4.2. Hotelling Model

Matilde Machado

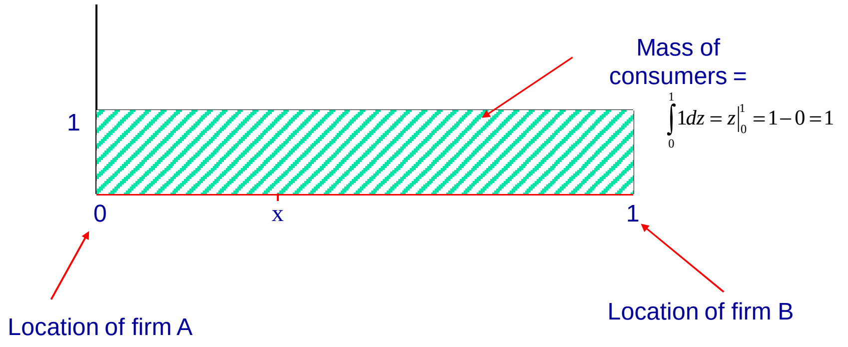
4.2. Hotelling Model

The model:

1. "Linear city" is the interval [0,1]
2. Consumers are distributed uniformely along this interval.
3. There are 2 firms, located at each extreme who sell the same good. The unique difference among firms is their location.
4. c= cost of 1 unit of the good
5. t= transportation cost by unit of distance squared. This cost is up to the consumer to pay. If a consumer is at a distance d to one of the sellers, its transportation cost is td^2. This cost represents the value of time, gasoline, or adaptation to a product, etc.
6. Consumers have unit demands, they buy at most one unit of the good {0,1}

4.2. Hotelling Model

Graphically



4.2. Hotelling Model

The transportation costs of consumer x:

- Of buying from seller A are tx^2
- Of buying from seller B are $t(1-x)^2$

- *s* ≡ gross consumer surplus - (i.e. its maximum willingness to pay for the good)
- Let's assume *s* is sufficiently large for all consumers to be willing to buy (this situation is referred to as "the market is covered"). The utility of each consumer is given by:
- U = s-p-td^2 where p is the price paid.

4.2. Hotelling Model

We first take the locations of the sellers as given (afterwards we are going to determine them endogenously) and assume firms compete in prices.

1. Derive the demand curves for each of the sellers

2. The price optimization problem given the demands

4.2. Hotelling Model

In order to derive the demands we need to derive the consumer $\tilde{x}$ that is just indifferent between buying from A or from B:

$\tilde{x}$ is defined as the location where $U_{\tilde{x}}(A) = U_{\tilde{x}}(B)$

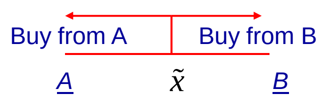
$$\Leftrightarrow s - p_A - t\tilde{x}^2 = s - p_B - t(1-\tilde{x})^2$$

$$\Leftrightarrow p_A + t\tilde{x}^2 = p_B + t(1-\tilde{x})^2$$

$$\Leftrightarrow p_A + t\tilde{x}^2 = p_B + t + t\tilde{x}^2 - 2t\tilde{x}$$

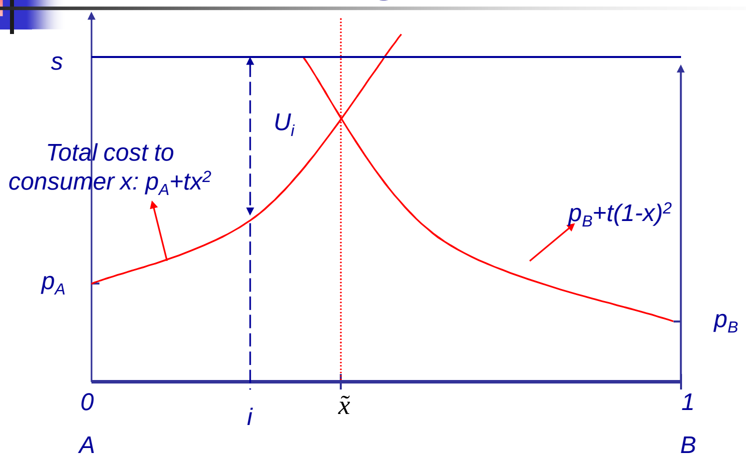
$$\Leftrightarrow 2t\tilde{x} = p_B - p_A + t$$

$$\Leftrightarrow \boxed{\tilde{x} = \frac{p_B - p_A + t}{2t}}$$



If (p_B=p_A) then the indifferent consumer is at half the distance between A and B. If (p_B-p_A)↑ the indifferent consumers moves to the right, that is the demand for firm A increases and the demand for firm B decreases.

