Unobserved Heterogeneity, Exit Rates, and Re-employment Wages *

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Abstract

Exit rates from unemployment and re-employment wages decline over an unemployment period after controlling for worker observable characteristics. We study the role of unobserved heterogeneity in an economy with asymmetric information and directed search. We show that the unique equilibrium is separating and skilled workers have more job opportunities and higher wages. The composition of the unemployed varies with the duration of unemployment, so average exit rates and wages fall with time. The separating equilibrium relies on performance-related pay schemes and the ability of firms to commit to renting an input that is complementary to the worker skills.

Keywords: Asymmetric Information, Directed Search, Unemployment Duration.

JEL Codes: J01, J21, J42, J64.

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I Introduction

It is well documented that exit rates from unemployment and re-employment wages decline with the duration of unemployment after controlling for worker observable characteristics. Since Lancaster (1979) and Heckman and Singer (1984), the empirical literature has emphasized that ex-ante unobserved heterogeneity (or sorting) accounts for a large part of the decline in exit rates. According to this explanation, workers differ in terms of certain time-invariant characteristics, which are unobservable to the econometrician and potentially to recruiting firms, and this translates into differences in employment prospects. As a result, the average exit rates fall with the duration of unemployment because of variation in the composition of the unemployed. Despite its quantitative importance, very little theoretical research has been performed to address this explanation. Lockwood (1991) was a remarkable early exception, but re-employment wages were assumed to be constant in this study. Gonzalez and Shi (2010) and Fernández-Blanco and Preugschat (2014) provided theoretical explanations of the duration dynamics of one of these two variables, but they were not conclusive about the other variable. Furthermore, the explanation of sorting in these three studies was somewhat circular because worker heterogeneity was modeled based on exogenous differences in the likelihood of obtaining a job.

We consider a sorting mechanism based on productivity differences across workers, which is consistent with falling exit rates and wages. We analyze a frictional labor market in which workers are informed privately about skills and search is directed. These skills are treated as abilities that recruiters cannot grasp either from a CV or an interview, but they can be assessed on the job. An adverse selection problem arises if unskilled workers crowd out skilled applicants. Thus, firms design self-selection schemes in equilibrium to separate worker types in different submarkets. The sorting mechanism relies on two ingredients. First, in the absence of screening devices, performance-contingent compensation plans are a central feature of these sorting schemes, as noted in the literature in personnel economics (e.g., Lazear and Shaw (2007)). In our study, performance-related pay is derived from the assumption that rents generated by a job-worker pair are split according to some exogenous rule. Second, firms have the ability to commit to renting an input that is complementary in terms of production to the skills of workers. We refer to that production input as capital or machine quality.[1] These linear renting costs together with the concavity of the production technology make the surplus generated by a firm-worker pair be hump-shaped in capital.

Firms are ex-ante identical. In equilibrium, a mass of firms commits to renting the machine quality that maximizes the surplus derived from a match with an unskilled worker. Likewise, a continuum of firms commits to machines with higher quality because they and skilled workers anticipate that their match would produce a larger surplus, and thus higher wages and profits. Higher expected profits attract a relatively higher number of firms, so they are associated with higher exit rates from unemployment. Therefore, the average exit rate from unemployment and the average re-employment wage fall with duration because of the variation in the composition of the unemployment pool.

It should be noted that due to the asymmetric information assumption and the hump shape of the surplus function, unskilled workers face a trade-off between the more job opportunities that the skilled submarket offers and the higher wages they may obtain in the unskilled submarket. In this setting, ex-ante commitment and directed search are crucial for market segmentation because firms that target skilled workers may increase the machine quality if necessary to discourage the unskilled from applying. This occurs because capital-skill complementarity ensures that the losses associated with the capital increase are larger for unskilled workers, thereby leaving the skilled as the only applicants.

The present study makes novel contributions in several areas. We build on Peters (1991) and Moen (1997) by extending the competitive search literature to an economy in which firms commit to machine quality instead of wage contracts. Moreover, this paper analyzes the equilibrium allocation and its efficiency in an economy with asymmetric information, e.g.,

[1] We interpret machine quality in a generic manner as any set of inputs that are complementary in terms of production to unobservable skills. For example, this set could contain co-workers, physical capital and software (to the extent that the documented complementarities with the observable characteristics of the labor input, mainly education, may be extended to the unobservable ones), or intermediate goods.

similar to Michelacci and Suarez (2006) and Guerrieri, Shimer, and Wright (2010). Related research is discussed in Section II.

The remainder of this paper is organized as follows. In Section III, the economy is described. The equilibrium is characterized in Section IV. In Section V, the equilibrium outcomes are analyzed. In Section VI, the conclusions are presented. Finally, the proofs are provided in the Appendix and supplementary material is presented in an online Appendix.

II Related Literature

As mentioned above, other studies have also utilized a sorting mechanism to address the negative duration dependence of exit rates and re-employment wages. In Lockwood (1991), workers are also privately informed about their types, but firms have access to an imperfect screening technology. To the best of our knowledge, this random search model was the first investigation of a pure sorting mechanism in exit rates, although wages were assumed to remain constant in duration. Gonzalez and Shi (2010) and Fernández-Blanco and Preugschat (2014) considered economies with symmetric incomplete information regarding worker skills and learning from unemployment duration in a competitive search framework. In these settings, the mechanisms at work combined state-dependence and sorting effects. The exit rates did not necessarily fall with duration in the former, whereas wages might not be monotonic in the latter. Alternative mechanisms analyzed in previous studies belong to the category known as state dependence. For example, Pissarides (1992) and Ljungqvist and Sargent (2008) modeled human capital depreciation and search discouragement, while the deterioration in social networks during unemployment was analyzed in Calvó-Armengol and Jackson (2004).

We build on the seminal work of Peters (1991) and Moen (1997) by analyzing a model of the labor market where firms commit to machine quality instead of wage contracts to attract applicants, and workers direct their own search. Acemoglu and Shimer (1999) also analyzed an economy in which firms announce capital investment and wages are bargained during meetings, but with identical workers.

The present study contributes to research into asymmetric information in the labor market by exploring the dynamic consequences of capital-skill complementarity as well as performance-based pay schemes. In a static setting, Michelacci and Suarez (2006) allowed firms to post either single-wage job offers or wage-bargaining vacancies. We do not allow firms to commit to single wages in the benchmark, but we inspect alternative contracting spaces and constrained efficiency in the online Appendix. In Gale (1992) and Guerrieri, Shimer, and Wright (2010), workers self-selected according to the disutility of work. The former found that good workers work more hours and are better paid in the absence of frictions. The latter showed that this result does not extend directly to a frictional labor market because workers with a lower disutility from working have a higher employability rate, although wages and working time comparisons were based on additional assumptions.

Furthermore, the set of assumptions employed by Guerrieri, Shimer, and Wright (2010) does not hold in our model. In particular, their sorting assumption disqualifies any pooling contract from being part of an equilibrium allocation since firms can always deviate and deliver an arbitrarily small utility gain to the skilled workers and a strictly lower utility to the unskilled. Instead, in our setting, pooling contracts do not participate in the equilibrium because firms can increase profits by deviating to a higher machine quality, thereby attracting only skilled applicants even if the skilled also obtain utility losses.

In terms of constrained efficiency, under perfect information, Acemoglu and Shimer (1999) found that maximum welfare is attained in the market economy when firms commit to capital and wages are negotiated if the Hosios condition holds, whereas Shi (2001) showed that the equilibrium is constrained efficient if firms can post type-contingent offers. By contrast, with asymmetric information, the equilibrium was not constrained efficient when the labor market was segmented in Michelacci and Suarez (2006). In agreement with Guerrieri, Shimer, and Wright (2010), we find that unskilled workers are detrimental to skilled workers if firms over-invest to discourage unskilled workers from applying for skilled jobs, which pushes down the skilled wages and exit rates. A similar outcome was also reported in Moen (2003) and Moen and Rosen (2005). The former analyzed a labor market with individual- and match-specific productivity where firms only observed the overall productivity. In this setting, if firms cannot commit to productivity-contingent wages, there are too few skilled jobs posted in equilibrium and although the wage premium for skilled workers is larger than optimal, their welfare reduces. The latter study considered a frictionless economy with asymmetric information where firms offer performance-based pay to make workers invest greater effort in their jobs. They showed that the equilibrium is separating, but not efficient, because too much effort is induced in skilled jobs to separate types.

In contrast to previous research into assortative matching, the capital-skill complementarity assumption is sufficient to derive positive assortative matching (PAM) in equilibrium, which is crucial for achieving falling wages in equilibrium. Shimer and Smith (2000) and Eeckhout and Kircher (2010) showed that PAM requires stronger complementarity in a frictional economy. This was also the case for Shi (2001) in a setting similar to ours, although with perfect information on agent types and the capital investments made prior to matching. In Section V, it is argued that a crucial difference compared with these previous studies is that capital costs are incurred only while producing.

III Model

In this section, a frictional economy with heterogeneous workers and asymmetric information about the worker type is described.

