Altruistically-Motivated Transfers Under Uncertainty

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Abstract

This paper aims to provide a tractable model of imperfectly-altruistic agents in a fully-dynamic setting. We study a Bewley-type environment with two altruistically-linked agents without commitment. In addition to the standard consumption-savings decision, players can give transfers to each other. In equilibrium, transfers only flow when the recipient's borrowing constraint binds. This is in line with stylized facts from the empirical literature. Moral-hazard problems arise that go beyond the Samaritan's dilemma: when one of the agents expects transfers, *both* agents over-consume long before transfers actually occur. We provide a flexible and stable solution algorithm. It is straightforward to adapt this algorithm to overlapping-generations and finite-horizon settings, as well as to add more choice variables.

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1 Introduction

There are many authors in the literature who call for the exploration of a fully-dynamic theory of altruistic agents. Laitner (1988) emphasizes the need for a multi-period model with altruistic agents and no-commitment. Fuster et al. (2007) point out the importance of studying dynamic models in which agents are imperfectly altruistic, do not share their assets and behave strategically – but remark upon the difficulty this would entail. Altig & Davis (1993) rule out strategic considerations in their analysis of social security, but emphasize that this leaves aside some potentially important aspects. McGarry (2006) argues in her discussion on altruism models that "[. . .] evidence suggests that dynamic models can provide insights into transfer behavior that are impossible to obtain in a static context."[1]

Our research agenda aims to fill this gap in the literature by providing a tractable theory for the behavior of imperfectly-altruistic agents in a fully-dynamic setting without commitment. The main goal of this paper is to provide a model that is flexible and stable enough to be used as a building-block in more complex settings, such as heterogeneous-agents models in macroeconomics, but also in microeconomic models of the family and in models of development. The analysis here builds on Barczyk & Kredler (2011), in which the simplest-possible dynamic altruism setting is studied. Players have no flow labor income, so that the problem boils down to a cake-eating problem with transfers. They identify the fundamental tensions in this setting that seem to have obstructed progress in the literature. The authors argue that the introduction of shocks resolves these tensions and point to one type of equilibrium that is empirically plausible and possesses desirable stability properties.

Here, we extend the environment in Barczyk & Kredler (2011) by adding an exogenous stochastic labor income stream. We still need to assume that there is a risky return to savings. This is because the labor-income process is assumed to be upper-bounded, so that labor income risk becomes negligible for very wealthy families (they rely mainly on their wealth). For these families, the tensions studied by Barczyk & Kredler (2011) resurface if the return to savings is safe.

[1]There are some two-period models in the literature in which strategic interactions occur. However, these models make simplifying assumptions in that they restrict transfers to flow in certain situations and focus on specific timing protocols. This limits their usefulness for extension. See the introduction of Barczyk & Kredler (2011) for a more detailed discussion.

To summarize, we study a standard Bewley-type environment inhabited by two altruistically-linked agents. In addition to the standard consumption-savings decision, agents can give transfers to each other. We study Markov-perfect equilibria, focusing on the equilibrium that arises from backward induction in a finite game.

The first important contribution is that we provide a stable numeric algorithm. The algorithm can easily be extended to more complex settings, such as overlapping-generations settings, finite-horizon settings, time or age dependence of value functions, etc. (see our online appendix for details). The algorithm has already been used by Barczyk (2011) in an overlapping-generations model to study deviations from Ricardian Equivalence in response to a deficit-financed tax cut.

We now provide a brief description of the properties of the equilibrium. Broadly speaking, the state space[2] can be broken up into three regions: transfer regions, over-consumption regions and self-sufficient (SS) regions.

The *transfer regions* are defined by a positive transfer flow. Transfers flow from relatively asset- and labor income-rich donors to poorer recipients. As in the transfer-when-constrained equilibrium in Barczyk & Kredler (2011), the donor only gives transfers once the recipient has exhausted his assets. In this way, the donor can control the consumption behavior of the recipient. Here, we add that the model predicts that transfers only flow to *constrained* recipients – that is, they cannot flow if the recipient is making positive savings. Positive labor income shocks to the recipient (and negative labor income shocks to the donor) cause transfers to decline or stop entirely.

The *over-consumption regions* are cones that emanate from the transfer regions. In these regions, one agent is poor relative to the other. In anticipation of transfers, the poorer player spends down her wealth and the economy heads towards a transfer region. A defining feature of equilibrium policies in these regions is *moral hazard*. The poor agent behaves recklessly by over-consuming, counting on the benevolence of the altruistic donor. The recipient's consumption path exhibits a downward discontinuity upon entering the transfer region. We refer to this phenomenon as a *party*.[3] Furthermore, when introducing a portfolio choice into the setting, we find that the poorer agent engages in excessive risk-taking. This occurs because some

[2]In our model, the state is 4-dimensional: Wealth and labor income for both players.

[3]The inefficiencies would be avoided if agents could contract upon consumption decisions.

of the downside risk is borne by the future donor, while the upside risk is only enjoyed by the future recipient. Meanwhile, the richer agent is also over-consuming. The resulting inefficiencies are similar to the tragedy of the commons. Both agents ultimately use a common resource (the donor's assets), which leads both agents to over-consume long before transfers actually occur. We call this result the *dynamic Samaritan's dilemma*.

The *self-sufficient region* comprises the rest of the state space, in which the wealth distribution is relatively balanced. The possibility of future transfers is remote and agents' consumption policies resemble the ones that agents would choose in the absence of a second altruistic agent. The allocation is close to efficient in this region.

The fact that transfers only flow to constrained recipients confirms a conjecture made by previous papers in the literature that take a stand on the timing of inter-vivos transfers. Fuster et al. (2007), Laitner (2001), McGarry (1999), and Nishiyama (2002), among others, assume that transfers only flow when recipients are constrained.[4] This is the outcome in our model and it confirms these conjectures. The features of this transfer-when-constrained equilibrium are also in line with the equilibrium typically studied in two-period models. It displays what Lindbeck & Weibull (1988), Bernheim & Stark (1988), and Bruce & Waldman (1990) refer to as the Samaritan's dilemma: the donor delays transfers to the second period and the recipient's first-period savings are inefficiently low.

In terms of the observable implications, the model is successful in generating transfer behavior broadly in line with the empirical evidence on inter-vivos transfers. Cox (1990) and Cox & Jappelli (1990) find that transfers flow primarily to liquidity-constrained individuals. Furthermore, our model predicts that transfers are increasing in the donor's wealth and labor income and decreasing in the recipient's labor income. Again, these predictions are borne out in the data, see McGarry & Schoeni (1995), McGarry & Schoeni (1997) and Berry (2008).

In the applied microeconomics literature, the so-called *unitary* and *collective* models are commonly used to model intra-household behavior. In these models, household behavior can be represented by a single utility function; the household

[4]Fuster et al. (2007) study a dynastic economy, which implicitly assumes that altruism is perfect. This leads to an indeterminacy in transfers. In order to resolve this indeterminacy, the authors choose the protocol that transfers only flow once the poor agent has exhausted her assets. The other two papers study settings with imperfect altruism.

is assumed to be able to commit to an efficient allocation. For a treatment, see the book by Browning et al. (2011). The authors stress (chapter 6.4.4) that it is important to consider relaxing the commitment assumption in a dynamic setting. They cite Mazzocco (2007), who extends the collective model to allow for no-commitment. He rejects the commitment assumption using data from the Consumer Expenditure Survey and calls for replacing the standard unitary model with no-commitment alternatives. Our model uses a non-cooperative approach to do this. A strength of our model is that, unlike the standard collective model, it has predictions on agent's separate asset positions. This is especially attractive for modeling between-household interactions, (e.g. parents and adult children), where data on households' separate asset positions are available.

Our model also speaks to a strand of the literature that places emphasis on the *transfer-income derivative* (TID) in altruism models. In the most basic altruism models, this derivative can be shown to equal unity. Altonji et al. (1997) estimate a TID of 0.13 on cross-sectional transfer data from the PSID and interpret this as evidence against the altruism model. However, as McGarry (2006) argues, this prediction does not hold if current income contains new information about future income. Our model generates TDIs that can be both higher and lower than unity, depending on the persistence in agents' labor income processes. A novel implication of our environment is the distinction between the transfer-income derivative and the transfer-*wealth* derivative. See section 6 for a detailed discussion.

The remainder of the paper is organized as follows. In section 2 we provide the physical environment, the equilibrium definition, and characterize the set of Pareto-efficient allocations. This is followed by a brief description in section 3 of the incentives the players face in the cases when they are unconstrained and constrained. Section 4 presents our main results and section 5 studies the sensitivity and robustness of these results. We discuss the observable implications of our model in section 6 and conclude with section 7.

2 Setting

2.1 Physical environment

Time t is continuous. There are two infinitely-lived players referred to as "she" and "he". We will denote variables pertaining to her as plain lower-case letters, e.g. c, and variables pertaining to him with prime-superscripts, e.g. c'.

Agents obtain an exogenous endowment stream $\{y, y'\}$. Each agent's endowment stream follows a Poisson with common support $y, y' \in \{y_1, y_2\}$, where $y_1 < y_2$. The Poisson rates of transitioning from high to low and low to high are ξ for both players.[5] There is one risky asset for each agent with expected return r and variance σ^2. Returns to the two assets are uncorrelated. Both agents are subject to a no-borrowing constraint.