4.2. Hotelling Model



The equilibrium of the Hotelling model

4.2. Hotelling Model

We say the market is covered if all consumers buy. Since the consumer with the lowest utility is the indifferent consumer (because it is the one who is further away from any of the sellers), we may say that the market is covered if the indifferent consumer buys i.e. if:

$$s - p_A - t\left(\frac{p_B - p_A + t}{2t}\right)^2 \geq 0$$

This condition is equivalent to say that *s* has to be high enough

4.2. Hotelling Model

Once we know the indifferent consumer, we may define the demand functions of A and B.

$$D_A(p_A,p_B)=\int_0^{\tilde{x}}1dz=z\Big|_0^{\tilde{x}}=\tilde{x}=\frac{p_B-p_A+t}{2t}=\frac{p_B-p_A}{2t}+\frac{1}{2}$$

$$D_B(p_A,p_B)=\int_{\tilde{x}}^{1}1dz=z\Big|_{\tilde{x}}^{1}=1-\tilde{x}=1-\left(\frac{p_B-p_A}{2t}+\frac{1}{2}\right)=\left(\frac{p_A-p_B}{2t}+\frac{1}{2}\right)$$

Demand of firm A depends positively on the difference (p_B-p_A) and negatively on the transportation costs. If firms set the same prices pB=pA then transportation costs do not matter as long as the market is covered, firms split the market equally (and the indifferent consumer is located in the middle of the interval ½).

4.2. Hotelling Model

The maximization problem of firm A is:

$$\underset{p_A}{Max}\,\Pi^A(p_A,p_B)=(p_A-c)D_A(p_A,p_B)=(p_A-c)\frac{p_B-p_A+t}{2t}$$

$$\text{FOC: }\frac{\partial\Pi^A}{\partial p_A}=0\Leftrightarrow\frac{p_B-p_A+t}{2t}-\frac{1}{2t}(p_A-c)=0$$

$$\Leftrightarrow p_B-2p_A+t+c=0\Leftrightarrow \boxed{p_A=\frac{p_B+t+c}{2}}$$

Firm A's reaction curve

Because the problem is symmetric $\Rightarrow p_A=p_B=p^*$

$$p^*=\frac{p^*+t+c}{2}\Leftrightarrow\frac{p^*}{2}=\frac{t+c}{2}\Leftrightarrow\boxed{p^*=t+c}$$

Note that if t=0 (no product differentiation) we go back to Bertrand p=c; Π*=0*

4.2. Hotelling Model

Once the equilibrium prices are determined, we may determine the other equilibrium quantities:

$$\tilde{x}^* = \frac{1}{2} \text{ (the indifferent consumer is in the middle because prices are equal)}$$

$$D_A(p_A^*, p_B^*) = \tilde{x}^* = \frac{1}{2}$$

$$D_B(p_A^*, p_B^*) = 1 - \tilde{x}^* = \frac{1}{2} = D_A(p_A^*, p_B^*)$$

$$\Pi^{A*} = \Pi^{B*} = \left(p^* - c\right) D_A^* = \left(t + c - c\right) \tilde{x}^* = \frac{t}{2}$$

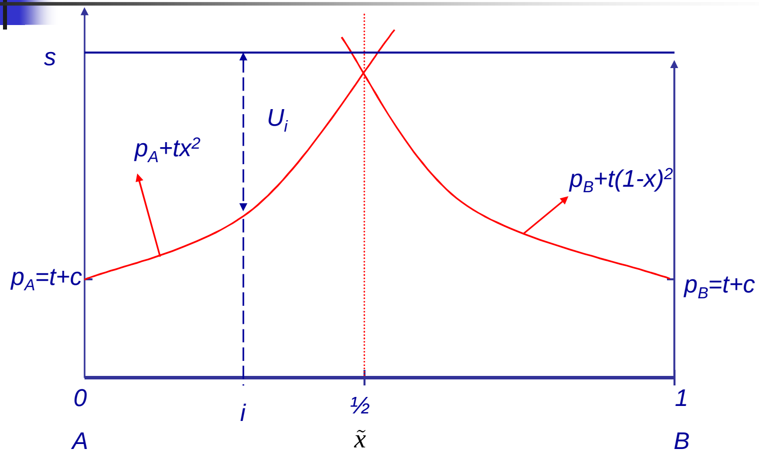
Note: The higher is t , the more differentiated are the goods from the point of view of the consumers, the highest is the market power (the closest consumers are more captive since it is more expensive to turn to the competition) which allows the firms to increase prices and therefore profits. When t=0 (no differentiation) we go back to Bertrand