Time is continuous. The economy is populated by a measure one of workers and a large continuum of ex-ante identical firms. Free entry determines the mass of active firms in equilibrium. Workers differ in their market skills. They can be either unskilled (type ℓ) or skilled (type h). There is a mass $\mu(s)$ of type s workers for $s \in \{\ell, h\}$ and $\mu(\ell) + \mu(h) = 1$.

The worker type is unobservable to the recruiting firms prior to production, but it can be assessed on the job. Workers and firms are risk neutral and future payoffs are discounted at a common rate r. The flow utility of unmatched workers is normalized to 0. Because of the focus on the steady state equilibrium, time indices are suppressed for notational simplicity. Let u_s denote the unemployment rate of type s workers at any instant.

Production. At any point in time, job-worker pairs with a machine of quality k produce $F(k,s) - ck$ units of net output, where $c > 0$ is the operating cost per quality unit. The following assumptions are imposed on the production technology. First, capital is an essential input, $F(0,s) = 0$. Second, the function F is increasing and concave in k. Note that the monotonicity and concavity of function F together with the linear variable costs make the net output a hump-shaped function of k. Third, skilled workers are more productive than unskilled workers for any machine quality, i.e., $F(k,\ell) < F(k,h)$ for $k > 0$. Fourth, the following standard limit conditions hold: $\lim_{k\to 0} F_k(k,s) = \infty$ and $\lim_{k\to\infty} F_k(k,s) = 0$. Fifth, capital and skills are complementary in production, i.e., $F_k(k,\ell) < F_k(k,h)$.

To allow agents to engage in search and production activities, it is assumed that some value k exists such that the net output $F(k,\ell) - ck$ is sufficiently large relative to the effective vacancy costs. Later, it is useful to refer to $\overline{k}_s$ as the solution of the equation $F_k(k,s) = c$, for $s \in \{\ell, h\}$. The assumptions on the production function F ensure the existence and uniqueness of a solution to that equation, and that $\overline{k}_\ell < \overline{k}_h$. Note that $\overline{k}_s$ maximizes the net output produced by a type s worker.

Search and Matching: Bargaining. Firms create a job at cost $c_0 > 0$. If a vacancy is filled, a surplus $\mathcal{S}$ is generated due to the matching frictions detailed below. We consider that wages are set ex-post through Nash-bargaining, but it is sufficient for now to assume a constant labor share of the surplus, α. As in Michelacci and Suarez (2006), we treat this as an institutional feature, and thus it is not contractible.[2] However, in contrast to

[2] We discuss alternative contracting environments in the online Appendix.

their setting, firms have the ability to commit to renting a machine quality k to attract workers. Therefore, vacancies are identified by the machine quality. There are infinitely many submarkets $k \in \mathcal{K}$.[3] A submarket where the firms and workers seek a trading partner is said to be active. Let $K \subset \mathcal{K}$ denote the subset of active submarkets.

Let $q(k)$ denote the ratio of workers relative to the vacancies created in submarket k. This is refered to as the expected queue length for submarket k. This ratio must be consistent with the optimizing behavior of the agents. Job-seekers and vacant firms rationally anticipate the wage linked to the committed capital k. Search is directed in the sense that workers are fully informed about vacancies and they apply for jobs that maximize their expected utility. Agents rationally anticipate that higher wages are associated with higher job-filling rates and lower job-finding rates. Note that due to the inverted U shape of the net output, higher machine qualities do not translate monotonically into higher wages.

A firm fills its vacancy at the Poisson rate $\eta(q(k))$. In submarket k, workers of either type find jobs at rate $\nu(q(k))$ because of the assumption of asymmetric information regarding the applicants' skills. The equality $\nu(q(k)) = \frac{\eta(q(k))}{q(k)}$ results from the fact that there is the same number of newly employed workers as newly filled jobs at any instant. To improve readability, the reference to the submarket is omitted, unless it is necessary. A number of standard assumptions on the matching rates are made. Function η is increasing in q, which indicates that it is easier to fill jobs in submarkets with more applicants per vacancy. Symmetrically, ν is decreasing in queue length to capture the intuition that finding a job is more difficult when workers are more abundant relative to vacancies. Furthermore, η and ν are twice continuously differentiable functions, and the elasticity of the job-finding probability, $\zeta(q)$, is an increasing function of the ratio. To guarantee the existence of an equilibrium, the following standard boundary conditions are assumed: $\lim_{q\to\infty} \eta(q) = \lim_{q\to 0} \nu(q) = \infty$ and $\lim_{q\to 0} \eta(q) = \lim_{q\to\infty} \nu(q) = 0$.

Furthermore, $\gamma(k, s)$ is defined as the share of applicants of type s who search for jobs in submarket $k \in \mathcal{K}$. Let $\Gamma(k) \equiv (\gamma(k,\ell), \gamma(k,h))$ be a point of the simplex Δ^1.

[3] As an abuse of language, but for the sake of notational simplicity, the letter k is used to refer indistinctly to the machine quality and its associated submarket.

Timing. The timing of the events is as follows. At the beginning of each instant, firms hold one job, which can be either vacant or filled. Workers are either employed or unemployed. There is potentially a continuum of submarkets indexed by a machine quality k. Vacant firms enter a submarket to locate their vacancy. Then, unemployed workers choose a submarket to search for a job. Matching, production, and wage-negotiation occur. Job-termination shocks are idiosyncratic and they hit active pairs at the Poisson rate λ. The worker becomes unemployed and the firm vanishes when shocks occur.

Value Functions. An unemployed worker of type s chooses the submarket that maximizes their utility. They become employed at rate $\nu(q)$ and obtain the value $\alpha\mathcal{S}_s(k)$.

$$rU_s = \max_k \nu(q(k))\alpha\mathcal{S}_s(k) \tag{1}$$

The surplus $\mathcal{S}_s(k)$ comprises the flow output net of the operating costs, as well as the net of the search option if they are unemployed. This satisfies the following functional equation.

$$(r+\lambda)\mathcal{S}_s(k) = F(k,s) - ck - rU_s \tag{2}$$

Firms choose a submarket to post their vacancies and maximize the expected profits. A firm incurs a one-time cost of c_0 when posting a vacancy in submarket k. The job is filled by a worker of type s with probability $\eta(q(k))\gamma(k,s)$. Then, the returns from a filled vacancy are $(1-\alpha)\mathcal{S}_s(k)$. Thus, the value of a vacant firm is defined by

$$rV = -c_0 + \eta(q(k))\sum_s \gamma(k,s)(1-\alpha)\mathcal{S}_s(k). \tag{3}$$

Furthermore, the expected profits must be zero in any active submarket in equilibrium because of free entry, $V=0$.

IV Equilibrium

We now define the steady state equilibrium, which is a natural extension of the equilibrium concept defined in the competitive search literature. We build upon the study of Guerrieri, Shimer, and Wright (2010). Note that rational expectations for job offers off the equilibrium path must also be set to help recruiting firms maximize their profits. Thus, the queue length $Q(k)$ and type distribution $\Gamma(k)$ must be defined for any submarket $k \in \mathcal{K}$.

Definition 1 *A steady state equilibrium consists of utility values $\{U_s\}_s$, surplus functions $\mathcal{S}_s : \mathcal{K} \to \mathcal{R}_+$, a set of active submarkets $K \subset \mathcal{K}$, a distribution G of vacancies across submarkets with support K, a queue length function $Q : \mathcal{K} \to \mathcal{R}_+$, and a type-share function $\Gamma : \mathcal{K} \to \Delta^1$ such that the following apply.*

1. *The surplus value $\mathcal{S}_s$ satisfies the functional equation (2).*

2. *Profit-maximization condition and free-entry condition.*

$$\forall k \in \mathcal{K},\ \eta(Q(k)) \sum_s \gamma(k,s)(1-\alpha)\mathcal{S}_s(k) \leq c_0;\ \textit{and with equality if } k \in K.$$

3. *Workers search optimally for jobs. For all $s \in \{\ell, h\}$,*

$$rU_s = \max_{k \in K} \nu(Q(k))\alpha\mathcal{S}_s(k),\ \textit{and } \nu(Q(k))\alpha\mathcal{S}_s(k) \leq rU_s, \forall k \in \mathcal{K}.$$

Furthermore, if k is such that $\gamma(k,s) > 0$ and $Q(k) > 0$, then $\nu(Q(k))\alpha\mathcal{S}_s(k) = rU_s$. If $\mathcal{S}_s(k) < 0$, then either $Q(k) = 0$ or $\gamma_s(k) = 0$.

4. *Resource constraint for labor.*

$$\int_{\mathcal{K}} \gamma(k,s)Q(k)dG(k) = u_s\mu(s),\ \forall s \in \{\ell, h\}$$

Firms set a job and choose a machine quality to maximize their profits. Because of free entry, the expected discounted profits are zero in equilibrium. Workers direct their search to maximize their expected discounted utility. The last condition ensures that, for a given type, the sum of job seekers across submarkets equals the total mass of unemployed.