At each point in time, agents choose a consumption rate, $c_t \geq 0$, and a non-negative transfer rate, $g_t \geq 0$, to the other agent (g stands for "gift"). These choices imply the following laws of motions for wealth:

$$dw_t = (\underbrace{rw_t + y_t - c_t - g_t + g'_t}_{\equiv \dot{w}_t\text{:her savings rate}})dt + w_t \sigma dB_t \tag{1}$$

$$dw'_t = (\underbrace{rw'_t + y'_t - c'_t - g'_t + g_t}_{\equiv \dot{w}'_t\text{:his savings rate}})dt + w'_t \sigma dB'_t, \tag{2}$$

where w stands for wealth and B_t and B'_t are uncorrelated standard Brownian motions. When $w = 0$, we must have that she does not spend more than she receives, i.e. $c + g \leq y + g'$ (and equivalently for him, of course). Also, note that in this case the Brownian motion does not enter the law of motion.

Her preferences are represented by

$$E_0 \int_0^\infty e^{-\rho t}[u(c_t) + \alpha u(c'_t)]dt.$$

The discount rate is$\rho > 0$ and $\alpha \in [0, 1]$ is the parameter which measures the intensity of altruism.

[5]It is straightforward to generalize our results to other Poisson processes. We focus on this simple case for expositional ease only.

He is a mirror-symmetric copy of her, but might have a different altruism parameter $\alpha' \in [0,1]$ from hers. His preferences are represented by

$$E_0 \int_0^\infty e^{-\rho t}[u(c_t') + \alpha' u(c_t)]dt.$$

We assume that agents do not differ in their discount rates and in the form of the felicity function $u(\cdot)$. This formulation encompasses he case of selfishness ($\alpha = \alpha' = 0$), perfect altruism ($\alpha = \alpha' = 1$) and one-sided altruism ($\alpha > 0, \alpha' = 0$).

2.2 Equilibrium definition

We focus on Markov-perfect equilibria. The payoff-relevant state is given by $x \equiv (w, w', y, y')$. A Markovian strategy is a pair of non-negative functions $\{c(x), g(x)\}$ for her and a pair $\{c'(x), g'(x)\}$ for him.

We leave strategies unrestricted, but enforce feasibility of consumption plans by setting "realized consumption" when she has run out of assets to

$$c^*(0, w', y, y') = \min\big\{c(0, w', y, y'), g'(0, w', y, y') + y\big\}. \tag{3}$$

This says that she cannot eat more than her labor income plus what he gives to her when she is broke, but she can announce plans to do so. In all other cases, realized consumption equals the announced strategy $c(0, w', y, y')$ because she faces no constraint. We define realized consumption $c'^*(w, 0, y, y')$ for him in same manner.

When the other player's strategy is Markov, the best-response problem of each player is a dynamic-programming problem and best responses will be Markov, too. Let $v(x)$ and $v'(x)$ be the value to her and him, respectively, when the state is given by x. Given his equilibrium strategy $\{c'(x), g'(x)\}$, her value function and its derivatives satisfy the following partial differential equation, known as the Hamilton-Jacobi-Bellman equation (HJB):

$$\begin{aligned} \rho v = & \max_{c,g} \big\{u(c) + \alpha u(c') + \dot{w}' v_{w'} + \dot{w} v_w\big\} + \\ & + \xi\big[v(\cdot, \tilde{y}) - v(\cdot, y)\big] + \xi\big[v(\cdot, \tilde{y}') - v(\cdot, y')\big] + \tfrac{\sigma^2}{2}(w^2 v_{ww} + w'^2 v_{w'w'}). \end{aligned} \tag{4}$$

The squiggle on labor income y indicates the labor income state the agents are currently not in. Subscripts denote partial derivatives, e.g. $v_w = \partial v/\partial w$. We suppress the dependence of the function v on x for better readability. His problem is characterized by a mirror-symmetric HJB. Throughout the paper we will only state her equations as long as the counterpart for him is obvious; for convenience, we collect his equations in appendix A.1. The interpretation of her HJB is provided in section 3. Finally, since the presence of Brownian motion smoothes the problem, we look for solutions to value functions in the space of twice-differentiable functions.

We now have everything in place to define a recursive equilibrium.

Definition 1 *A* Markov-perfect equilibrium (MPE) *is a collection of functions* $\{v(\cdot), c(\cdot), g(\cdot)\}$ *for her and* $\{v'(\cdot), c'(\cdot), g'(\cdot)\}$ *for him such that*

1. $\{v(\cdot), c(\cdot), g(\cdot)\}$ *solve her problem given* $\{c'(\cdot), g'(\cdot)\}$*, i.e. they solve* (4)*, and*
2. $\{v'(\cdot), c'(\cdot), g'(\cdot)\}$ *solve his problem given* $\{c(\cdot), g(\cdot)\}$.

Since players' strategies are required to be optimal for all points in the state space, players have to be best-responding at any node of the game tree, even the ones off the equilibrium path. As is well known, Markov perfection thus implies subgame perfection.

2.3 Pareto-optimal allocations

For what follows it is useful to establish the set of Pareto-optimal allocations. In order to solve the Pareto planner's problem, individual levels of wealth and labor income can be pooled. Define $W_t = w_t + w'_t$ as the total wealth of the family, and similarly $Y_t = y_t + y'_t$ as total labor income.[6]

Consider the allocations a benevolent planner would choose who places a weight η on her life-time value and a weight $(1-\eta)$ on his. Given an initial wealth level W_0,

[6]Under our assumptions on the endowment processes, Y_t contains enough information to forecast future Y_{t+s}, $s > 0$, perfectly. Under more general Poisson processes for y_t and y'_t, the planner may need to keep track of y_t and y'_t individually in order to forecast Y_{t+s} correctly. It is straightforward to adapt the arguments in this section to this case.

the planner chooses consumption policies c_t, c'_t to maximize

$$E_0 \left[\eta \int_0^\infty e^{-\rho t} \big[u(c_t) + \alpha u(c'_t) \big] dt + (1-\eta) \int_0^\infty e^{-\rho t} \big[u(c'_t) + \alpha' u(c_t) \big] dt \right] \tag{5}$$

s.t.

$$dW_t = (rW_t + Y_t - c_t - c'_t)dt + W_t \sigma_W dB_{W,t}$$

$W_t \geq 0$, $\forall t$, all histories,

where $\sigma_W = \sigma/\sqrt{2}$ and $B_{W,t}$ is a standard Brownian motion. The standard deviation for the planner is smaller than for the individual players, since the planner will allocate one-half of wealth in each risky asset (see section A.2 in the appendix for the derivation of the planner's optimal portfolio and the resulting law of motion). Varying $\eta \in [0, 1]$ yields all allocations on the Pareto frontier.

Let $V^\eta(W, Y)$ be the value to the planner with wealth W and labor income Y when putting weight η on her life-time values. The planner's value function satisfies the HJB

$$\begin{aligned} \rho V^\eta = & \max_{c,c'} \{ [\eta + \alpha'(1-\eta)] u(c) + [\alpha\eta + (1-\eta)] u(c') + (rW + Y - c - c') V_W^\eta + \\ & + \xi [V^\eta(\cdot, \tilde{y} + y') - V^\eta(\cdot, Y)] + \xi [V^\eta(\cdot, \tilde{y}' + y) - V^\eta(\cdot, Y)] + \frac{\sigma_W^2}{2} W_t^2 V_{WW}^\eta \}. \end{aligned}$$

Intra-temporal optimality requires that the two margins of allotting consumption to him and to her are equalized:

$$\big[\eta + \alpha'(1-\eta)\big] u_c(c_t) = \big[(1-\eta) + \alpha\eta\big] u_c(c_t) \qquad \forall t, \text{all histories.}$$

Solving for $u_c(c_t)$ yields

$$u_c(c_t) = \frac{(1-\eta) + \alpha\eta}{\eta + \alpha'(1-\eta)} u_c(c'_t) \qquad \forall t, \text{all histories.} \tag{6}$$

As in standard insurance problems, marginal utilities are proportional over states and time. Here, the factor of proportionality is a function of the planner's weight η on her and the altruism parameters α and α'. It is instructive to consider the extreme cases where $\eta = 0$ or $\eta = 1$. Placing all weight on her yields $u_c(c_t) = \alpha u_c(c'_t)$, whereas placing all weight on him yields $u_c(c_t) = \frac{1}{\alpha'} u_c(c'_t)$. Thus, just as in the static altruism setting, the ratio of marginal utilities is restricted to the interval $[\alpha, \frac{1}{\alpha'}]$. The more altruistic both agents are, the smaller is the consumption inequality tolerated by the Pareto planner. These bounds approach zero and infinity as altruism goes to zero, until reaching the standard case with selfish agents. For perfect altruism ($\alpha = \alpha' = 1$) there is a unique Pareto-optimal allocation and both agents consume the same amount always.