4.2. Hotelling Model

Observations:

- Each firm serves half the market $D^*_A = D^*_B = 1/2$
- The Bertrand paradox disapears (note that firms compete in prices) $p_A = p_B > c$
- An increase in t implies more product differentiation. Therefore, firms compete less vigorously (set higher prices) and obtain higher profits.
- t=0 back to Bertrand

4.2. Hotelling Model



The equilibrium of the Hotelling model

4.2. Hotelling Model

How do prices change if the locations of A and B change?

- If A=0 and B=1 there is maximum differentiation
- Si A=B, there is no differentiation, all consumers will buy from the seller with the lowest price, back to Bertrand,

  $p_A=p_B=c$ y $\Pi_A=\Pi_B=0$.

4.2. Hotelling Model

General Case– Endogenous locations:

2 periods:

- In the first period, firms choose location
- In the second period firms compete in prices given their locations

We solve the game backwards, starting from the second period.

4.2. Hotelling Model

Second period:

- Denote by $a \in [0,1]$ the location of A
- Denote by (1-b) $\in [0,1]$ the location of B

Note: Maximum differentiation is obtained with *a*=0; and 1-b=1 (i.e. b=0)

Minimum differentiation (perfect substitutes) is obtained with a=1-b ⇔a+b=1

4.2. Hotelling Model

1. The indifferent consumer: $U_{\tilde{x}}(A) = U_{\tilde{x}}(B)$

$$p_A + t(\tilde{x} - a)^2 = p_B + t(\tilde{x} - (1-b))^2$$

$$\Leftrightarrow p_A + t\tilde{x}^2 + ta^2 - 2t\tilde{x}a = p_B + t\tilde{x}^2 + t(1-b)^2 - 2t\tilde{x}(1-b)$$

$$\Leftrightarrow 2t\tilde{x}(1-b-a) = p_B - p_A + t(1-b)^2 - ta^2$$

$$\Leftrightarrow \tilde{x} = \frac{p_B - p_A + t(1-b)^2 - ta^2}{2t(1-b-a)} = \frac{p_B - p_A + t\left((1-b)^2 - a^2\right)}{2t(1-b-a)}$$

$$\Leftrightarrow \tilde{x} = \frac{p_B - p_A}{2t(1-b-a)} + \frac{(1-b-a)(1-b+a)}{2(1-b-a)}$$

$$\Leftrightarrow \tilde{x} = \frac{p_B - p_A}{2t(1-b-a)} + \frac{(1-b+a)}{2} = \frac{p_B - p_A}{2t(1-b-a)} + \frac{(1-b-a)}{2} + a$$

Hence if $p_A = p_B$, A's demand is a+(1-b-a)/2

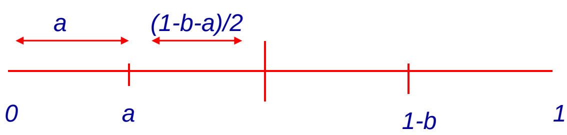
4.2. Hotelling Model

Demands are:

$$D_A(p_A, p_B) = \tilde{x} = \frac{p_B - p_A}{2t(1-b-a)} + \frac{(1-b-a)}{2} + a$$

$$D_B(p_A, p_B) = 1 - \tilde{x} = 1 - \frac{p_B - p_A}{2t(1-b-a)} - \frac{(1-b-a)}{2} - a$$

$$= \frac{p_A - p_B}{2t(1-b-a)} + \frac{1-b-a}{2} + b$$



4.2. Hotelling Model

Interpretation of the demand functions:

if $p_A = p_B$

$$D_A(p_A, p_B) = \underbrace{a}_{\text{captive consumers to the left (own backyard)}} + \underbrace{\frac{(1-b-a)}{2}}_{\text{half of the consumers between a and 1-b}}$$