The third equilibrium condition determines the rational expectations about the probability of filling a vacancy off the equilibrium path. The expected number of applicants per firm in an inactive submarket is determined by the maximum queue length that allows each type of worker to obtain their market utility U_s. Let us use a trembling-hand type of argument to provide an intuition. Consider an arbitrarily small mass of firms, which deviate from the equilibrium allocation and post their vacancies in submarket $k \notin K$. These firms form rational expectations about $Q(k)$, $\gamma(k,\ell)$, and $\gamma(k,h)$. Suppose that skilled workers are better off by applying to submarket k relative to all of the other active submarkets with the queue length that allows the unskilled to obtain their market utility U_ℓ at k. Then, there would be a larger inflow of skilled applicants to submarket k. This would reduce the expected utility obtained at k by the unskilled below U_ℓ. As a result, skilled workers would be the only applicants to these deviating firms. Note that this notion is a natural extension of the subgame perfection condition assumed in the competitive search framework with homogeneous workers.

Characterization of the Equilibrium

Next, the equilibrium allocation is characterized. First, it is shown that an equilibrium where both types of worker apply to the same vacancies cannot exist. Then, the existence and uniqueness of a separating equilibrium is proved. All of the proofs are given in the Appendix.

Consider a submarket k where skilled workers are willing to apply. The following lemma states that the surplus generated by skilled workers with machines of quality k is larger than that generated by the unskilled.

Lemma 1 *Let k be a submarket such that $\gamma(k,\ell) \geq 0$, $\gamma(k,h) > 0$ and $Q(k) > 0$. Then, $\mathcal{S}_\ell(k) < \mathcal{S}_h(k)$.*

From $\mathcal{S}_\ell(k) < \mathcal{S}_h(k)$, it follows that the firm value obtained from hiring a type ℓ worker in this submarket is lower than that when filling the vacancy with a type h worker. As a result, Lemma 1 partially resembles the monotonicity assumption in Guerrieri, Shimer, and Wright (2010), according to which the principals (firms) always prefer to trade with higher types.

Proposition 2 states that workers of different types cannot apply for the same jobs in equilibrium.

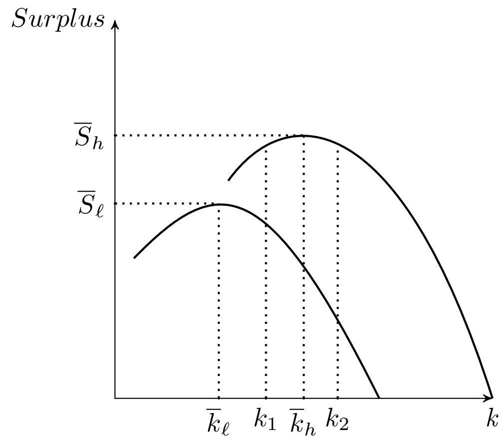

Figure 1: Surplus as a function of capital

Proposition 2 *There is no equilibrium in which workers of different types search for jobs in the same submarket. Thus, for all active submarkets $k \in K$, either $\gamma(k, \ell) = 1$ and $\gamma(k, h) = 0$, or vice versa.*

To understand this result, Figure 1 shows the surplus as a function of capital for each type of worker. Recall that the surplus has an inverted U-shape in k because of the monotonicity and concavity of the flow output as well as the linear costs that firms incur when operating. The previous lemma implies that the skilled surplus stands above its unskilled counterpart for the subset of capital qualities of interest. If a submarket k_1 is active at equilibrium with $\gamma(k_1, \ell), \gamma(k_1, h) > 0$, then committing to capital k' arbitrarily above k_1 would be a profitable deviation. By entering submarket k', firms can eliminate the unskilled applicants, thereby achieving a discrete jump in profits. This is the case because unskilled workers would need to be compensated with a higher job-finding rate in k' relative to k_1 due to their surplus fall, whereas the expected queue length that makes the skilled indifferent is indeed higher in k' than in k_1. Thus, this argument is similar to the sorting assumption in Guerrieri, Shimer, and Wright (2010), where firms can always find a way to locally increase the utility of the skilled and make the unskilled worse off to sort them out. However, this sorting assumption does not hold for a submarket $k_2 > \overline{k}_h$. Note that firms make both types either worse or better off when deviating from submarket k_2. However, the intuition derived from the equilibrium definition is that the relative rather than the absolute utility gains and losses are important. Thus, by deviating to submarket k' arbitrarily above k_2, firms can again screen out the unskilled because capital-skill complementarity implies that the surplus fall is relatively larger for the unskilled, and thus they must be compensated with a shorter queue length to make them indifferent.

As a result, if an equilibrium exists, it must be separating. The existence and uniqueness of a separating equilibrium is demonstrated. Given the pair (U_ℓ, U_h), problem (P_s) s defined as

$$\begin{aligned} \sup_{q,k} \quad & \nu(q)\alpha\mathcal{S}_s(k) \\ \text{s. to} \quad & \eta(q)(1-\alpha)\mathcal{S}_s(k) \geq c_0, \\ & \nu(q)\alpha\mathcal{S}_{-s}(k) \leq rU_{-s}, \end{aligned}$$

where function $\mathcal{S}_s$ is defined by (2). The solution to this problem maximizes the unemployment value of the type s worker subject to firms making non-negative profits, as in the standard characterization of the competitive search equilibrium allocation. However, the separating feature of the equilibrium requires one extra constraint. The last inequality is the non-participation condition for type $-s$ workers because they must have no incentive to apply to submarket of type s. As is shown in Section V, the non-participation condition for type h workers in problem (P_ℓ) does not bind. Therefore, the problem (P_ℓ) can be rewritten without this constraint. Proposition 3 shows that an equilibrium allocation is a solution of the set of problems $(P_s)_s$, and vice versa.

Proposition 3 *A unique separating equilibrium exists. The equilibrium set of active submarkets $K \equiv \{k_\ell, k_h\}$ and the respective queue lengths q_ℓ and q_h are determined by the following conditions*

$$k_\ell = \overline{k}_\ell, \tag{4}$$

$$c_0 = \eta(q_\ell)(1-\alpha)\frac{F(k_\ell, \ell) - ck_\ell}{r + \lambda + \alpha\nu(q_\ell)}, \tag{5}$$

$$c_0 = \eta(q_h)(1-\alpha)\frac{F(k_h, h) - ck_h}{r + \lambda + \alpha\nu(q_h)}, \tag{6}$$

and

$$k_h = \overline{k}_h, \text{ if } \nu(q_h)\alpha\mathcal{S}_\ell(k_h) \leq rU_\ell; \tag{7}$$

otherwise,

$$k_h > \overline{k}_h \text{ and } \nu(q_h)\alpha\mathcal{S}_\ell(k_h) = rU_\ell. \tag{8}$$

Proposition 3 states that a unique equilibrium exists. There are two possible types of equilibrium allocations, which are shown in Figure 2. In both submarkets, firm entry is determined by the corresponding zero-profit conditions (5) and (6). We plot the firms' zero-profit curves and the indifference curves for unemployed workers. Lemma 1 and the zero-profit condition imply that the zero-profit curve for skilled workers remains above that of its unskilled counterpart if the skilled are willing to apply to the submarket in question. Likewise, the capital-skill complementarity assumption is a single-crossing property, and thus the indifference curves cross only once.

First, consider the case of unskilled workers. According to equilibrium condition (4), unskilled workers apply to the submarket characterized by the machine quality $\overline{k}_\ell$. This result is intuitive because $\overline{k}_\ell$ maximizes the surplus created by a match with an unskilled worker, and both firms and unskilled workers benefit from the maximum surplus. Graphically, the equilibrium allocation corresponds to the tangency point between the zero-profit curve and the indifference curve because the non-participation condition does not bind in the unskilled case. In other words, unskilled workers sign up for the same job offer, as in the scenario with no informational frictions.

