initial conditions X_0 such that the trajectory converges to the fixed point $X^0 = (0, 0)$. This can be seen once we find the solution

$$X_t = PD^tP^{-1}X_0.$$

The eigenspaces are $S(2) =< (3,1) >$ and $S(-1/2) =< (2,-1) >$, thus

$$P = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 1/5 & 2/5 \\ 1/5 & -3/5 \end{pmatrix}.$$

The solution is

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 2^t \frac{3}{5}(x_0 + 2y_0) + 2^{1-t}\frac{1}{5}(x_0 - 3y_0) \\ 2^t \frac{1}{5}(x_0 + 2y_0) - 2^{-t}\frac{1}{5}(x_0 - 3y_0) \end{pmatrix}.$$

If the initial conditions are linked by the relation $x_0 + 2y_0 = 0$, then the solution converges to $(0,0)$. For this reason, the line $x + 2y = 0$ is called the *stable manifold*. Notice that the stable manifold is in fact the eigenspace associated to the eigenvalue $\lambda_2 = -1/2$, since

$$S(-1/2) =< (2,-1) >= \{x + 2y = 0\}.$$

For any other initial condition $(x_0, y_0) \notin S(-1/2)$, the solution does not converge.

Example 5.7 (Dynamic Cournot adjustment)**.** The purpose of this example is to investigate under what conditions a given adjustment process converges to the Nash equilibrium of the Cournot game. Consider a Cournot duopoly in which two firms produce the same product and face constant marginal costs $c_1 > 0$ and $c_2 > 0$. The market price P_t is a function of the total quantity of output produced $Q = q_1 + q_2$ in the following way

$$P = \alpha - \beta Q, \qquad \alpha > c_i, \quad i = 1,2, \quad \beta > 0.$$

In the Cournot duopoly model each firm chooses q_i to maximize profits, taking as given the production level of the other firm, q_j. At time t, firm i's profit is

$$\pi_i = q_i P - c_i q_i.$$

As it is well–known, taking $\partial \pi^i / \partial q_i = 0$ we obtain the best response of firm i, which depends on the output of firm j as follows[1]

$$\mathrm{br}_1 = a_1 - q_2/2, \qquad \mathrm{br}_2 = a_2 - q_1/2,$$

where $a_i = \dfrac{\alpha - c_i}{2\beta}$, $i = 1,2$. We suppose that $a_1 > a_2/2$ and that $a_2 > a_1/2$ in order to have positive quantities in equilibrium, as will be seen below.
The Nash equilibrium of the game, (q_1^N, q_2^N), is a pair of outputs of the firms such that none firm has incentives to deviate from it unilaterally, that is, it is the best response against itself. This means that the Nash equilibrium of the static game solves

$$\begin{aligned} q_1^N &= \mathrm{br}_1(q_2^N), \\ q_2^N &= \mathrm{br}_2(q_1^N). \end{aligned}$$

In this case

$$\begin{aligned} q_1^N &= a_1 - q_2^N/2, \\ q_2^N &= a_2 - q_2^N/2. \end{aligned}$$

Solving, we have

$$\begin{aligned} q_1^N &= \frac{4}{3}\left(a_2 - \frac{a_1}{2}\right), \\ q_2^N &= \frac{4}{3}\left(a_1 - \frac{a_2}{2}\right), \end{aligned}$$

[1] Actually, the best response map is $\mathrm{br}_i = \max\{a_i - q_j/2, 0\}$, since negative quantities are not allowed.

which are both positive by assumption. As a specific example, suppose for a moment that the game is symmetric, with $c_1 = c_2 = c$. Then, $a_1 = a_2 = \dfrac{\alpha - c}{2\beta}$ and the Nash equilibrium is the output

$$q_1^N = \frac{\alpha - c}{3\beta},$$
$$q_2^N = \frac{\alpha - c}{3\beta}.$$

Now we turn to the general asymmetric game and introduce a dynamic component in the game as follows. Suppose that each firm does not choose its Nash output instantaneously, but they adjust gradually its output q_i towards its best response br_i at each time t as indicated below