When using the functional form of power utility in equation (6), we see that the planner will choose the consumption rate of him as a fixed proportion of her consumption rate. So given that the planner wants to devote expenditures $C_t = c_t + c'_t$ on both players' combined consumption in a given state, it is now easy to determine how consumption should be split between the two agents. From this rule we can then write an indirect utility function of form $U_\eta(C_t) = \frac{H_\eta}{1-\gamma} C_t^{1-\gamma}$ for the planner, where H_η is a constant that depends on η. Since U_η represents the same preferences for all η, this implies that the planner will choose the same aggregate consumption plan C_t regardless of η; only the division of C_t between the agents will depend on η. Furthermore, it implies that the planner always runs down aggregate wealth K_t at the same rate in any efficient allocation. The reasoning just laid out leads us to the following solution strategy for the planner's problem(s). First, solve a standard Bewley problem in C_t for a planner who faces constraints of the form in (5); then find the two agent's consumption plans (c_t, c'_t) given C_t according to the sharing rule implicit in (6).

We now proceed to the inter-temporal optimality conditions. Note that equation (6) gives us c'_t as a function of c_t,[7] so that the planner's problem collapses to a conventional consumption-savings problem with a modified objective function (to see this, substitute out c'_t in the objective (5) using (6)). It follows that the Euler equation from the standard one-person consumption-savings problem must hold. Otherwise, the planner would re-allocate resources inter-temporally for one agent

[7] When $\eta = 1$ and $\alpha = 0$, equation (6) does not give us c'_t as a function of c_t any more – in this case, however, it is obviously optimal for the planner to set $c'_t = 0$ for all t. Analogously, $\eta = 0$ and $\alpha' = 0$ implies that $c_t = 0$.

maintaining the present value of resources allocated to this agent. We must have:

$$\underbrace{\mathcal{A}u_c(c)}_{\text{expected growth of } u_c} \leq (\rho - r)u_c(c), \tag{7}$$

where equality has to hold whenever $W_t > 0$, i.e. whenever the planner is unconstrained. The operator $\mathcal{A}$ (the *infinitesimal generator*) is defined for any twice-differentiable function $f(x)$ on the state as the "expected time derivative":

$$\begin{aligned}\mathcal{A}f(x_t) \equiv & \lim_{\Delta t \to 0} \tfrac{1}{\Delta t} E_t\big[f(x_{t+\Delta t}) - f(x_t)\big] = \\ & = f_w \dot{w}_t + f_{w'} \dot{w}_t + \eta\big[f(\cdot, \tilde{y}) - f(\cdot, y)\big] + \eta\big[f(\cdot, \tilde{y}') - f(\cdot, y')\big] + \\ & + \tfrac{\sigma^2}{2}(w_t^2 f_{ww} + {w'}_t^2 f_{w'w'}).\end{aligned}$$

The Euler equation (7) says that marginal utility should grow at the expected rate $(\rho - r)$ when the planner is unconstrained, which is the same as in the standard (selfish) savings problem.

As was pointed out in the introduction, the collective model of the family builds on the above Pareto problem. It assumes that family members can coordinate on an efficient allocation. This implicitly assumes that family members have the ability to commit, an assumption that we relax in our model.

We will later see that equilibrium allocations are usually not efficient in our framework. However, it is still instructive to think about arrangements that would implement efficient allocations. In a simpler setting without labor income uncertainty, Barczyk & Kredler (2011) show that any efficient allocation can be implemented by assigning appropriate shares of initial assets to the agents and then shutting down transfers forever. In the setting of the current paper, the analogous arrangement is that players commit to share their joint flow labor income $Y_t = y_t + y_t'$ for all t and initial assets $W_0 = w_0 + w_0'$ according to a fixed sharing rule. Furthermore, transfers are ruled out for all t. If utility $u(\cdot)$ is homothetic, then agents' consumption rules will equal the planner's rule since the agents' problems are just scaled version of the planner's problem. If $\sigma = 0$, indeed, this mechanism yields exactly the same consumption allocation as the planner's problem since there are no gains from insuring players against investment risk. If $\sigma > 0$, however, there are also gains from portfolio diversification. As we saw before, the planner will then hold exactly half of total assets in each account and has to rely on transfers to implement the efficient allocation, causing the equivalence to the resource-sharing mechanism to break down.

3 Understanding players' incentives

We now return to her HJB, equation (4), and shed light on her optimal choices in response to his consumption and transfer strategy. We will see that obtaining her best responses is akin to solving a standard consumption-savings problem.

For convenience we reproduce her HJB, additionally writing out her and his savings policies, $\dot{w}$ and $\dot{w}'$, and group terms according to whether they can be controlled or not:

$$\begin{aligned} \rho v = & \underbrace{\alpha u(c') + (rw' + y' - g' - c')v_{w'}}_{\text{his decisions}} + (rw + y + g')v_w + \\ & + \underbrace{\xi[v(\cdot,\tilde{y}) - v(\cdot,y)]}_{\text{shock to her } y} + \underbrace{\xi[v(\cdot,\tilde{y}') - v(\cdot,y')]}_{\text{shock to his } y'} + \underbrace{\frac{\sigma^2}{2}(w^2 v_{ww} + w'^2 v_{w'w'})\}}_{\text{shocks to wealth}} + \\ & + \max_{g\geq 0}\{g[\underbrace{v_{w'} - v_w}_{\equiv\mu:\text{ transfer motive}}]\} + \max_{c\geq 0}\{\underbrace{u(c) - cv_w}_{\text{consumption-savings tradeoff}}\}. \end{aligned} \tag{8}$$

The key simplification of continuous time with respect to discrete time is that the players' four decisions do not contemporaneously interact with each other. The max-operator for the transfer decision can be separated from the one for the consumption decision, and the other player's decisions do not enter in these max-operators. Of particular importance is the fact that his contemporaneous consumption decision c' does not affect her optimal choice c. In other words, her best-response function over an infinitesimal amount of time is a constant. The simplification occurs because immediate strategic considerations are of second order[8] and the agent's instantaneous

[8] To see why immediate strategic interactions are of second order, consider the following first-order approximation of the marginal value of saving in a small neighborhood of the current state (w, w'):

$$v_w(w_{t+\Delta t}, w'_{t+\Delta t}) = v_w(w_t, w'_t) + \underbrace{v_{ww'}(w_t, w'_t)\big[rw'_t - c'_t \ldots\big]\Delta t}_{\text{of second order}} + \ldots$$

We see that as Δt becomes small, the changes in the marginal value of saving induced by c'_t over the planning period Δt become negligible. So even when he chooses a very high c', we can still payoff $u(c) + \alpha u(c')$ is separable in c and c'. The first-order condition (FOC) for consumption is given by

$$u_c(c) = v_w,$$

which says that the marginal utility of current consumption is set equal to the marginal value of saving in the optimum.

At first glance it seems striking that his current decisions should not matter to her. But note this is only true for the decisions taken at the same instant of time. In general, his decisions do matter for her, which will become evident from her Euler equation in subsection 3.1. For now recall that the effects of his future decisions on her are all contained in the partial derivative v_w, which encodes the savings incentives stemming from the entire continuation of the game.

From (4) we see that the maximization problem with respect to the transfer g is linear. If the term $\mu \equiv (v_{w'} - v_w)$, which we will refer to as the "transfer motive", is negative, then transfers are set to zero – after all, she cannot force him to give transfers to her. If $\mu = 0$, then any transfer flow is consistent with optimality. Should the transfer motive be *positive*, however, then the agent wants to choose g as large as possible. In fact, since the agent is allowed to make transfers that are mass points, she will choose to follow the vector $(-1, 1)$ in (w, w')-space as long as the directional derivative $v_{w'} - v_w = \mu$ is positive.[9] In other words, a player can induce a jump in the time path of the state. For the equilibrium this paper focuses on it will be the case that $\mu < 0$ throughout the state space since transfers will only occur when one of the players is borrowing constrained. We refer the reader to our previous paper for a more detailed discussion on mass transfers.

The second line on the right-hand side of equation (8) contains all sources of uncertainty that she faces. The first term embodies the possibility that her labor income may change from its current value of y to the realization $\tilde{y}$. The difference in

find Δt small enough so that v_k is (almost) constant over the planning horizon. It is therefore valid to disregard the interaction effects between the consumption decisions. Economically speaking, an agent who reconsiders her savings decisions on a daily basis need not worry about the daily savings decisions of her counterpart, since the latter will not have a large impact on the other's assets – it is enough to keep an eye on the other's bank account to be sufficiently informed. Her Euler equation will make it transparent that his consumption decision does of course enter v_k over time.

[9]Note that in this case, the HJB ceases to be a valid characterization for his problem since – as a PDE – it only contains *local* information on the value function. When making a mass transfer, however, it is crucial to consider the continuation value v *globally*.

value that would arise from this labor income change is measured by $v(\cdot, \tilde{y}) - v(\cdot, y)$, the change occurring at rate ξ. The second term is entirely analogous except that it is about a change in *his* labor income. The third term arises due to the investment risk that both players face.

The first line on the right-hand side of equation (8) collects the remaining terms from the savings policies and its valuations as well as flow utility obtained from the consumption of the other player. Finally, the left-hand side of the HJB is her flow value of the optimal program.