Next the skilled submarket is considered. First, if unskilled workers are not willing to apply to submarket $\overline{k}_h$, firms that target skilled workers aim to maximize the surplus from the match. Similar to the unskilled case, this allocation corresponds to the tangency point depicted in the left panel of Figure 2. This equilibrium allocation coincides with that with

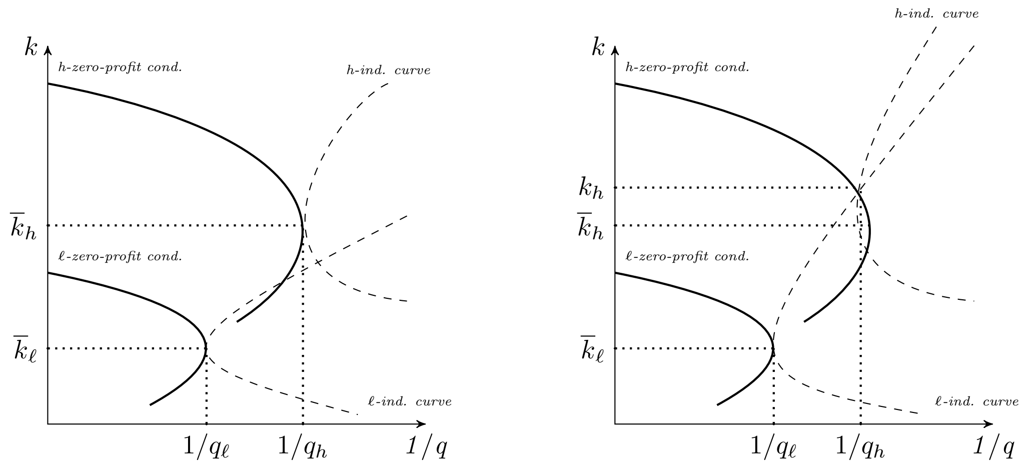

Figure 2: The two possible separating equilibrium allocations.

perfect information and type-contingent offers. Second, if the non-participation condition in problem (P_h) binds, condition (8) implies that the equilibrium capital $k_h > \overline{k}_h$. The right panel of Figure 2 shows this second case. As in Rothschild and Stiglitz (1976), Wilson (1977), and Guerrieri, Shimer, and Wright (2010), this is the minimum cost required to screen out unskilled applicants because it makes them indifferent with respect to the equilibrium submarket $\overline{k}_\ell$. The forgone welfare is accounted for by the capital difference $k_h - \overline{k}_h$, which we refer to in the following as *over-investment*, and the firm entry difference. Likewise, the forgone expected wages can be treated as the burden created by the presence of unskilled workers on the skilled. This result agrees with Moen (2003) and Moen and Rosen (2005).

The result showing the existence of a separating equilibrium is not new and it relies on the matching frictions as well as the capital-skill complementarity assumption. In particular, since screening out unskilled workers may require skilled agents to forgo some of their potential returns to matching, a deviation by firms (to a pooling submarket) that leads to a reduction in these forgone returns might be tempting and it could break the separating equilibrium. This was the case described in Rothschild and Stiglitz (1976). However, similar to Guerrieri, Shimer, and Wright (2010), these deviations are not profitable in the present case. To understand this, we simply need to determine what type of applicants might apply to the deviating firms in a submarket $k' \in (\overline{k}_h, k_h)$. Note that unskilled workers are indifferent between equilibrium submarket $\overline{k}_\ell$ and k_h, i.e., the queue length q_h and capital k_h maximize the search value of both types. Therefore, we need to determine who benefits the most from applying to submarket k'. By differentiating expression (1) with respect to k and evaluating it at k_h, we obtain:

$$\frac{\partial q_s}{\partial k}|_{k=k_h} = \frac{\nu(q_h)}{\nu'(q_h)} \frac{-F_k(k_h, s) + c}{\mathcal{S}_s(k_h)(r+\lambda)}, \text{ for } s \in \{\ell, h\}. \tag{9}$$

Lemma 1 implies that $\mathcal{S}_\ell(k_h) < \mathcal{S}_h(k_h)$. Furthermore, $F_k(k_h, \ell) - c < F_k(k_h, h) - c < 0$ due to capital skill complementarity. As a result, $0 > \frac{\partial q_h}{\partial k}|_{k=k_h} > \frac{\partial q_\ell}{\partial k}|_{k=k_h}$ when deviating to k'. Thus, the skilled workers benefit the least (their surplus increases relatively less) if they apply to submarket k'. Therefore, the queue length must increase relatively less to make them indifferent with respect to k_h. As a result, only the unskilled apply to the deviating submarket k'; hence, this cannot be a profitable deviation.

What leads to over-investment? As described above, we showed that over-investment $(k_h - \overline{k}_h)$ can occur at equilibrium. This result is interesting because it is an inefficient outcome, as is shown in the online Appendix. Some of the key forces that underlie over-investment are now considered.

Proposition 3 states that type h firms over-invest in machine quality at equilibrium if and only if the unskilled are better off searching for a job with machine quality $\overline{k}_h$. Thus, over-investment occurs at equilibrium if and only

$$\nu(\overline{q}_h)\mathcal{S}_\ell(\overline{k}_h) > \nu(\overline{q}_\ell)\mathcal{S}_\ell(\overline{k}_\ell), \tag{10}$$

where $\overline{q}_s$ is defined by the equation $c_0 = \eta(\overline{q}_s)(1-\alpha)\frac{F(\overline{k}_s, s) - c\overline{k}_s}{r+\lambda+\alpha\nu(\overline{q}_s)}$. As Proposition 5 states below, this may not be the case because of higher wages, but instead because of the more job opportunities workers have in the skilled submarket. Whether this inequality holds or not depends on the functional forms and the parameter values. In particular, it depends on the skill difference, the capital elasticity of the production function, and the job-finding probability function. The following lemma states that firms over-invest at equilibrium if the

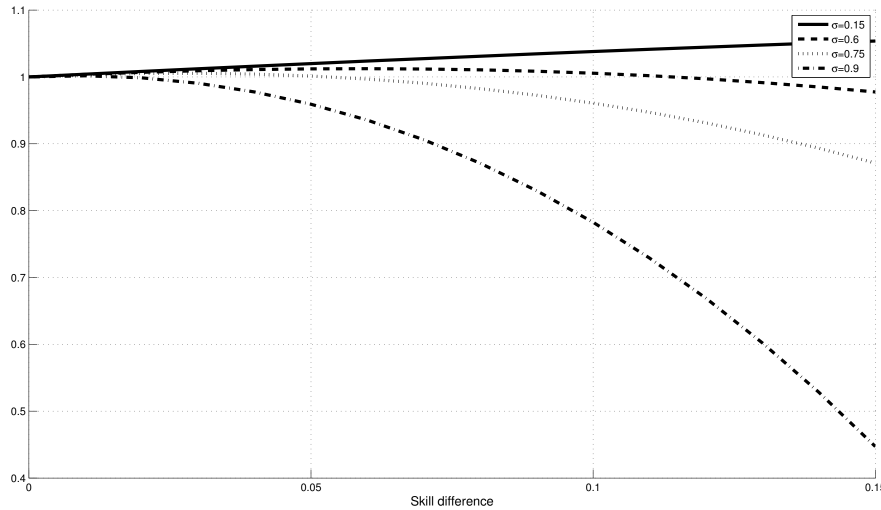

Figure 3: The equilibrium condition for over-investment for an economy with Cobb–Douglas production technology and an urn-ball matching function.

skill difference is sufficiently small because the potential employment gains from applying for type h jobs are of a higher order than the associated wage losses.[4]

Lemma 4 *Over-investment occurs at equilibrium if the skill difference is sufficiently small.*

More generally, Figure 3 illustrates the case for a Cobb–Douglas production technology, $F(k, a) = k^{\sigma} a^{1-\sigma}$, where $\sigma \in (0, 1)$ and $a \in \{a_\ell, a_h\}$ denotes the market skills of the employee, and a Cobb–Douglas matching function, $\nu(q) = q^{\alpha-1}$. We obtain $\overline{k}_s = a_s \left(\frac{\sigma}{c}\right)^{\frac{1}{1-\sigma}}$, for $s \in \{\ell, h\}$. The machine quality $\overline{k}_s$ increases with the skills and decreases with the operating cost, whereas the effects of the capital elasticity σ are uncertain. Condition (10) becomes

$$\frac{\nu(\overline{q}_h)(r+\lambda) + \alpha\nu(\overline{q}_\ell)\nu(\overline{q}_h)}{\nu(\overline{q}_\ell)(r+\lambda) + \alpha\nu(\overline{q}_\ell)\nu(\overline{q}_h)} \frac{F(\overline{k}_h, \ell) - c\overline{k}_h}{F(\overline{k}_\ell, \ell) - c\overline{k}_\ell} > 1. \tag{11}$$

The first factor in this expression is greater than 1 because skilled jobs are relatively more abundant, whereas the second factor is lower than 1 because $\overline{k}_\ell$ is the surplus-maximizing level. Whether or not this product is above 1 depends on the parameter values. Figure 3 plots this product for several values of the skill difference $a_h - a_\ell$ and capital elasticity σ. Over-investment occurs mostly for low values of the capital elasticity and skill differences. Our

[4]The authors thank a referee for drawing our attention to this point.

intuition regarding capital elasticity is that the wage penalization associated with applying for jobs of quality $\overline{k}_h$ increases as the firms' reaction through the capital margin becomes more important. Furthermore, the occurrence of over-investment at equilibrium is generally not monotonic in the skill difference, as shown for $\sigma = 0.6$.

V Exit Rates and Re-employment Wages

The equilibrium outcomes are now considered, particularly the dynamics of the exit rates from unemployment and re-employment wages over the unemployment duration. All of the proofs are given in the Appendix.

After matching, the wages are Nash-bargained. Let α denote the bargaining power of the worker. In the Appendix, the equilibrium wages are derived, which are

$$w_s = \alpha\big(F(k_s, s) - ck_s\big) + (1-\alpha)rU_s. \tag{12}$$

In addition to their continuation value of unemployment, workers obtain a share α of the surplus. The following proposition states that skilled workers have better employment prospects on all dimensions compared with their unskilled counterparts.