$$\begin{cases} q_{1,t+1} = q_{1,t} + d_1(\mathrm{br}_{1,t} - q_{1,t}) = q_{1,t} + d_1(a_1 - \frac{1}{2}q_{2,t} - q_{1,t}), \\ q_{2,t+1} = q_{1,t} + d_2(\mathrm{br}_{2,t} - q_{2,t}) = q_{2,t} + d_2(a_2 - \frac{1}{2}q_{1,t} - q_{2,t}), \end{cases} \tag{5.3}$$

where d_1 and d_2 are positive constants. The objective is to study whether this *tattonement* process converges to the Nash equilibrium.
To simplify notation, let us rename $x = q_1$ and $y = q_2$. Then, rearranging terms in the system (5.3) above, it can be rewritten as

$$\begin{cases} x_{t+1} = (1 - d_1)x_t - \frac{d_1}{2}y_t + d_1 a_1, \\ y_{t+1} = (1 - d_2)y_t - \frac{d_2}{2}x_t + d_2 a_2. \end{cases}$$

It is easy to find the equilibrium points by solving the system

$$\begin{cases} x = (1 - d_1)x - \frac{d_1}{2}y + d_1 a_1, \\ y = (1 - d_2)y - \frac{d_2}{2}x + d_2 a_2. \end{cases}$$

The only solution is precisely the Nash equilibrium,

$$(x^N, y^N) = \left(\frac{4}{3}\left(a_2 - \frac{a_1}{2}\right), \frac{4}{3}\left(a_1 - \frac{a_2}{2}\right)\right).$$

Under what conditions this progressive adjustment of the produced output does converge to the Nash equilibrium? According to the theory, it depends on the module of the eigenvalues being smaller than 1. Let us find the eigenvalues of the system. The matrix of the system is

$$\begin{pmatrix} 1 - d_1 & -\frac{d_1}{2} \\ -\frac{d_2}{2} & 1 - d_2 \end{pmatrix}$$

To simplify matters, let us suppose that the adjustment parameter is the same for both players, $d_1 = d_2 = d$. The eigenvalues of the matrix are

$$\lambda_1 = 1 - \frac{d}{2}, \qquad \lambda_2 = 1 - \frac{3d}{2},$$

which only depend on d. We have

$$|\lambda_1| < 1 \text{ iff } 0 < d < 4,$$
$$|\lambda_2| < 1 \text{ iff } 0 < d < 4/3,$$

therefore $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff $0 < d < 4/3$. Thus, $0 < d < 4/3$ is a necessary and sufficient condition for convergence to the Nash equilibrium of the one shot game from any initial condition (g.a.s. system).

6. The nonlinear first order equation

We investigate here the stability of the solutions of an autonomous first order difference equation

$$x_{t+1} = f(x_t), \qquad t = 0, 1, \dots,$$

where $f : I \to I$ is nonlinear and I is an interval of the real line. Recall that a function f is said to be *of class* C^1 in an open interval, if f' exists and it is continuous in that interval. For example, the functions x^2, $\cos x$ or e^x are C^1 is the whole real line, but $|x|$ is not differentiable at 0, so is not C^1 in any open interval that contains 0.

Theorem 6.1. *Let $x^0 \in I$ a fixed point of f, and suppose that f is C^1 in an open interval around x^0, $I_\delta = (x^0 - \delta, x^0 + \delta)$.*

(1) If $|f'(x^0)| < 1$, then x^0 is locally asymptotically stable;
(2) If $|f'(x^0)| > 1$, then x^0 is unstable.

Proof. Since f' is continuous in I_δ and $f'(x^0) < 1$, there exists some open interval $I_\delta = (x^0 - \delta, x^0 + \delta)$ and a positive number $k < 1$ such that $|f'(x)| \le k$ for any $x \in I_\delta$.