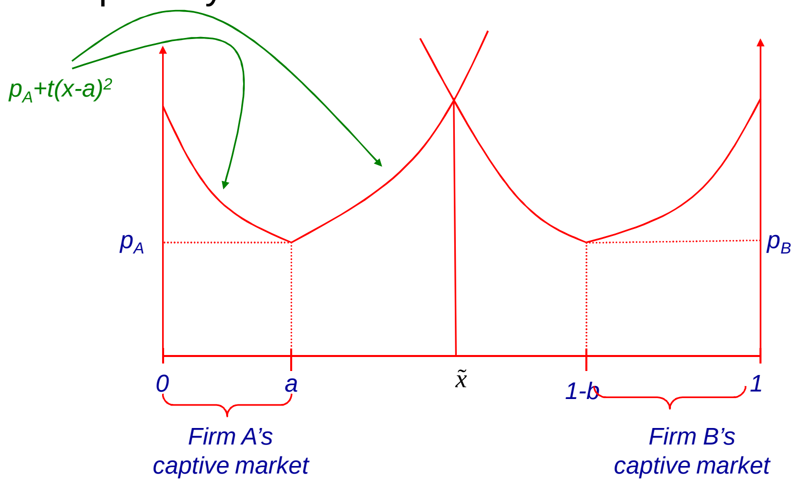
$$D_B(p_A, p_B) = \underbrace{\frac{1-b-a}{2}}_{\text{half of the consumers between a and 1-b}} + \underbrace{b}_{\text{captive consumers to the right (own backyard)}}$$

if $p_A \neq p_B$

$$D_A(p_A, p_B) = a + \frac{(1-b-a)}{2} + \underbrace{\frac{p_B - p_A}{2t(1-b-a)}}_{\text{sensitivity of the demand to price difference}}$$

4.2. Hotelling Model

Graphically



4.2. Hotelling Model

2. Finding the reaction functions

$$\underset{p_A}{Max}\,\Pi^A = (p_A - c)D_A(p_A, p_B) = (p_A - c)\left(a + \frac{(1-b-a)}{2} + \frac{p_B - p_A}{2t(1-b-a)}\right)$$

$$\text{FOC:} \frac{\partial \Pi^A}{\partial p_A} = 0 \Leftrightarrow a + \frac{(1-b-a)}{2} + \frac{p_B - p_A}{2t(1-b-a)} + (p_A - c)\left(-\frac{1}{2t(1-b-a)}\right) = 0$$

$$\Leftrightarrow \frac{2p_A}{2t(1-b-a)} = a + \frac{(1-b-a)}{2} + \frac{p_B + c}{2t(1-b-a)}$$

$$\Leftrightarrow \frac{p_A}{t(1-b-a)} = a + \frac{(1-b-a)}{2} + \frac{p_B + c}{2t(1-b-a)}$$

$$\Leftrightarrow p_A = at(1-b-a) + \frac{t(1-b-a)^2}{2} + \frac{p_B + c}{2}$$

Firm A's reaction function

4.2. Hotelling Model

2. Finding the reaction functions

$$\underset{p_B}{Max}\,\Pi^B = (p_B - c)D_B(p_A, p_B) = (p_B - c)\left(b + \frac{(1-b-a)}{2} + \frac{p_A - p_B}{2t(1-b-a)}\right)$$

$$\text{FOC:} \frac{\partial \Pi^B}{\partial p_B} = 0 \Leftrightarrow$$

$$\Leftrightarrow b + \frac{(1-b-a)}{2} + \frac{p_A - p_B}{2t(1-b-a)} + (p_B - c)\left(-\frac{1}{2t(1-b-a)}\right) = 0$$

$$\Leftrightarrow b + \frac{(1-b-a)}{2} + \frac{p_A - 2p_B + c}{2t(1-b-a)} = 0$$

4.2. Hotelling Model

2. 2. Finding the reaction functions

$$b+\frac{(1-b-a)}{2}+\frac{-2p_B+c}{2t(1-b-a)}+\frac{1}{2}\left(a+\frac{1-b-a}{2}+\frac{p_B+c}{2t(1-b-a)}\right)=0$$

$$\Leftrightarrow \frac{-3p_B+3c}{4t(1-b-a)}+b+\frac{(1-b-a)}{2}+\frac{1}{2}a+\frac{1-b-a}{4}=0$$

$$\Leftrightarrow \frac{3p_B}{4t(1-b-a)}=\frac{3c}{4t(1-b-a)}+\frac{b}{4}+\frac{3}{4}-\frac{a}{4}$$

$$\Leftrightarrow p_B=c+\frac{t(3+b-a)(1-b-a)}{3}$$

$$=c+t(1-b-a)\left(1+\frac{b-a}{3}\right) \text{ y } p_A=c+t(1-b-a)\left(1+\frac{a-b}{3}\right)$$