Proposition 5 *The equilibrium machine quality, job-finding rate, surplus, unemployment value, and wage are higher for skilled workers.*

The returns obtained by skilled workers from searching in the type ℓ submarket are strictly lower than the returns from searching in the skilled submarket. Therefore, the non-participation constraint in problem (P_ℓ) does not bind, as had been speculated when characterizing the equilibrium allocation. By contrast, unskilled workers face a trade-off when considering whether to apply for skilled jobs. First, skilled jobs are relatively more abundant, and second, the net output produced by operating with a machine quality is above $\overline{k}_\ell$, and thus wages are lower. Firms that target skilled workers use the capital margin to discourage unskilled workers from applying, if necessary. This may imply that there is a burden on skilled workers in terms of excessive capital, but their employment prospects are always relatively better.

Wage differences. The equilibrium wage difference can be derived from expression (12). Note that wage differences among observationally identical workers are not derived simply from differences in the worker unobservable characteristics. An amplification mechanism occurs at equilibrium due to the reaction of ex-ante identical profit-maximizing firms to worker differences. First, firms commit to a higher machine quality to attract (only) skilled workers. Second, the expected returns from posting vacancies in the skilled submarket outweigh those in the unskilled submarket, so firm entry is greater in the former. As a result, the exit rates from unemployment, and thus the outside options of the skilled workers, are higher.

Dynamics over the duration of unemployment. The steady state unemployment rate of workers of type $s \in \{\ell, h\}$ is determined by

$$u_s = \frac{\lambda}{\lambda + \nu(q_s)}. \tag{13}$$

Proposition 5 implies that the unemployment rate of skilled workers is lower. The dynamics of the equilibrium variables over the duration of unemployment are now investigated. Let the length of an unemployment spell be denoted by $\tau \in (0, \infty)$. The average job-finding rate at duration τ is derived as the ratio of the mass of workers who find a job immediately after a period of length τ relative to the total mass of unemployed for a period of this length. The average wage for new matches conditional on an unemployment period of length τ is obtained in an analogous manner.

Proposition 6 *The average job-finding rates and re-employment wages decline with the duration of unemployment, and they flatten out for long periods. If $\mu(h) \leq 0.5$, then the rate at which exit rates fall also declines with duration.*

Proposition 6 states that these two variables decline over an unemployment period despite constant individual rates and wages. This falling trend is due to a pure sorting effect. Ex-ante differences across workers affect output directly. Due to the performance-related pay schemes, firms can separate different types of workers into different submarkets by committing to an input that is complementary in terms of production to the workers skills. Skilled workers are more likely to exit unemployment as well as obtaining higher wages. Therefore, the average worker is more likely to be unskilled when the unemployment duration is longer. Two other implications are derived from the pure sorting mechanism. First, both the average exit rates and re-employment wages flatten out for sufficiently long periods. Second, because they are functions of the unemployment duration, these variables are either convex or they begin as concave and then become convex after some duration, where the inflection point depends on the parameter values. In particular, a sufficient condition for the rate of the decline of the exit rate to decrease steadily over the duration of unemployment is that the mass of skilled workers is less than half of the total population in the economy.

The empirical evidence for exit rates is robust in terms of these three features: they fall with the duration of unemployment at a declining rate, and flatten out for long spells, e.g., see Shimer (2008) and Farber and Valletta (2013). A number of studies have established the negative relationship between re-employment wages and unemployment duration, e.g., see Addison, Portugal, and Centeno (2004) and Fernández-Blanco and Preugschat (2014). However, to the best of our knowledge, there is no clear evidence for wages regarding the other two features.

Although alternative mechanisms may also be consistent with the empirical falling trends in these two variables, they may be at odds with the decreasing decline in exit rates. For example, although little is known about the precise process that rules the depreciation of human capital over the duration of unemployment, in Machin and Manning (1999), it was stated that: "an educated guess might be that productivity falls slowly initially but there is then a period in which deterioration is rather rapid and then it bottoms out." Under this assumption, the exit rates and re-employment wages will follow productivity. A similar pattern is likely to occur if the mechanism at work is the deterioration of the social network of an unemployed worker, e.g., see Figure 6 in Calvó-Armengol and Jackson (2004).

Positive Assortative Matching and Wages

The unobserved heterogeneity explanation is consistent with both falling exit rates and wages over the duration of unemployment if the workers who leave unemployment sooner also obtain higher wages. Therefore, PAM is a key equilibrium outcome. However, it is a robust result in previous research into assignment, where capital-skill complementarity is not sufficient to produce PAM in a frictional economy. In a random search framework, Shimer and Smith (2000) showed that PAM requires log-supermodularity. Eeckhout and Kircher (2010) found that a weaker (root-supermodularity) condition is necessary and sufficient for PAM in a competitive search economy. Shi (2001) also demonstrated that fairly strong complementarity is needed in a world closer to ours, but where there is perfect information about worker types and firms differentiate themselves by investing in capital, which is a sunk cost at the meeting stage. The requirement for strong complementarity in competitive search models is due to the trade-off between the complementarity in production and the complementarity in securing the match. Thus, high-value workers (buyers) find that it is optimal at equilibrium to match with low value firms (sellers) if there is no complementarity in production (trade). This is the case because the former accept a low wage (high price) to increase the matching chances, whereas the latter are more willing to make business through wages (prices) at the risk of remaining idle. Simple supermodularity is not sufficient to outweigh these preferences.

The fundamental difference in our setting is that the sunk costs c_0 do not depend on the machine quality to which firms commit and the operating costs c are only incurred if the job is filled.[5] To understand this, we consider the perfect information scenario analyzed in

[5] As stated in the Introduction, capital is treated as any input that is complementary in production to worker skills such as co-worker skills, machine quality, or the quantity and quality of intermediate goods. Therefore, the associated costs are variable instead of fixed. However, because of its centrality, alternative

Shi (2001), but with our cost assumptions. The capital decision is then determined by the first order condition $F_k(k, s) = c$. It follows that capital-skill complementarity is sufficient to obtain PAM because of the assumption that the cost scheme breaks the link between machine quality and search decisions. Thus, there are no high or low type firms ex-ante, but there are different match values ex-post, and the capital decision maximizes the joint value of the worker-firm pair. Clearly, adverse selection may restore the link between capital and the search for skilled jobs. However, in this case, Proposition 3 states that the machine quality is always higher for skilled jobs to prevent unskilled workers from applying, and Proposition 5 shows that private information does not reverse the order for exit rates and wages.

VI Conclusions

This paper models a sorting mechanism based on adverse selection to explain the declining exit rates from unemployment and re-employment wages over the duration of unemployment. The central assumptions of our model are as follows. First, workers are informed privately about their skills. Second, recruiting firms can commit to an input (e.g., capital) that is complementary in terms of production with worker skills. Third, firms have the ability to set performance-related pay schemes. Fourth, the search is directed.

In this setting, different types of workers search for a job in different submarkets at equilibrium. Skilled workers look for jobs with higher machine quality, experience more job opportunities, and obtain higher wages than their unskilled counterparts, i.e., a separating equilibrium exists with PAM. Therefore, both the average exit rate from unemployment and the entry wage fall with the duration of unemployment because the composition of the pool of unemployed workers deteriorates over time. Furthermore, separating worker types may be costly, where firms that target skilled workers may have to over-invest in order to discourage applications from unskilled workers. In this case, the equilibrium allocation is not constrained efficient.

cost schemes are studied in the online appendix.

Another interpretation of the sorting explanation is that firms observe the applicant's type and base their recruiting decision on this information. However, if a worker's type is not contractible, our results would still hold in this alternative setting.

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Appendix

Surplus and Wage Determination.

To determine wages as a Nash-bargaining solution, we need to add some details to our setting. A type s employed worker obtains the flow wage $w_s(k)$ and becomes displaced at rate λ. The employment value is defined as

$$rE_s(k) = w_s(k) + \lambda\big(U_s - E_s(k)\big). \tag{A1}$$

Analogously, the value of a filled vacancy is determined by the following functional equation.

$$rJ_s(k) = F(k,s) - ck - w_s(k) - \lambda J_s(k) \tag{A2}$$

The net surplus is defined as $\mathcal{S}_s(k) \equiv J_s(k) + E_s(k) - U_s$. Using the above expressions for the value functions, the functional equation (2) for the surplus value function is obtained.

After matching, a firm and a worker set the wage through Nash-bargaining. Let α denote the bargaining power of the worker. The equilibrium wage is the solution of the following problem.

$$\max_w \left(E_s(k) - U_s\right)^{\alpha} \left(J_s(k) - V\right)^{1-\alpha} \tag{A3}$$

This solution is characterized by $J_s(k) = (1-\alpha)\mathcal{S}_s(k)$. By using the latter equation to replace the value J_s in the above Bellman equation (A2), it is concluded that the equilibrium wages are determined as

$$w_s(k) = \quad \alpha\big(F(k,s) - ck\big) + (1-\alpha)rU_s. \tag{A4}$$

Thus, workers obtain a proportion α and firms receive the remaining $1-\alpha$ of the surplus of the match, in addition to the unemployment value rU_s.