(1) By the mean value theorem (also called Theorem of Lagrange), there exists some c between x_0 and x^0 such that

$$f(x_0) - f(x^0) = f'(c)(x_0 - x^0),$$

or

$$x_1 - x^0 = f'(c)(x_0 - x^0),$$

since $x^0 = f(x^0)$ by definition of fixed point. Consider an initial condition $x_0 \in I_\delta$. Then any c between x_0 and x^0 belongs to I_δ and thus taking absolute values in the equality above we get

$$|x_1 - x^0| = |f'(c)||x_0 - x^0| \le k|x_0 - x^0|. \tag{6.1}$$

Also note that $|x_1 - x^0| \le k\delta < \delta$, thus $x_1 \in I_\delta$. Reasoning as above, one gets

$$|x_2 - x^0| = |f(x_1) - x^0| = |f'(c)||x_1 - x^0| \le k|x_1 - x^0| \le k^2|x_0 - x^0|.$$

where c is a number between x_1 and x^0 that belongs to I_δ (and thus $|f'(c)| \le k$). Continuing in this fashion we get after t steps

$$|x_t - x^0| \le k^t|x_0 - x^0| \to 0, \qquad \text{as } t \to \infty.$$

So x_t converges to the fixed point x^0 as $t \to \infty$, and x^0 is l.a.s.

(2) Now suppose that $|f'(x^0)| > 1$. Again by continuity of f', there exists $\delta > 0$ and $K > 1$ such that $|f'(x)| > K$ for any $x \in I_\delta$. By equation (6.1) one has

$$|x_1 - x^0| = |f'(c)||x_0 - x^0| > K|x_0 - x^0|$$

and after t steps

$$|x_t - x^0| > K^t|x_0 - x^0|.$$

Since K^t tends to ∞ as $t \to \infty$, then x_t departs more and more from x_0 at each step, and the fixed point x^0 is unstable.

□

Remark 6.2. If $|f'(x)| < 1$ for every point $x \in I$, then the fixed point x^0 is globally asymptotically stable.

Example 6.3 (Population growth models)**.** In the Malthus model of population growth it is postulated that a given population x grows at constant rate r,

$$\frac{x_{t+1} - x_t}{x_t} = r, \qquad \text{or} \qquad x_{t+1} = (1 + r)x_t.$$

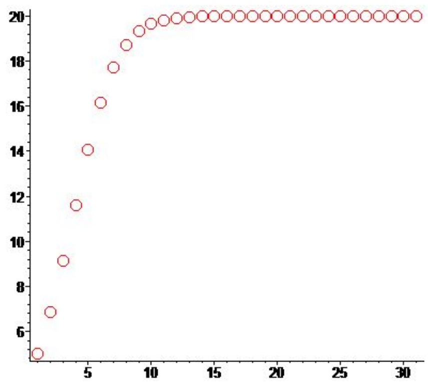
This is a linear equation and the population grows unboundedly if the per capita growth rate r is positive[2]. This is not realistic for large t. When the population is small, there are ample environmental resources to support a high birth rate, but for later times, as the population grows, there is a higher death rate as individuals compete for space and food. Thus, the growth rate should be decreasing as the population increases. The simplest case is to take a linearly decreasing per capita rate, that is

$$r\left(1-\frac{x_t}{K}\right),$$

where K is the carrying capacity. This modification is known as the *Verhulst' law.* Then the population evolves as

$$x_{t+1}=x_t\left(1+r-\frac{r}{K}x_t\right),$$

which is not linear. The function f is quadratic, $f(x)=x(1+r-rx/K)$. In Fig. 6.3 is depicted a solution with $x_0=5$, $r=0.5$ and $K=20$.



We observe that the solution converges to 20. In fact, there are two fixed points of the equation, 0 (extinction of the population) and $x^0=K$ (maximum carrying capacity). Considering the derivative of f at these two fixed points, we have

$$f'(0)=1+r-2\frac{r}{K}x\bigg|_{x=0}=1+r>1,$$
$$f'(K)=1+r-2\frac{r}{K}x\bigg|_{x=K}=1-r.$$

Thus, according to Theorem 6.1, 0 is unstable, but K is l.a.e. iff $|1-r|<1$, or $0<r<2$.

[2]The solution is $x_t=(1+r)^t x_0$, why?