4.2. Hotelling Model

2. 2. Finding the reaction functions

$$p_B^*(a,b)=c+t(1-b-a)\left(1+\frac{b-a}{3}\right) \text{ and } p_A^*(a,b)=c+t(1-b-a)\left(1+\frac{a-b}{3}\right)$$

Note that prices are maximum when differentiation is maximum (a=b=0; $p_A=p_B=c+t$) and minimum when there is no differentiation (a+b=1 (same location) and $p_A=p_B=c$)

4.2. Hotelling Model

3. 1st period, simultaneous choice of a and b

Profits are functions of (a, b) alone:

$$\Pi^A(a,b) = \left(p_A^*(a,b) - c\right) D_A(a,b,\, p_A^*(a,b),\, p_B^*(a,b))$$
$$\Pi^B(a,b) = \left(p_B^*(a,b) - c\right) D_B(a,b,\, p_A^*(a,b),\, p_B^*(a,b))$$

Replace $p_A^*(a,b),\, p_B^*(a,b),\, D_A^*(a,b),\, D_B^*(a,b)$ and we get a function of a and b alone. Take the FOC as always with respect to a and b.

4.2. Hotelling Model

3. 1st period, simultaneous choice of a and b

$$\Pi^A(a,b) = \left(c + t(1-a-b)\left(1+\frac{a-b}{3}\right) - c\right)\left(\frac{p_B^* - p_A^*}{2t(1-a-b)} + \frac{1-b-a}{2} + a\right)$$

but $p_B^* - p_A^* = 2t(1-a-b)\left(\frac{b-a}{3}\right)$

which simplifies:

$$\Pi^A(a,b) = \left(t(1-a-b)\underbrace{\left(1+\frac{a-b}{3}\right)}_{=\left(\frac{3+a-b}{3}\right)}\right)\left(\underbrace{\frac{b-a}{3} + \frac{1-b+a}{2}}_{\frac{3-b+a}{6}}\right) = t(1-a-b)\frac{(3-b+a)^2}{18}$$

4.2. Hotelling Model

3. 1st period, simultaneous choice of a and b

$$\underset{a}{Max}\,\Pi^A(a,b) = t(1-a-b)\frac{(3-b+a)^2}{18}$$

$$\text{FOC:}\frac{\partial \Pi^A(a,b)}{\partial a} = -t\frac{(3-b+a)^2}{18} + t(1-a-b)\frac{2(3-b+a)}{18}$$

$$= -\frac{t}{18}(3-b+a)(1+b+3a) < 0 \Rightarrow a^* = 0$$

4.2. Hotelling Model

3. 1st period, simultaneous choice of a and b

$$\underset{b}{Max}\,\Pi^B(a,b) = t(1-a-b)\frac{(3+b-a)^2}{18}$$

$$\text{FOC:}\frac{\partial \Pi^B(a,b)}{\partial b} = -t\frac{(3+b-a)^2}{18} + t(1-a-b)\frac{2(3+b-a)}{18}$$

$$= -\frac{t}{18}(3+b-a)(1+3b+a) < 0 \Rightarrow b^* = 0 \Leftrightarrow 1-b^* = 1$$

4.2. Hotelling Model

Conclusion: Firms choose to be in the extremes i.e. they choose maximum differentiation.

For firm A, for example, an increase in a (movement to the right):

- Has a positive effect because it moves towards where the demand is (demand effect)
- Has a negative effect (competition effect)
- If transportation costs are quadratic, the competition effect is stronger than the demand effectand firms prefer maximum differentiation.

4.2. Hotelling Model

The social optimum solution is the one that minimizes costs (or maximizes utility) and it would be a=1/4 and 1-b=3/4. Therefore, from a social point of view the market solution leads to too much differentiation.