Proofs

Proof of Lemma 1. Consider a submarket k such that $\gamma(k,\ell) \geq 0$, $\gamma(k,h) > 0$, and $Q(k) > 0$. Using the functional equations (1) and (2), the following expression is obtained.

$$\left(r + \lambda + \alpha\nu(Q(k))\right)\left(\mathcal{S}_h(k) - \mathcal{S}_\ell(k)\right) \geq F(k,h) - F(k,\ell) \tag{A5}$$

The right-hand side of this inequality is strictly positive because skilled workers are more productive. Therefore, the surplus gap is positive, $\mathcal{S}_h(k) > \mathcal{S}_\ell(k)$.‖

Proof of Proposition 2. The proof is obtained by contradiction. Suppose that there is an equilibrium where a submarket k is active and $\gamma(k,\ell), \gamma(k,h) > 0$. According to Lemma 1, $\mathcal{S}_\ell(k) < \mathcal{S}_h(k)$. Both types of workers enter submarket k, so we have

$$rU_s = \alpha\nu(Q(k))\mathcal{S}_s(k), \text{ for } s \in \{\ell, h\}. \tag{A6}$$

The expected profits of firms are defined by expression (3) and the average ex-post profits across worker types.

Consider now the alternative submarket k', where $k' \equiv k + \epsilon$ and ϵ are arbitrarily small and positive. By differentiating expression (A6) with respect to k, we obtain

$$\frac{\partial q_s}{\partial k} = \frac{\nu(Q(k))}{\nu'(Q(k))} \frac{-(F_k(k,s) - c)}{\mathcal{S}_s(k)(r+\lambda)}, \text{ for } s \in \{\ell, k\}, \tag{A7}$$

where $\frac{\partial q_s}{\partial k}$ measures how much the queue length needs to change to keep a worker of type s indifferent between submarkets k and k'. The above expression considers that

$$(r+\lambda)\frac{d\mathcal{S}_s(k)}{dk} = F_k(k,s) - c. \tag{A8}$$

The capital-skill complementarity assumption implies that $\frac{d\mathcal{S}_\ell(k)}{dk} < \frac{d\mathcal{S}_h(k)}{dk}$. Recall that ν is a decreasing function. Consider two cases, as follows.

Case 1: Suppose that $k \in [\overline{k}_\ell, \overline{k}_h]$. Then, $-(F_k(k,h) - c) \leq 0 \leq -(F_k(k,\ell) - c)$. Therefore, $\frac{\partial q_h}{\partial k} \geq 0 \geq \frac{\partial q_\ell}{\partial k}$, with has at least one strict inequality. Unlike the skilled workers, unskilled workers must be compensated with a higher job-finding rate to make them indifferent between k and k'.

Case 2: Suppose now that $\overline{k}_h < k$. Then, $0 < -(F_k(k,h) - c) < -(F_k(k,\ell) - c)$. It follows that $\frac{\partial q_\ell}{\partial k} < \frac{\partial q_h}{\partial k} < 0$. The queue length must decrease more for unskilled workers so they are indifferent between k and k'.

In both cases, firms that deviate to submarket k' achieve a discrete jump in profits because low-ability workers do not apply to k' and $\mathcal{S}_\ell(k) < \mathcal{S}_h(k)$. Therefore, we find a contradiction in either case; hence, submarket k cannot be active at equilibrium.$\|$

Proof of the Existence of the Equilibrium.

We proceed according to several steps. First, we focus on the necessary conditions that the equilibrium allocation must satisfy for type ℓ workers and we then consider the skilled. Second, we establish that these conditions are also sufficient and we state the existence of an equilibrium.

First, we state and prove the following result.

Claim 7 *Let k be a submarket such that either $\gamma(k,\ell) > 0$ or $\gamma(k,h) > 0$. Then,*

$$\sum_s \gamma(k,s)(1-\alpha)\mathcal{S}_s(k) \geq (1-\alpha)\mathcal{S}_\ell(k). \tag{A9}$$

Proof Expression (A9) is equivalent to

$$(1-\gamma(k,\ell))\mathcal{S}_h(k) \geq (1-\gamma(k,\ell))\mathcal{S}_\ell(k). \tag{A10}$$

We analyze two cases. If $\gamma(k,\ell) = 1$, then expression (A10) holds with equality. Otherwise, if $\gamma(k,\ell) < 1$, then the inequality holds because of Lemma 1.‖

Now, we consider problem (P_ℓ).

Proposition 8 *Let $\{G, K, Q, \Gamma, \{U_s, \mathcal{S}_s\}_s\}$ be an equilibrium. If an active submarket $k_\ell \in K$ exists such that $q_\ell \equiv Q(k_\ell) > 0$ and $\gamma(k_\ell, \ell) > 0$, then the vector (q_ℓ, k_ℓ) is the unique solution of problem (P_ℓ) and the system of equations (4) and (5).*

Proof First, it is shown that for any $k \in K$ such that $q_\ell \equiv Q(k) > 0$ and $\gamma(k,\ell) > 0$, the pair $(q_\ell, \mathcal{S}_\ell(k))$ solves the following problem (A11). Then, we prove that the solution of the problem is uniquely characterized by conditions (4) and (5):

$$\begin{aligned} \max_{q\in[0,\infty], S\in[0,\overline{S}]} \quad & \nu(q)S \\ \text{s. to} \quad & \eta(q)(1-\alpha)S \geq c_0, \end{aligned} \tag{A11}$$

where $\overline{S} \equiv \max_{k'} \mathcal{S}_\ell(k')$ is the maximum surplus. First, note that Weierstrass Theorem appliesto ensure the existence of a solution to this maximization problem; therefore, it is well-defined. Similarly, Proposition 2 ensures that if k is an active submarket such that $\gamma(k,\ell) > 0$, then $\gamma(k,\ell) = 1$. As a result, the second equilibrium condition implies that the constraint on problem (A11) evaluated for the pair $(q_\ell, \mathcal{S}_\ell(k))$ holds with equality. We prove that this is a maximizer by contradiction. Suppose that $(q', S') \in [0, \infty] \times [0, \overline{S}]$ exists such that $\nu(q')\alpha S' > \nu(q_\ell)\alpha \mathcal{S}_\ell(k)$, and $\eta(q')(1-\alpha)S' \geq c_0$. The continuity, monotonicity, and concavity of the value function $\mathcal{S}_\ell$ are inherited from the production technology. Further, the surplus reaches its maximum at $\overline{k}_\ell$; hence, $\overline{S} = \mathcal{S}_\ell(\overline{k}_\ell)$. By continuity, $k' \in [0, \overline{k}_\ell]$ must exist such that $\mathcal{S}_\ell(k') = S'$. By the definition of the equilibrium beliefs, $Q(k') \geq q'$. Therefore,

$$\eta(Q(k'))\sum_s \gamma(k', s)(1-\alpha)\mathcal{S}_s(k') \geq \eta(Q(k'))(1-\alpha)\mathcal{S}_\ell(k') > \eta(q')(1-\alpha)\mathcal{S}_\ell(k') \geq c_0,$$

where the first inequality comes from Claim 7, and the second inequality results from the monotonicity of function η. This implies that a deviation to submarket k' would be profitable. This contradicts the assumption that k is an active submarket in equilibrium; therefore, the pair $(q_\ell, \mathcal{S}_\ell(k))$ is a solution to the problem (A11).

Now, we exploit the fact that the constraint must hold with equality to rewrite the problem (A11) as

$$\max_{q \in [b, \infty]} 1/q,$$

where $\eta(b)$ equals $\frac{c_0}{(1-\alpha)\overline{S}}$. Thus, the maximizer q is b, and k must be $\overline{k}_\ell$. Thus, conditions (4) and (5) hold at equilibrium, and the solution to the problem (P_ℓ) is unique. $\|$

Now, consider the problem (P_h).

Proposition 9 *Let $\{G, K, Q, \Gamma, \{U_s, \mathcal{S}_s\}_s\}$ be an equilibrium. Consider an active submarket k_h such that $q_h \equiv Q(k_h) > 0$ and $\gamma(k_h, h) > 0$, then the equilibrium vector (q_h, k_h) is the unique solution to the problem (P_h) given U_ℓ, which must satisfy the following conditions:*

$$c_0 = \eta(q)(1-\alpha)\frac{F(k,h) - ck}{r + \lambda + \alpha\nu(q)} \tag{A12}$$

and

$$k = \overline{k}_h, \text{ if } \nu(q)\alpha\mathcal{S}_\ell(k) \leq rU_\ell; \tag{A13}$$

otherwise,

$$k > \overline{k}_h \text{ and } \nu(q)\alpha\mathcal{S}_\ell(k) = rU_\ell. \tag{A14}$$

Proof First, arguing by contradiction, it is shown that for any active submarket $k \in K$ such that $q \equiv Q(k) > 0$ and $\gamma(k, h) > 0$, the vector (q, k) solves problem (P_h) given the value U_ℓ.