6.1. **Phase diagrams.** The stability of a fixed point of the equation

$$x_{t+1} = f(x_t), \qquad t = 0, 1, \ldots,$$

can also be studied by a graphical method based in *the phase diagram*. This consists in drawing the graph of the function $y = f(x)$ in the plane xy. Note that a fixed point x^0 corresponds to a point (x^0, x^0) where the graph of $y = f(x)$ intersects the straight line $y = x$.
The following figures show possible configurations around a fixed point. The phase diagram is at the left (plane xy), and a solution sequence is shown at the right (plane tx). Notice that we have drawn the solution trajectory as a continuous curve because it facilitates visualization, but in fact it is a sequence of discrete points. In Fig. 4, $f'(x^0)$ is positive, and the sequence $x_0, x_1, \ldots$ converges monotonically to x^0, whereas in Fig. 5, $f'(x^0)$ is negative and we observe a cobweb–like behavior, with the sequence $x_0, x_1, \ldots$ converging to x^0 but alternating between values above and below the equilibrium. In Fig. 6, the graph of f near x^0 is too steep for convergence. After many iterations in the diagram, we observe an erratic behavior of the sequence $x_0, x_1, \ldots$. There is no cyclical patterns and two sequences generated from close initial conditions depart along time at an exponential rate (see Theorem 6.1 above). It is often said that the equation exhibits chaos. Finally, Fig. 7 is the phase diagram of an equation admitting a cycle of period 3.