3.1 Savings incentives: The Euler equation

Despite the fact that consumption decisions are made without explicitly considering the contemporaneous choices of the other player it is of course the case that the policies of the other player are taken into account and do influence decision making. This can best be seen by studying the Euler equation, which in turn provides insights into savings incentives that a player faces.

In order to obtain her Euler equation, take the derivative of her HJB (4) with respect to w, use the FOC for consumption and re-arrange:

$$\mathcal{A}u_c(c) = \underbrace{(\rho - r)u_c(c)}_{\text{standard}} + \underbrace{[v_{w'} - \alpha u_c(c')]c'_w}_{\text{altruistic-strategic distortion}} + \underbrace{[u_c(c) - v_{w'}]g'_w}_{\text{transfer-induced incentives}} . \tag{9}$$

Here, c'_w denotes the partial derivative of his consumption policy with respect to her assets and g'_w is the partial derivative of his transfer function (assuming it is a rate, i.e. of flow-type) with respect to her assets. We note that the right-hand side of the Euler equation looks identical to the one in a deterministic environment. Barczyk & Kredler (2011) extensively describe this case (see equation (15) in their paper and its discussion). Here it will thus suffice to provide a summary of their main insights.

The terms referred to as altruistic-strategic distortion and transfer-induced incentives do not appear in the planner's Euler equation (7) and are therefore distortions. For the equilibrium of this paper the term referred to as transfer-induced incentives is zero; transfers do not flow within the state space. The interpretation for the altruistic-strategic distortion is, however, key to understand our results. Suppose his consumption increases when her wealth increases, i.e. $c'_w > 0$. On the one hand this is a benefit to her, given by $\alpha u_c(c')$, which induces her to save more and therefore enters with the same sign as the interest rate does. On the other hand there is a cost to her, given by $v_{w'}$, since an increase in his consumption comes with a decrease in his wealth, which induces her to save less and therefore enters with the same sign as the discount rate does.

Regions of the state space in which the there are positive altruistic-strategic distortions are characterized by the agent consuming at a faster rate than is socially optimal, a phenomenon we will refer to as *over-consumption*. Over-consumption will play an important role in the equilibrium and behind the intuition of why in the equilibrium transfers only occur when one of the players is constrained.

3.2 Optimality when broke

The characterization and determination of optimality so far was confined to the case where both agents have positive levels of wealth. We will now turn to the cases where one or both agents are without wealth. This turns out to be a very important part of the state space since in our equilibrium transfers only flow when the recipient is broke. The analysis of these cases is an additional contribution with respect to our previous paper, Barczyk & Kredler (2011). In the previous paper, there was no labor income and an Inada condition in the donor's utility, which led to transfers always flowing when the recipient was broke. We will see that this will not hold any more when the recipient receives labor income.

Consider her problem when he has zero assets (i.e. $w' = 0$) but she has positive assets ($w > 0$).[10] To simplify matters, we leave out all terms in agents' problems that are not influenced by agents' contemporaneous decisions and focus on the Hamiltonians from the HJBs:

$$\max_{c\geq 0, g\geq 0} H(c,g) = \max_{c\geq 0, g\geq 0} \Big\{ u(c) + \alpha u(c'^*(c'_0, g)) + (rw + y - c - g)v_w + \\ + (y' - c'^*(c'_0, g) + g)v_{w'} \Big\}, \tag{10}$$

$$\max_{c'\geq 0} H'(c') = \max_{c'\geq 0} \big\{ u(c') + (y' - c;^*(c', g) + g)v'_{w'} \big\}. \tag{11}$$

We set $g' = 0$ because no transfers flow towards agents with positive wealth in the

[10]The case $w' > 0$ and $w = 0$ is entirely symmetric.

type of equilibrium we consider. Recall that c' is his consumption strategy; realized consumption c'^* for him is obtained from g and c' as prescribed in equation (3):

$$c'^*(c', g) = \begin{cases} c' & \text{if } y' + g \geq c' \\ y' + g & \text{if } y' + g < c'. \end{cases}$$

We will now argue that agents' optimal consumption strategies are given by the "unconstrained levels" (c_0, c'_0), which are implicitly defined as

$$\begin{aligned} u_c(c'_0) &= v'_{w'}, \\ u_c(c_0) &= v_w. \end{aligned} \tag{12}$$

Obviously, c_0 maximizes H and is thus the optimal strategy independent of c'_0. As for him, observe that the unconstrained maximum of H' is reached at c'_0. So setting c'_0 is definitely optimal if it is feasible. Also, since H' is increasing in c' for $c' < c'_0$, announcing c'_0 leads to the constrained-optimal outcome $c'^* = y + g$ in the case that $c'_0 > y + g$. This means that announcing c'_0 is a dominant strategy for him.

To analyze her transfer decision, it will be convenient to introduce a variable g_{dict} that tells us what she would like to give to him (or take from him) if she could dictate his consumption in the current instant. We define

$$\begin{aligned} g_{dict} &\equiv \arg \max_{-\infty < \tilde{g} < \infty} H(c_0, \tilde{g}) \\ \Rightarrow \quad \alpha u_c(y' + g_{dict}) &= u_c(c_0) \end{aligned}$$

For the case of CES utility we have $g_{dict} = \alpha^{1/\gamma} c_0 - y'$.

We will now study the properties of the Hamiltonian $H(c, g)$ in equation (10). Since utility is separable in c and c', H is additive in its c- and g-terms. This greatly simplifies our analysis. We see that H is concave in c and that

$$\frac{\partial H}{\partial c} = u_c(c) - v_w \begin{cases} \geq 0 & \text{if } c \leq c_0 \\ < 0 & \text{if } c > c_0. \end{cases}$$

H is continuous in g, but there is a kink at the point where he starts saving the additional transfer instead of consuming it:

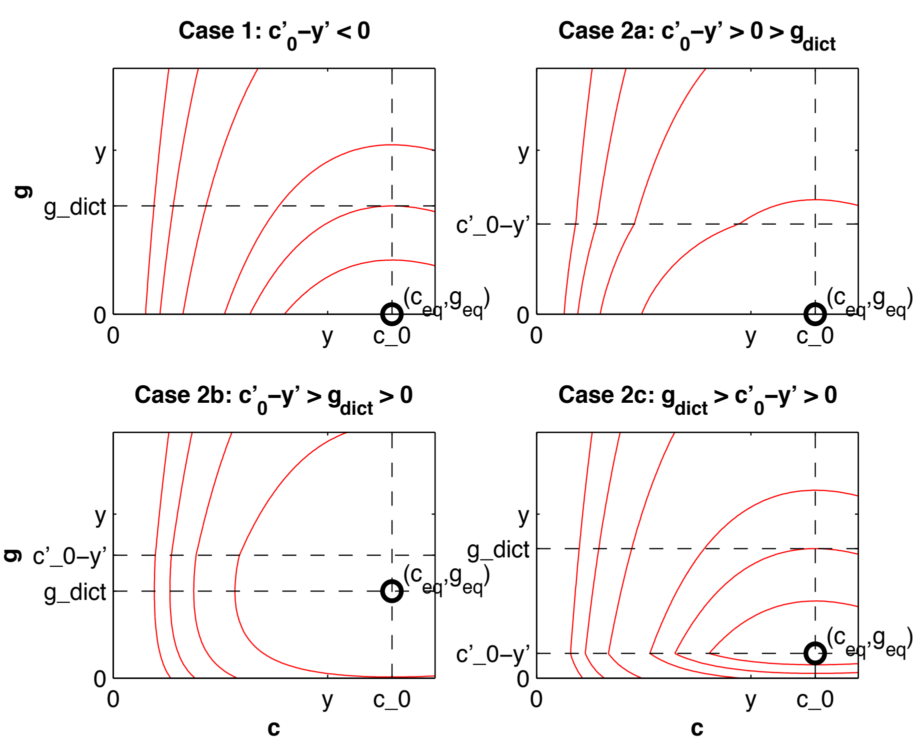
$$\frac{\partial H}{\partial g} = \begin{cases} \alpha u_c(y' + g) - v_w & \text{if } g < c_0' - y' \\ v_{w'} - v_w & \text{if } g > c_0' - y'. \end{cases}$$

The jump in the derivative reflects that her transfers go directly into his consumption until the satiation point c_0' is reached. On this lower part, $H(c, \cdot)$ is concave and might (or might not) reach a local maximum. This local maximum – if it exists – occurs at g_{dict}. From the satiation point on he starts saving, which is marginally valued by her at $v_{w'}$. We are looking for an equilibrium in which transfers are never optimal within the state space, so we assume $v_{w'} < v_w$ for now. This implies that she never desires to raise g to the point where he saves additional transfers.

We will distinguish between two cases: In case 1, he saves even when transfers are zero, i.e. $c_0' < y$. In case 2, he consumes the marginal transfer at $g = 0$, i.e. $c_0' > y$. We will denote her optimal transfer choice by g_{unc} (*unc* is for "unconstrained", in contrast to the case where she is also constrained).

Figure 1 provides an illustration of these two cases. It plots level lines of her Hamiltonian $H(c, g)$ as curves and the optimal choice as a circle. The graph in the upper-left corner depicts the case 1. She sets $g_{unc} = 0$ since all transfer units would go into savings, but $v_{w'} < v_w$. We may say that her *transfer margin* (the marginal benefit of increasing transfers) is lower than the *savings* and the *consumption margins* (the marginal benefits of savings and consumption, respectively).