Proposition 2 ensures that if k is such that $\gamma(k, h) > 0$, then $\gamma(k, h) = 1$. As a result, the second equilibrium condition implies that the first constraint on problem (P_h) holds with equality. The third equilibrium condition ensures that the second constraint also holds given the equilibrium value U_ℓ when evaluated for the pair (q, k). Therefore, the pair (q, k) belongs to the constraint set. Now, suppose that (q', k') exists such that $\nu(q')\alpha\mathcal{S}_h(k') > \nu(q)\alpha\mathcal{S}_h(k)$, and the two constraints on problem (P_h) hold. ν is an decreasing continuous function, so $\widetilde{q} > q'$ must exist such that $\nu(\widetilde{q})\alpha\mathcal{S}_h(k') = \nu(q)\alpha\mathcal{S}_h(k)$. By the definition of the equilibrium beliefs, $Q(k') \geq \widetilde{q}$. Therefore, the type-ℓ workers must be strictly worse off in submarket k' because $\nu(Q(k'))\alpha\mathcal{S}_\ell(k') < \nu(q')\alpha\mathcal{S}_\ell(k') \leq rU_\ell$, and it follows that $\gamma(k', \ell) = 0$. Therefore,

$$\eta(Q(k'))\sum_s \gamma(k', s)(1-\alpha)\mathcal{S}_s(k') = \eta(Q(k'))(1-\alpha)\mathcal{S}_h(k') > \eta(q')(1-\alpha)\mathcal{S}_h(k') \geq c_0$$

This result contradicts the assumption that submarket k is active in equilibrium because the deviation of a firm to submarket k' would be profitable. Thus, the equilibrium pair (q, k) is a solution to problem (P_h).

Now, the existence and uniqueness of a solution to Problem (P_h) is demonstrated. Due to the continuity of the objective function and the compactness of the constraint set of the problem, the Weierstrass Theorem is applied to ensure the existence of a solution. Indeed, the constraint set is non-empty because the pair $q = \infty$ and $k = \overline{k}_h$ (for example) satisfies both constraints.

Later, it is shown that any solution satisfies the first constraint with equality. Then, by using the constraint to replace the surplus from the objective function, as in the proof of Proposition 8, the problem can be rewritten as maximizing function $1/q$ subject to the two constraints on problem (P_h). Therefore, the maximizer is the minimum feasible queue length. Hence, a unique solution exists for q and k. It is straightforward to show that the solution k must be greater than $\overline{k}_\ell$; otherwise, we would obtain a contradiction by increasing k by an arbitrarily small amount. This is the case because it can be argued along the same lines as the proof of Lemma 1 that $\mathcal{S}_h(k) > \mathcal{S}_\ell(k)$.

To characterize the solution of problem (P_h), we define the Lagrangian as

$$\mathcal{L} = \nu(q)\alpha\mathcal{S}_h(k) + \xi_1\left(\eta(q)(1-\alpha)\mathcal{S}_h(k) - c_0\right) - \xi_2\left(\nu(q)\alpha\mathcal{S}_\ell(k) - rU_\ell\right),$$

where ξ_1 and ξ_2 denote the non-negative multipliers associated with the first and second constraints, respectively. The Kuhn Tucker (necessary) conditions are

$$\nu'(q)\alpha\left(\mathcal{S}_h(k) - \xi_2\mathcal{S}_\ell(k)\right) + \xi_1\eta'(q)(1-\alpha)\mathcal{S}_h(k) \leq 0,\ q \geq 0 \tag{A15}$$

and $$q\nu'(q)\alpha\left(\mathcal{S}_h(k) - \xi_2\mathcal{S}_\ell(k)\right) + q\xi_1\eta'(q)(1-\alpha)\mathcal{S}_h(k) = 0.$$

$$\tfrac{d\mathcal{S}_h(k)}{dk}\left(\nu(q)\alpha + \xi_1\eta(q)(1-\alpha)\right) - \xi_2\nu(q)\alpha\tfrac{d\mathcal{S}_\ell(k)}{dk} \leq 0,\ k \geq 0 \tag{A16}$$

and $$k\tfrac{d\mathcal{S}_h(k)}{dk}\left(\nu(q)\alpha + \xi_1\eta(q)(1-\alpha)\right) - k\xi_2\nu(q)\alpha\tfrac{d\mathcal{S}_\ell(k)}{dk} = 0.$$

$$\eta(q)(1-\alpha)\mathcal{S}_h(k) \geq c_0,\ \xi_1 \geq 0 \text{ and } \xi_1\left[\eta(q)(1-\alpha)\mathcal{S}_h(k) - c_0\right] = 0. \tag{A17}$$

$$\nu(q)\alpha\mathcal{S}_\ell(k) \leq rU_\ell,\ \xi_2 \geq 0 \text{ and } \xi_2\left[\nu(q)\alpha\mathcal{S}_\ell(k) - rU_\ell\right] = 0. \tag{A18}$$

This set of complementary slackness conditions allows the characterization of the solution to the problem. The case where $\xi_2 = 0$ is analogous to that studied in the proof of Proposition 8, i.e., $k = \overline{k}_h$ and the necessary condition (A12) holds. Furthermore, the second constraint on the problem (P_h) must also hold for the solution pair (q, x).

Now, suppose that $\xi_2 > 0$, then the second constraint must be binding according to the complementary slackness condition (A18). This implies that $q \in (0, \infty)$. Next, it is shown by contradiction that the first constraint also cannot be slack in this case. Suppose that $\xi_1 = 0$ and $\eta(q)(1-\alpha)\mathcal{S}_h(k) > c_0$. It follows from condition (A15) that $\mathcal{S}_h(k) = \xi_2\mathcal{S}_\ell(k)$, and from condition (A16), we obtain $\frac{d\mathcal{S}_h(k)}{dk} = \xi_2\frac{d\mathcal{S}_\ell(k)}{dk}$. As we argued above, $k > \overline{k}_\ell$, so $\frac{d\mathcal{S}_\ell(k)}{dk} < 0$ implies that $\frac{d\mathcal{S}_h(k)}{dk} < 0$ and $\xi_2 < 1$. Therefore, $\mathcal{S}_h(k) < \mathcal{S}_\ell(k)$, which is not true. As a result, condition (A12) holds. Finally, the necessary condition (A16) together with $\frac{d\mathcal{S}_\ell(k)}{dk} < 0$ leads to $\frac{d\mathcal{S}_h(k)}{dk} < 0$, i.e., $k > \overline{k}_h$.‖

Now, it is shown that a solution to the set of problems $(P_s)_s$ participates in an equilibrium allocation, and the existence and uniqueness of the equilibrium allocation is proven. The proof of Proposition 3 then follows.

Proof of Proposition 3.

Let (q_s, k_s) be the vector solution of problem (P_s), where $\mathcal{S}_s(k_s) = \frac{F(k_s,s)-ck_s}{r+\lambda+\alpha\nu(q_s)}$ and $rU_s = \nu(q_s)\alpha\mathcal{S}_s(k_s)$. We prove that it participates in an equilibrium allocation. An equilibrium is a tuple $\{G, K, Q, \Gamma, \{U_s, \mathcal{S}_s\}_s\}$ that satisfies the equilibrium definition.

We define $K \equiv \{k_\ell, k_h\}$. The surplus function is determined by the functional equation (2) and the above unemployment values. The remaining equilibrium objects are set in a manner consistent with the equilibrium definition:

$$dG_\psi(k) \equiv \begin{cases} (1-\mu)u_\ell/q_\ell, \text{ if } k = k_\ell \\ \mu u_h/q_h, \text{ if } k = k_h \\ 0, \text{ otherwise,} \end{cases}$$

where $u_s \equiv \frac{\lambda}{\lambda+\nu(q_s)}$. By construction, the distribution G trivially ensures that the last equilibrium condition holds. The beliefs for all off-the-equilibrium k are defined as

$$Q(k) \equiv \max_s \hat{q}_s(k),$$

$$\gamma(k,\ell) = \begin{cases} 1, \text{ if } \hat{q}_\ell(k) > \hat{q}_h(k) \\ 0, \text{ otherwise} \end{cases}, \text{ and } \gamma(k,h) = \begin{cases} 1, \text{ if } \hat{q}_h(k) \geq \hat{q}_\ell(k) \\ 0, \text{ otherwise,} \end{cases}$$

where

$$\hat{q}_s(k) \equiv \begin{cases} q \text{ such that } \nu(q)\alpha\mathcal{S}_s(k) = rU_s, \text{ if } \mathcal{S}_s(k) > 0 \\ 0, \text{ otherwise.} \end{cases}$$

Note that the beliefs are well defined given the assumptions on the matching function ν and there is no submarket k such that both types of workers enter simultaneously.

Next, it isshown that firms maximize their profits, which are zero at equilibrium in all active submarkets. It is then proven that workers search optimally for jobs.

The proofs of Propositions 8 and 9 show that the free entry condition holds in each active submarket. By contradiction, it is proven that firms maximize their profits by entering either of the active submarkets in K. We distinguish two cases.