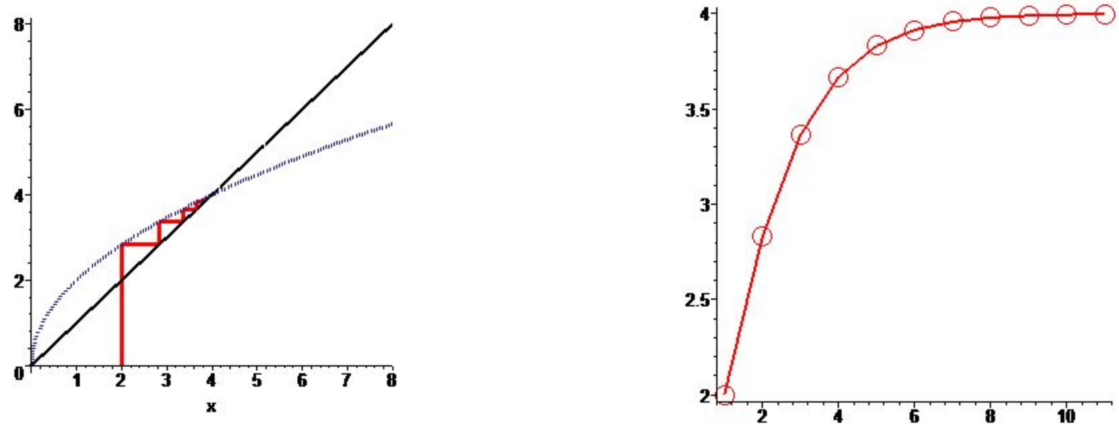


FIGURE 4. x^0 stable, $f'(x^0) \in (0, 1)$

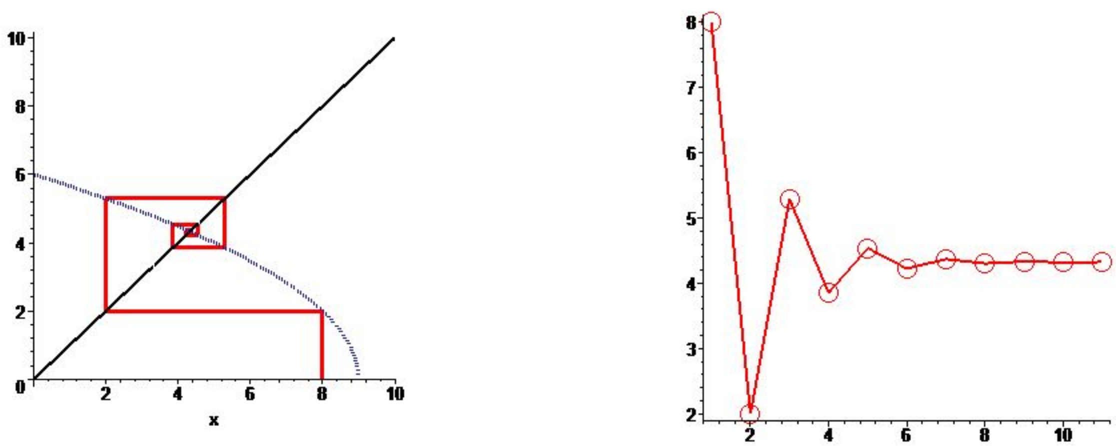


FIGURE 5. x^0 stable, $f'(x^0) \in (-1, 0)$

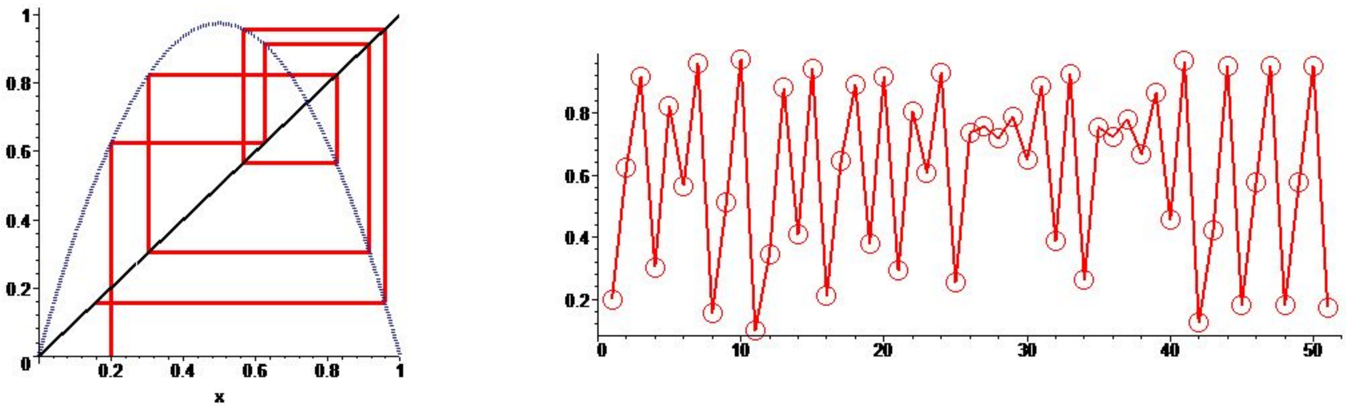


FIGURE 6. x^0 unstable, $|f'(x^0)| > 1$

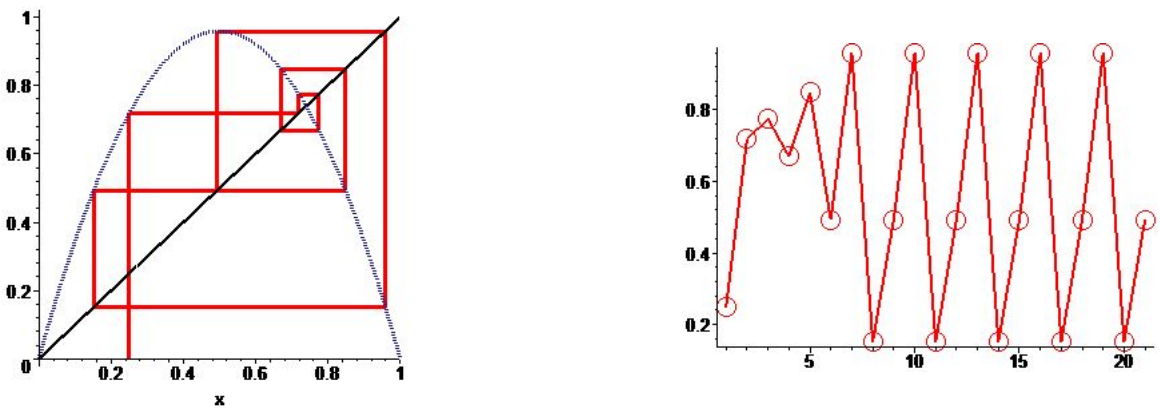


FIGURE 7. A cycle of period 3