Figure 1: Transfer decision when $w > 0$



The remaining graphs refer to the second case, which is split up into further subcases. In case 2a ($g_{dict} \leq 0 < c_0' - y'$) she is already unwilling to give transfers to him at $g = 0$, i.e. $\partial H/\partial g|_{g=0} \leq 0$. Then the optimal transfer is zero since H is decreasing in g throughout: $g_{unc} = 0$. Again, her transfer margin is lower than the consumption and savings margins.

In case 2b ($0 < g_{dict} < c_0' - y'$) there is an interior solution where he consumes the entire transfer and she sets the c- and g-margin equal, so she can implement her desired consumption for him. In this case, she equalizes the consumption, savings and transfer margins and we end up with her favorite allocation (at least over the next instant Δt).

Finally, case 2c shows that another corner solution can occur where she increases transfers until reaching his satiation point and he would start saving the transfer, which she dislikes. In this case she gives $c_0' - y'$, and we have $0 < c_0' - y' \leq g_{dict}$ so that he is just indifferent between saving and consuming the marginal transfer unit. The transfer margin is now *lower* than the consumption and savings margins when considering a local *increase* in transfers at the optimum, and *higher* than the consumption and savings margins when considering a marginal *decrease* in transfers at the optimum. Graphically, this shows ups as kinks in the level lines of the Hamiltonian.

Computationally, we find that in equilibrium all cases apart from 2c occur. We can summarize all cases by defining her optimal transfer by the formula

$$g_{unc} = \max\left\{0; \min\{g_{dict}, c_0' - y'\}\right\}. \tag{13}$$

The maximizers for problem (10) are thus $(c, g)_{eq} = (c_0, g_{unc})$. His equilibrium consumption is given by using his policy rule for her optimal choice g_{unc}:

$$c'^{*}_{eq} = \min\{c_0', y' + g_{unc}\} = y' + g_{unc}, \tag{14}$$

where the second equality follows because she never gives a transfer that would flow into his savings.

The case where both players are broke introduces the additional complication that also the donor faces the constraint. Specifically, the donor's consumption plus transfers cannot exceed labor income. However, the different cases and the intuition are very similar to the unconstrained case treated here. We thus relegate the discussion of the case where both players are constrained to section A.3 in the appendix.

4 Equilibrium

It is well-known that there are potentially many Markov-perfect equilibria in dynamic games. Indeed, Barczyk & Kredler (2011) find a continuum of equilibria in a setting as ours with the modification that flow labor income is zero and that there are no shocks to assets. However, they argue that this type of equilibrium is fragile. It ceases to exist when shocks are introduced, when in-kind transfers are made available to agents and when the game is restricted to a finite horizon. The authors find that a more stable equilibrium exists when a shock is introduced into their simple deterministic setting. This equilibrium obtains naturally when iterating backward on value functions in a finite-horizon game. In this equilibrium, transfers flow only when the recipient is borrowing-constrained. This enables the donor to keep the consumption of the recipient in check. The equilibrium is also empirically plausible. Inter-vivos transfers in the data tend to flow when the recipient appears to be liquidity-constrained, and they go from well-off to poorer family members.

We now turn to studying this *transfers-when-constrained equilibrium* in our setting. As Barczyk & Kredler (2011), we find that this type of equilibrium exists in the case of imperfect altruism (at least for some minimum level of disturbance σ, see the detailed discussion in 5.4). It is indeed the unique equilibrium obtained as a limit of a finite game, which is a common equilibrium-selection criterion in the literature (see our online appendix for how we compute equilibrium in the finite-horizon case).

4.1 Algorithm

We now describe the broad features of the numerical algorithm we use to compute the equilibrium. More details are provided in our online appendix. There, we give an introduction to the Markov-chain approximation method for continuous-time control problems and how to adapt it to our framework. We also discuss how to include additional choice variables and to adapt it to overlapping-generations and finite-horizon settings.

In general, our algorithm is very much related to value-function iteration in discrete time. We first guess reasonable value functions and then iterate on them, updating the agents' policies on the way, until the value functions converge. We do this using a form of the Markov-chain approximation method (see Dupuis & Kushner (2001)). The state space is discretized on a linear grid, and the law of motion for the state is approximated by a Markov chain that is locally restricted to adjacent grid points. If the local properties of the Markov chain are chosen such that they are in line with the true continuous-time process (in first and second moments), then theorems ensure that the discretized solution converges to the continuous-time solution as the grid becomes finer, see Dupuis & Kushner (2001) for a proof. The algorithm can also be seen as a traditional finite-difference PDE approximation scheme for the two HJBs (see our appendix for the relationship between the two approaches).

We start the algorithm by specifying reasonable guesses for the value functions. We then iterate backward, computing optimal policies in the following manner. On grid points where he has positive assets, we set her transfers to zero and determine her consumption rate from her marginal value of assets. As pointed out before, it is very easy to find the equilibrium of the "stage game" when both players are unconstrained since best responses are constant in our continuous-time setting. On grid points where one player has zero assets, our algorithm follows the discussion in section 3.2. Once value functions have converged, it remains to be checked that the transfer motive is indeed negative for both players throughout the state space; if this is indeed the case, we have found an equilibrium.

We find that our algorithm is stable and works for a variety of initial guesses. Due to the simplicity of the optimal policy rules in the stage games, the algorithm is very fast and computation time is not an issue.

4.2 Equilibrium Policies

We now describe the workings of the model in a numerical example. To keep things as simple as possible, we consider the case in which only one of the agents faces labor income risk. His labor income is fixed at $y' = 25$, while her labor income fluctuates between $y \in \{20, 30\}$ with a transition hazard of $\xi = 0.1$ (i.e. her labor income changes from one state to the other every 10 periods on average). For the per-period utility function, we choose $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma = 2$. See table 1 for a summary of the remaining parameter values.

Table 1: Parameters in numerical example.

Parameter	Value	Parameter	Value
α	0.3	y_h	30
α'	0.1	y_l	20
γ	2	y'	25
ρ	3.4%	ξ	10%
r	3%	σ	3%

Players' equilibrium transfer policies are shown in figure 2. Her transfers are shown in the two upper panels, his in the two lower panels. The two figures on the left refer to the case where her labor income is low, and the two figures on the right to the case where her labor income is high. Players' assets are on the x- and y-axes and transfers are on the z-axis. Since she is more altruistic than he is, her transfers are in general on a higher level than his. We see that, as pointed out before, transfers only flow when the recipient is constrained. As is to be expected, transfers are increasing in the donor's assets and decreasing in the recipient's labor income (see the lower two panels). Once the donor's assets are too low, transfers are stopped altogether.

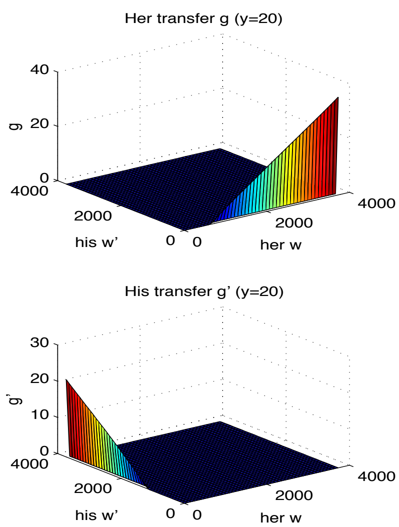


Figure 2: Transfer policies

Players' consumption policies are depicted in figure 3. We only show the consumption policies for the case where her labor income is low since policies for the other labor income state are qualitatively very similar (although they are higher in levels, especially in regions where wealth is low). As was to be expected, consumption is increasing in both own and the other player's assets. Note that the state space is clearly separated into three regions. In one region, her assets are relatively low with to respect to his assets (left-upper corner); in the middle region players assets are balanced; in the lower-right corner her assets are high with respect to his.

Consider first the region where she is relatively poor, which we will refer to as an *over-consumption* (OG) region. The first striking feature is that her consumption jumps downward when she goes from positive to zero assets. This jump occurs only when he is rich enough to provide transfers to her, compare to figure 2. Note that *his* consumption policy is continuous at these points since he is not receiving but giving transfers. As can be seen in the lower-right corners of the graphs, the situation is the opposite where she is rich and he receives transfers. We will discuss the discontinuity in consumption in depth in section 4.4. In general, we see that her consumption depends more on his assets than in her own in the entire triangle. This is because she can rely on his transfers when eventually broke. The economy is moving towards the region where she receives transfers. This may be seen from figure 4, which shows the law of motion for the economy. The blue arrows depict the law of motion (or drift $(\dot{w}, \dot{w}')$ for a given (w, w')) in the equilibrium of the altruistic model. For illustrative purposes, this law of motion is contrasted to the law of motion of an autarkic economy in which agents do not have altruistic preferences but all other parameters are the same (the red arrows). We see that in the upper-left corner, the altruistic economy heads rapidly towards her being broke, whereas she shows typical precautionary-savings behavior in the self-sufficient economy. She saves when her labor income is high (i.e. the red arrows point to the right in the right panel) and dis-saves when her labor income is low (i.e. the red arrows point to the left in the left panel). As for his consumption policy, it is worthwhile pointing out that he consumes less as her wealth decreases since he foresees that he will have to provide for her in the end.