Case 1. Suppose that a submarket k' exists such that $Q(k') > 0$, $\gamma(k', \ell) = 1$ and

$\eta(Q(k'))(1-\alpha)\mathcal{S}_\ell(k') > c_0$. To attract unskilled workers, it must be the case that $\nu(Q(k'))\alpha\mathcal{S}_\ell(k') = rU_\ell$. Because of the continuity and monotonicity of function η, $q' < Q(k')$ exists such that $\eta(q')(1-\alpha)\mathcal{S}_\ell(k') = c_0$. ν is a decreasing function of the queue length, so it follows that

$$\nu(q_\ell)\alpha\mathcal{S}_\ell(k_\ell) = \nu(Q(k'))\alpha\mathcal{S}_\ell(k') < \nu(q')\alpha\mathcal{S}_\ell(k'),$$

which contradicts the assumption that (q_ℓ, k_ℓ) is the maximizer of problem (P_ℓ). Therefore, there is no profitable deviation that attracts only unskilled workers.

Case 2. Suppose that a submarket k' exists such that $Q(k') > 0$, $\gamma(k', h) = 1$, and $\eta(Q(k'))(1-\alpha)\mathcal{S}_h(k') > c_0$. Skilled workers enter the submarket if

$$\nu(Q(k'))\alpha\mathcal{S}_h(k') = rU_h = \nu(q_h)\alpha\mathcal{S}_h(k_h).$$

Furthermore, the next inequality follows from the off-the-equilibrium beliefs defined above,

$$\nu(Q(k'))\alpha\mathcal{S}_\ell(k') \leq rU_\ell.$$

We distinguish two sub-cases. First, if the latter inequality is strict, then $q' < Q(k')$ must exist such that the two constraints on problem (P_h) hold at (q', k'). Due to the properties of the matching technology, it isfound that $\nu(q')(1-\alpha)\mathcal{S}_h(k') > rU_h$, which contradicts the assumption that (q_h, k_h) is the maximizer of problem (P_h). Second, if the inequality is indeed an equality, then it can be argued along the same lines as the proof of Proposition 2 that a submarket k'' exists, which differs from k' by an arbitrarily small amount, such that the first constraint on problem (P_h) holds, the second holds with strict inequality, and $\nu(Q(k''))(1-\alpha)\mathcal{S}_h(k'') = rU_h$, which leads us back to the first case.

It remains to be demonstrated that the equilibrium condition that workers search optimally. This is the case by the construction of the equilibrium expectations functions.$\|$

Proof of Lemma 4.

Let $\tilde{U}_\ell \equiv \nu(\overline{q}_s)\alpha\mathcal{S}_\ell(\overline{k}_s)$ denote the expected discounted utility of a type ℓ worker conditional on applying to submarket $(\overline{k}_s, \overline{q}_s)$. This pair is uniquely defined by the system of equations

$$\begin{aligned} \eta(\bar{q}_s)(1-\alpha)\frac{F(\bar{k}_s,s)-c\bar{k}_s}{r+\lambda+\alpha\nu(\bar{q}_s)} &= c_0 \\ F_k(k,s) &= c. \end{aligned}$$

The second equation determines a unique capital level $\bar{k}_s$ as a function of type s, so we plug this into the first condition to obtain a single equation, which relates $\bar{q}_s$ and s. By differentiating this with respect to s, we obtain

$$\frac{d\bar{q}_s}{ds} = -\frac{g(\bar{q}_s)}{g'(\bar{q}_s)}\frac{\frac{d\left(F(\bar{k}_s,s)-c\bar{k}_s\right)}{ds}}{F(\bar{k}_s,s)-c\bar{k}_s} < 0, \tag{A19}$$

where $g(q) \equiv \frac{\eta(q)}{r+\lambda+\alpha\nu(q)}$ is increasing in q. This derivative is negative because of the capital-skill complementarity.

To investigate the effects of an arbitrarily small skill difference on $\tilde{U}_\ell(s)$, we evaluate the total derivative at $s=\ell$, where

$$\frac{d\tilde{U}_\ell}{ds} = \frac{\partial\tilde{U}_\ell}{\partial\bar{q}_s}\frac{d\bar{q}_s}{ds} + \frac{\partial\tilde{U}_\ell}{\partial\bar{k}_s}\frac{d\bar{k}_s}{ds} = \nu'(\bar{q}_s)\alpha\mathcal{S}_\ell(\bar{k}_s)\frac{d\bar{q}_s}{ds} + \nu(\bar{q}_s)\frac{\partial\mathcal{S}_\ell(\bar{k}_s)}{\partial\bar{k}_s}\frac{d\bar{k}_s}{ds}.$$

Note that the last term vanishes as $\frac{\partial\mathcal{S}_\ell(\bar{k}_s)}{\partial\bar{k}_s} = 0$ when evaluated at $\bar{k}_\ell$. Therefore, the total derivative is positive and unskilled workers are better off applying for a type h job if firms do not over-invest when the skill differences are sufficiently small.$\|$

Proof of Proposition 5. Conditions (4)–(5) and (A12)-(A14) characterize the equilibrium outcome. According to these equilibrium conditions, it is always the case that $k_\ell < k_h$.

The conditions can be rewritten (4) and (A12) as

$$\eta(q_\ell)\mathcal{S}_\ell(k_\ell) = \eta(q_h)\mathcal{S}_h(k_h). \tag{A20}$$

If $k_h = \bar{k}_h$, then this expression can be written as $g(q_\ell)\big(F(k_\ell,\ell)-ck_\ell\big) = g(q_h)\big(F(k_h,h)-ck_h\big)$, where g is an increasing function. The properties of the production technology ensure that $F(k_\ell,\ell)-ck_\ell < F(k_h,h)-ck_h$. Therefore, $q_\ell > q_h$. Otherwise, if $k_h > \bar{k}_h$, then condition (A14) holds. As $\mathcal{S}_\ell(k_h) < \mathcal{S}_\ell(k_\ell)$, it follows that $q_\ell > q_h$.

The queue length gap together with expression (A20) imply that $\mathcal{S}_\ell(k_\ell) < \mathcal{S}_h(k_h)$. Then, it follows directly from the definition of the unemployment value (1) that $U_\ell < U_h$. Finally, by manipulating the wage and surplus equations, we can write the wages as $w_s = \big(r + \lambda + \nu(q_s)\big)\alpha \mathcal{S}_s(k_s)$, and thus $w_\ell < w_h$.‖

Proof of Proposition 6.

A type s worker is unemployed for an exact period of length τ with probability $e^{-\tau\nu(q_s)}\nu(q_s)$. The average exit rate from unemployment at duration τ amounts to:

$$\overline{\nu}(\tau) \quad = \quad \frac{\mu u_h e^{-\tau\nu(q_h)}\nu(q_h) + (1-\mu)u_\ell e^{-\tau\nu(q_\ell)}\nu(q_\ell)}{\mu u_h e^{-\tau\nu(q_h)} + (1-\mu)u_\ell e^{-\tau\nu(q_\ell)}}. \tag{A21}$$

We can proceed in an analogous manner for the average re-employment wage at duration τ. The average wage for new matches conditional on an unemployment period of length τ is determined by:

$$\overline{w}(\tau) \quad = \quad \frac{\mu u_h e^{-\tau\nu(q_h)}\nu(q_h)w_h + (1-\mu)u_\ell e^{-\tau\nu(q_\ell)}\nu(q_\ell)w_\ell}{\mu u_h e^{-\tau\nu(q_h)}\nu(q_h) + (1-\mu)u_\ell e^{-\tau\nu(q_\ell)}\nu(q_\ell)}. \tag{A22}$$

Next, we show the steps for the average exit rate but we skip the proofs for the average wages because they are analogous.

First, we differentiate expression (A21) with respect to the duration to obtain

$$\frac{d\overline{\nu}(\tau)}{d\tau} = -\frac{x\big(\nu(q_h) - \nu(q_\ell)\big)^2 e^{\tau(\nu(q_h)-\nu(q_\ell))}}{\big(1 + xe^{\tau(\nu(q_h)-\nu(q_\ell))}\big)^2},$$

where $x \equiv \frac{1-\mu}{\mu}\frac{u_\ell}{u_h}$. According to Proposition 5, the derivate is negative and thus the average exit rate falls over the unemployment period. To show that this variable flattens out for long spells, it is sufficient to compute the limit of the derivative as τ goes to infinity.

$$\lim_{\tau\to\infty}\frac{d\overline{\nu}(\tau)}{d\tau} = \lim_{z\to\infty} -\frac{z}{(1+xz)^2} = 0.$$

Finally, to analyze the curvature of the average exit rate, the second derivative is computed.

$$\frac{d^2\overline{\nu}(\tau)}{d\tau^2} = \frac{\big(\nu(q_h) - \nu(q_\ell)\big)^2\big(xe^{\tau(\nu(q_h)-\nu(q_\ell))} - 1\big)}{\big(1 + xe^{\tau(\nu(q_h)-\nu(q_\ell))}\big)^3}$$

The second derivative is positive if and only if $e^{\tau(\nu(q_h)-\nu(q_\ell))} > 1/x$. Therefore, there is one inflection point at most. A sufficient condition that the average exit rate is always convex is $\mu \leq 0.5$.‖