4.2. Hotelling Model

The social planner's problem is:

- Surplus of consumer x is:
  - $s-t(x-a)^2-p_A$ if he buys from A
  - $s-t(x-(1-b))^2-p_B$ if he buys from B
- For each consumer, the seller's profit is
  - p_A-c firm A
  - p_B-c firm B
- Prices are therefore pure transfers between consumers and sellers (note that here it is important the assumption that the market is covered that is that s is sufficiently high), the total surplus associated with a given consumer x is:
  - $s-t(x-a)^2-p_A+p_A-c= s-t(x-a)^2-c$ if he buys from A
  - $s-t(x-(1-b))^2-p_B+p_B-c= s-t(x-(1-b))^2-c$ if he buys from B

4.2. Hotelling Model

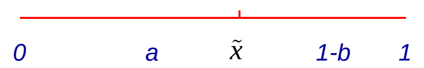
To derive the social optimum we must first derive the "indifferent" consumer :

$$s-t(\tilde{x}-a)^2-c=s-t(\tilde{x}-(1-b))^2-c$$
$$\Leftrightarrow (\tilde{x}-a)^2=(\tilde{x}-(1-b))^2$$
$$\Leftrightarrow \tilde{x}^2+a^2-2a\tilde{x}=\tilde{x}^2+(1-b)^2-2(1-b)\tilde{x}$$
$$\Leftrightarrow a^2-2a\tilde{x}=(1-b)^2-2(1-b)\tilde{x}$$
$$\Leftrightarrow 2\tilde{x}\left[1-b-a\right]=(1-b)^2-a^2$$
$$\Leftrightarrow \tilde{x}=\frac{(1-b-a)(1-b+a)}{2\left(1-b-a\right)}=\frac{(1-b+a)}{2}=\text{half the distance bweteen a and 1-b}$$

4.2. Hotelling Model

The planner has to max total surplus which is the same as minimize transportation costs

$$\underset{a,b}{Min} \underbrace{\int_0^a t(a-z)^2 dz + \int_a^{\tilde{x}=\frac{1-b+a}{2}} t(z-a)^2 dz}_{\text{buy from A}} + \underbrace{\int_{\tilde{x}=\frac{1-b+a}{2}}^{1-b} t((1-b)-z)^2 dz + \int_{1-b}^{1} t(z-(1-b))^2 dz}_{\text{buy from B}}$$



4.2. Hotelling Model

$$\underset{a,b}{Min} \underbrace{\int_0^a t(a-z)^2 dz + \int_a^{\tilde{x}=\frac{1-b+a}{2}} t(z-a)^2 dz}_{\text{buy from A}} + \underbrace{\int_{\tilde{x}=\frac{1-b+a}{2}}^{1-b} t((1-b)-z)^2 dz + \int_{1-b}^{1} t(z-(1-b))^2 dz}_{\text{buy from B}}$$

$$\Leftrightarrow \underset{a,b}{Min} \left[-\frac{(a-z)^3}{3}\Big|_0^a + \frac{(z-a)^3}{3}\Big|_a^{\frac{1-b+a}{2}} - \frac{(1-b-z)^3}{3}\Big|_{\frac{1-b+a}{2}}^{1-b} + \frac{(z-(1-b))^3}{3}\Big|_{1-b}^{1} \right]$$

$$\Leftrightarrow \underset{a,b}{Min} \left[\frac{a^3}{3} + \frac{1}{3}\left(\frac{1-b-a}{2}\right)^3 + \frac{1}{3}\left(\frac{1-b-a}{2}\right)^3 + \frac{b^3}{3} \right]$$

4.2. Hotelling Model

$$\underset{a,b}{Min}\left[\frac{a^3}{3}+\frac{1}{3}\left(\frac{1-b-a}{2}\right)^3+\frac{1}{3}\left(\frac{1-b-a}{2}\right)^3+\frac{b^3}{3}\right]$$

The FOC: $$\begin{cases}\frac{\partial}{\partial a}=0 \Leftrightarrow 4a^2-(1-b-a)^2=0 \ \ \text{(A)}\\ \frac{\partial}{\partial b}=0 \Leftrightarrow 4b^2-(1-b-a)^2=0 \ \ \text{(B)}\end{cases}$$

(A)-(B):

$$4a^2-4b^2=0 \Leftrightarrow a^2=b^2 \Leftrightarrow a=b$$

replacing in (A) implies that:

$$4a^2-(1-a-a)^2=0 \Leftrightarrow a^*=\frac{1}{4};(1-b^*)=\frac{3}{4}$$

4.2. Hotelling Model

The basic conclusion of the Hotelling model is the principle of differentiation: firms want to differentiate as much as possible in order to soften the price competition.

It may happen that some forces will lead firms to locate in the same location, usually the center (minimum differentiation):

1) Firms may want to locate where demand is (i.e. in the center)
2) In the case of no price competition (for example if prices are regulated) firms may want to locate in the center and split the market 50-50.