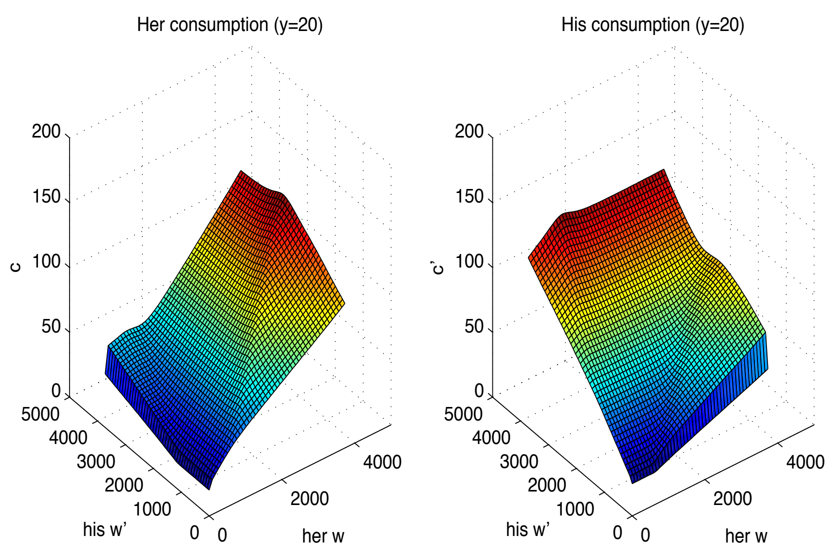


Figure 3: Consumption policies

We now turn to the middle cone, i.e. the region in which the wealth distribution is relatively balanced. We will refer to this region as the *self-sufficient* (SS) region. We see that in this part of the state space, player's consumption is governed purely by their own assets: her consumption function is increasing in her assets w and flat in his assets w' (and the opposite for him). Figure 4 shows that players' policies are practically the same as in the autarkic economy. This occurs because the chance of ending up in a region where one player becomes dependent on the other are relatively slim.

Finally, the third region (in the lower-right corner of the graphs) is – at least in qualitative terms – a mirror image of the first (upper-right) over-consumption region. However, this region is larger since her altruism is higher than his, which induces him to rely on her in a larger range of circumstances.

It is very interesting to observe what happens on the seams between the self-sufficient and the over-consumption regions. We will again illustrate the economic intuition using the upper-left region of the state space. On the boundary between the SS and the OG region, we see that both players' consumption functions have upward bumps just as we leave the SS region: for both him and her consumption is higher at this bump than we would expect when linearly extrapolating from within the SS region. The reason for this is that both players have incentives to over-consume as they enter the OC region. As already shown in Barczyk & Kredler (2011), the *dynamic Samaritan's dilemma* induces players to over-consume. In anticipation of transfers being given eventually, both players consume at inefficiently high rates since they essentially consume out of a common asset stock: the rich agent's assets. We will discuss this issue in more depth in section 4.5. For her, this may – in the extreme – lead to the consumption function being locally decreasing in her own wealth. This is equivalent to the value function being convex, meaning that she is locally risk-loving. Indeed, as we show in section 4.5, she may be induced to buy lotteries at this point of the state space.

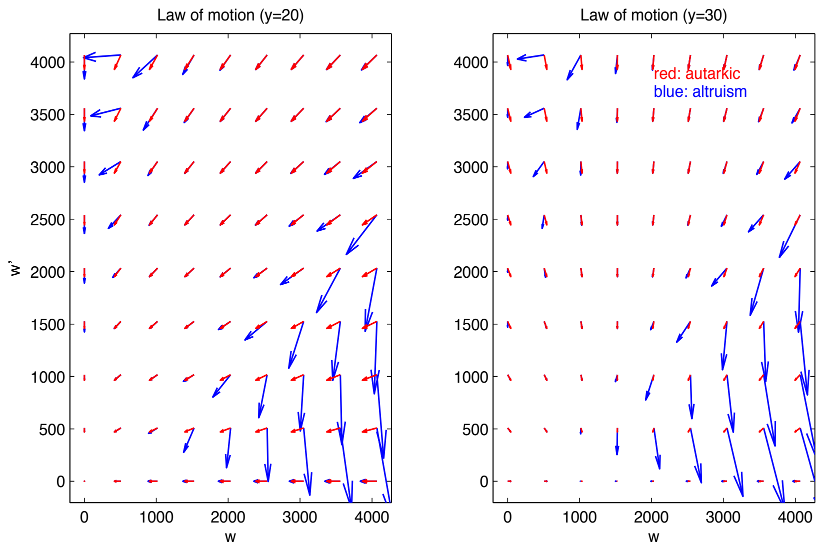


Figure 4: Law of motion: Drift of asset position

When assets are large for one or both agents, flow labor income becomes insignificant relative to the stock of wealth and interest payments. In this case, the model converges to a cake-eating problem as studied in Barczyk & Kredler (2011). In this paper, the authors exploit homogeneity of their setting and find an equilibrium in which policies are proportional to joint assets $W = w + w'$. Indeed, we find that the policies in the setting with flow labor income converge to the ones predicted by Barczyk & Kredler (2011) when joint assets W grow large.[11] This leads us to conjecture that in any setting where endowment processes (y, y') are upper-bounded, equilibrium policies converge to those in the cake-eating problem. Furthermore this shows, as claimed by Barczyk & Kredler (2011), that it is essential to understand the tensions in a cake-eating environment in order to solve a model with labor income risk.

A final note on figure 4 is in order. All arrows are pointing downward in both figures, which means that he is dis-saving at all points of the state space. This is to be expected since he bears no labor income risk and $\rho > r$. For her, however, matters are different. Since she does face labor income risk, we see typical precautionary-savings behavior. This leads to a non-degenerate ergodic distribution of agents over

[11] We have checked this for various parameter constellations.

the (w, w', y)-space, which has support on the line between $(0, 0)$ and $(150, 0)$ (her maximal savings when having a long history of high labor income, which are reached at the point where the arrows become dots in the right panel). If he also faced labor income risk, than the ergodic distribution becomes richer. Arrows then also point upward when he has high labor income, so both agents may have positive assets now. However, there are still mass regions on the lines where one of the agents is broke and a mass point at $(0, 0)$.[12]

4.3 A history

In order to highlight some of the attributes of the equilibrium it is useful to consider a particular history for players' assets, consumption, and transfers. Figure 5 displays such a history. The graph on the top displays her (red) and his (blue) wealth trajectories. We see that he starts out with high assets and that she is relatively poor at $t = 0$ (which corresponds to starting the economy in the upper-left corner of figures 2, 3 etc.). The dashed lines in the second panel show the labor income realizations for this history. She receives the low labor income for 13 years and then switches to the high labor income. Until the third year she consumes her wealth down, at which point she obtains transfers as shown in the bottom panel. Notice that her consumption path jumps downward at the point when she obtains transfers, which corresponds to the discontinuity of her consumption function in figure 3. He then provides her with transfers from year 3 to 9. These transfers decrease over time (since he is spending down his assets) and eventually stop. In year 13 she obtains the high labor income realization and as a result begins to accumulate wealth. Once she receives the high labor income, he stops transfers.

4.4 Parties: Characterization of incentives

We now return to the discontinuity in the recipient's consumption path when entering a transfer regime. Barczyk & Kredler (2011) derive a closed-form expression for the size of this jump in the same setting without flow labor income and with logarithmic preferences and argue that this discontinuity is the equivalent to the *Samaritan's*

[12] Transfers may or may not flow once the ergodic set is reached, depending on the model's parameters.

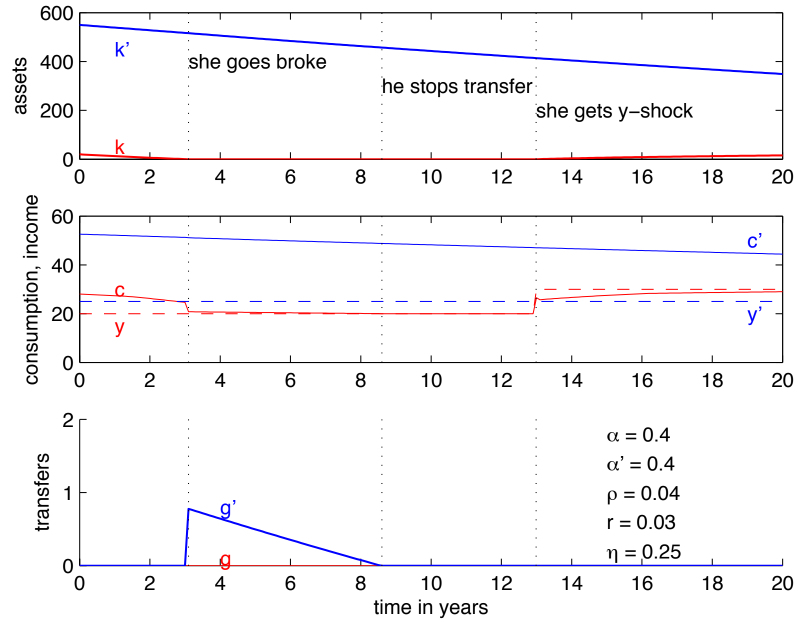


Figure 5: A history: Assets, consumption, and transfers over 20 years.

Dilemma in two-period models.[13] The size of the jump is shown to be decreasing in both altruism parameters. The more agents take into account the effects of their behavior on their counterpart, the smaller the inefficiency becomes. In our computations, we find that the same is true in the more general model here.

The intuition for why a jump in consumption is indeed optimal can already be obtained in the absence of altruism, as the following simple example from Barczyk & Kredler (2011) demonstrates. Consider a consumer with wealth $w_0 > 0$ and no flow labor income. A government provides a means-tested benefit in the form of a flow payment g_0 handed out conditional on the agent having zero wealth, i.e. $w_t = 0$. For simplicity assume that $\rho = r$, which implies that the optimal consumption path has to be constant while assets are positive: $c_t = \bar{c}$ for some constant $\bar{c}$. The agent will be able to consume $\bar{c}$ over an interval $t \in [0; T(\bar{c})]$, where $T(\bar{c}) \in (0; \infty]$ is the insolvency time implied by the consumption plan.

Consider the two consumption paths depicted in figure 6. A smooth consumption

[13]The Samaritans dilemma states the following in a two-period model: If an agent receives a transfer from an altruistic donor in period 2, then her consumption is inefficiently high in period 1. See Lindbeck & Weibull (1988).

path implies $\bar{c} = g_0$. As is obvious from the figure, any plan with $\bar{c} > g_0$ does better than this, so a smooth consumption plan cannot be optimal. In technical terms, the usual theorems from control theory fail because the law of motion $w_t = rw_t + c_t + g_0 I_{w_t=0}$ is discontinuous at zero. In terms of marginal cost-benefit analysis, there is an additional cost of saving here that is not present in a standard setting. When postponing bankruptcy, one diminishes the net present value of government transfers. For an altruistic recipient, the situation is similar. However, the situation is not quite as pronounced since she takes the effects of her behavior on the donor into account.

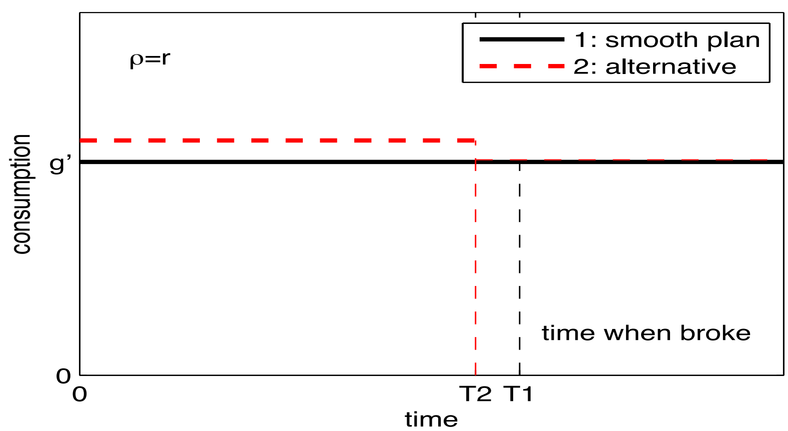


Figure 6: Means-tested benefit. Would you prefer plan 1 or plan 2?

We will now see which additional incentives are present for a transfer recipient facing an altruistic donor. Consider the following problem. Suppose she has left a small level of wealth Δw. Define Δt as the time it takes her to reach zero assets as a function of consuming at rate c. Her problem, taking his policy c' and thus $\dot{w}'$ as given, may be written as

$$\max_c \left\{ u(c)\Delta t + \alpha u(c')\Delta t + e^{-\rho\Delta t} V(0, w' + \dot{w}'\Delta t; y, y') \right\},$$

$$\text{where } \Delta t = \frac{\Delta w}{c - y - r\Delta w}.$$

In order to make the trade-offs apparent we take the first-order condition with respect to c:

$$\underbrace{u_c(c)\Delta t}_{\text{flow utility}} + \frac{d\Delta t}{dc}\Big[\underbrace{u(c) + \alpha u(c') - \rho V}_{A} + \underbrace{\dot{w}' v_{w'}}_{B}\Big] = 0$$

An increase in the consumption rate leads to additional flow utility, $u_c(c)\Delta t$, but her wealth is exhausted earlier, i.e. $d\Delta t/dc < 0$. Expression A says that exhausting wealth quicker replaces flow utility before broke, $u(c) + \alpha u(c')$, with the flow value of being broke, ρV. The term A will usually be positive, since consumption just before getting broke is higher than consumption when broke. So term A acts as an incentive to delay the point of getting broke, i.e it is an incentive to save. Depending on the sign of $\dot{w}'$, expression B either provides an incentive or disincentive to save. In our example history from before, he was dis-saving at the relevant point in the state space, i.e. $\dot{w}' < 0$. This gives her an additional disincentive to save. Consuming more makes her face a richer donor upon entering the transfer regime, which increases the transfers she can expect.

We have now discussed the most obvious deviation from efficiency in this model, which is a form of *moral hazard*. The poor agent behaves recklessly by over-consuming, counting on the benevolence of an altruistic donor. If agents' consumption and transfer decisions were contractible, this inefficiency could be avoided. It turns out that there are also other sources of inefficiency in our equilibrium. A second form of moral hazard that may arise is excessive risk-taking by the poor agent. Finally, there is an inefficiency stemming from the absence of lending contracts between the agents. We discuss these points in the following subsection.

4.5 Inefficiencies and moral hazard

We first introduce a measure for inefficiency of an allocation. By an *allocation* we mean a contingent plan for consumption and transfers for both agents for all time. Whenever an inefficient allocation is played departing from some point $x = (w, w', y, y')$ in the state space, then there exists is a continuum of efficient allocations indexed by Pareto weights η that both players prefer to the equilibrium allocation at x (see section 2.3 for a characterization of efficiency). The Pareto weights associated with these preferred allocations lie in a range $\eta \in [\underline{\eta}(x), \bar{\eta}(x)]$. Depending on η, these allocations share the improvements upon the equilibrium allocation in different ways between the agents: the $\underline{\eta}(x)$-allocation gives all gains to him, while the $\bar{\eta}(x)$-allocation gives all gains to her. In order to find a unique measure for the potential welfare gains, we will focus on the efficient allocation that provides the same gain to both players' welfare in the sense of consumption equivalent variation.

Formally, consider the following thought experiment. At a given point x in the state space, offer the efficient allocation with Pareto weight η to both agents. Compute the percentage increase $\gamma(x, \eta)$ in consumption in the equilibrium allocation (for all future t, for all states of the world and for both players) that she requires to be indifferent between the offered η-allocation and the equilibrium allocation. Equivalently, compute the percentage increase $\gamma'(x, \eta)$ that he would require in the equilibrium allocation to be indifferent to the offered efficient allocation. We can compute this measure for all efficient allocations $\eta \in [0; 1]$. See figure 7 for an illustration of the functions γ and γ' for one particular x. In this example, she is well-off enough under the equilibrium allocation to reject any efficient allocation that assigns weight lower than $\underline{\eta}(x) = 0.5$ to her. For him, matters are worse under the equilibrium. He would accept any allocation with $\eta < \bar{\eta}(x) = 0.8$. Note that the interval $[\underline{\eta}(x), \eta(\bar{x})]$ of allocations that make both players better off must always contain at least one point by the definition of Pareto efficiency.

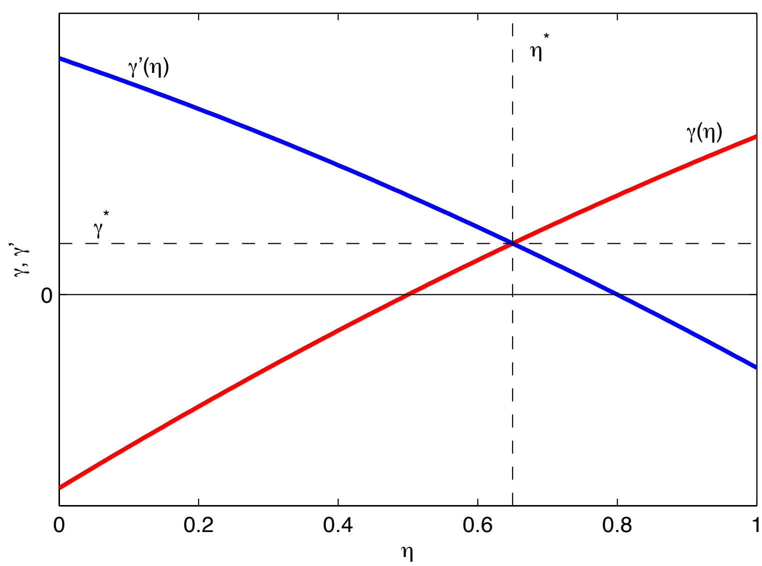


Figure 7: Potential welfare gains for a given state $x = (w, w'; y, y')$

Now, we want to select the efficient allocation that gives equal gains to both agents. To do this, we compute the η^* that solves $\gamma(x, \eta^*) = \gamma'(x, \eta^*)$ – obviously, the intersection of γ and γ' must be unique. The common welfare gain associated with this allocation is $\gamma^*(x) = \gamma(x, \eta^*) = \gamma'(x, \eta^*)$ and constitutes a measure for the inefficiency of the equilibrium allocation at x.

According to this procedure, we can now compute the potential (symmetric) welfare gains for all points in the state space. Figure 8 shows η^* and γ^* in our numerical example for low labor income y in the (w, w')-plane.[14] As expected, the allocation

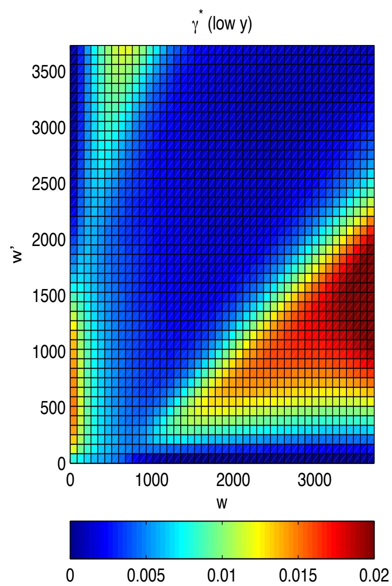


Figure 8: Quantifying distortions: Consumption equivalent variation.

is very close to efficient in the SS cone. Recall that players' policies are close to the SS policies, which satisfy the planner's Euler equation (7) and thus have consumption grow at the efficient rate for both agents. Indeed, we see in figure 9 – which plots the altruistic-strategic distortion to the Euler equation (9) – that Euler equations are essentially undistorted in the SS cone.[15]

For the over-consumption region, figure 9 shows that distortions to the Euler equations are large and positive for the poorer player. There are also positive distortions to the richer player's EE, but they are weaker. These distortions constitute what we call the *dynamic Samaritan's dilemma*. Inefficiencies feed back in time to long before transfer start to flow, and both players are over-consuming.[16] The intuition

[14]The figures for high y look very similar again.

[15]Note that since transfers are zero inside the state space, the transfer-induced incentive is zero and we can focus solely on the altruistic-strategic distortion when analyzing distortions to the EE.

[16]As mentioned before, in the two-period models studied before in the literature, only the recipi-

behind this type of inefficiency is akin to the tragedy of the commons. Both agents ultimately consume out of a common resource – the donor's assets. Agents take into account the adverse consequences of their behavior on the other person, but fail to do so completely because their altruism is imperfect.

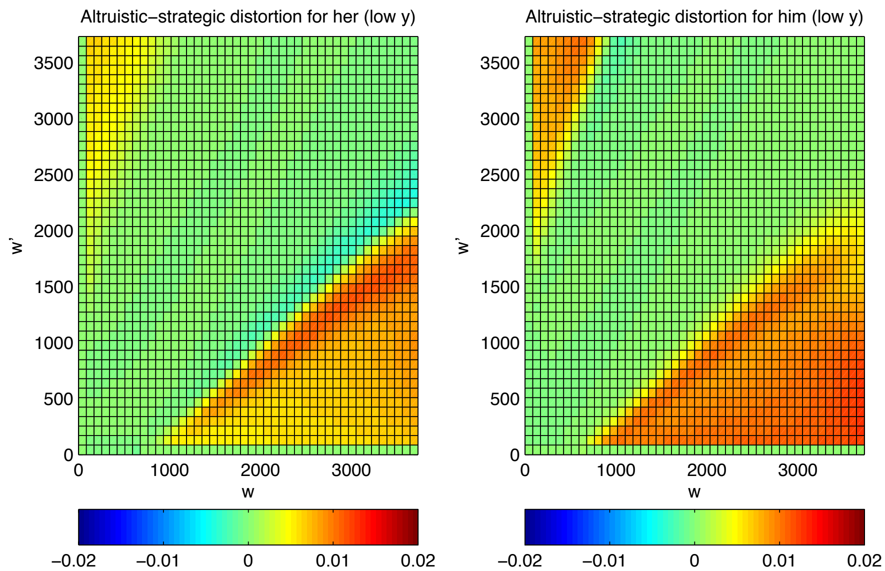


Figure 9: Altruistic-strategic distortions in generalized Euler equation.

Since both agents' altruistic-strategic distortions are positive in the over-consumption region, their marginal utility grows faster than in the efficient benchmark – they are "too impatient" in the sense that their consumption profile is excessively front-loaded. However, note that Euler equations are about *changes* but not *levels* in consumption. We will now answer the question if agents' consumption *levels* are inefficiently high in the party cone. To do this, first recall that the Pareto planner from section 2.3 spends down the common asset stock $W = w' + w$ at the same rate at a given state (W, Y) regardless of the Pareto weight η. While it is impossible to determine if a single agent's consumption level is inefficiently high or low – these levels vary with the Pareto weight η –, it is actually possible to decide if the two agents jointly over-consume. Figure 10 shows how the extraction of wealth $\dot{W} = \dot{w} + \dot{w}'$ in equi-

ent's savings decision is distorted and – by the two-period assumption – distortions are only present immediately before transfers flow.

librium compares to the efficient extraction.[17] We see that in the over-consumption region, agents indeed over-consume in levels. Over-consumption is especially severe close to the recipient's borrowing limit and is more pronounced when the recipient has low altruism and the donor's altruism is high.

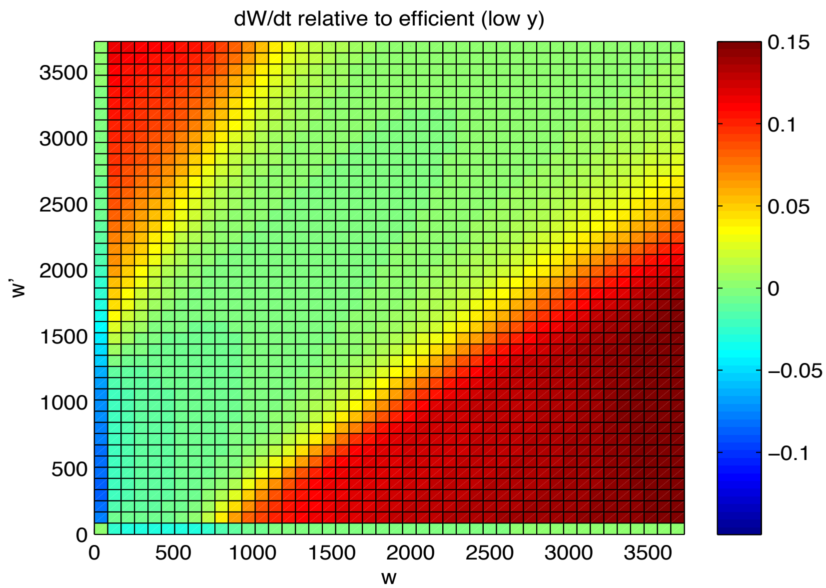


Figure 10: Consumption distortion in levels.

We now turn attention to the regions in which transfers flow (*transfer regions*). These are the lines along the w- and w'-axis that border the over-consumption regions. Figure 9 shows that agents' Euler equations are undistorted in these regions, and figure 10 shows that also the level of consumption is efficient. This is because the donor effectively acts as a family dictator here. Figure 8 tells us that indeed the allocation is very close to efficiency at these points and that the rich player obtains his/her preferred allocation. Of course, there is always the possibility that the economy enters inefficient stages again, but this possibility is unlikely and lies far in the future.

At this point, it is worthwhile return to figure 8. We see that inside the over-consumption region, γ^* grows larger the farther we go away from the transfer region. When the recipient has little wealth left, the allocation is very close to efficient. This is despite the fact that *current* consumption levels are too high, see figure 10. The

[17]The figure is again very similar for low y.

reason for this is that the party inefficiency is short-lived at this point – there is only a short time left until the economy enters a close-to-efficient regime. However, the criterion $\gamma^*(x)$ is about the consumption allocation for the entire future. When moving farther into the party cone, more time is spent in a distorted regime and γ^* increases.

As alluded to in section 4.2 before, a second form of moral hazard can arise in this model. Value functions can be locally convex for the poor player on the seam between the SS and the OC region, meaning that the poor player would be risk-loving. The economic intuition for this risk-lovingness is as follows. Suppose the risk-loving agent could participate in a fair lottery. Then his downside risk would be limited: if he lost the bet and entered the OC region, the rich agent would take on part of the losses by providing transfers eventually. The upside potential of the bet, however, is entirely enjoyed by the poor agent – winning a large amount would bring the economy into the SS region, where the possibility of transfers is very remote. In reality we could think of such lotteries as purchasing insufficient insurance for health or longevity, making risky investments in assets or a business, and risky career choices.

Indeed, risk-taking occurs in equilibrium when we modify the setting. In the online-appendix we extend our setting to two assets. Agents now face a portfolio choice between a safe and a risky asset. The risky assets is assumed to have the same expected return as the safe asset. Obviously, it is inefficient to use the risky asset since it induces unnecessary risk. However, in equilibrium the poor player invests in the risky asset between the SS and OC region and "gambles for resurrection". Figure 11 shows players' portfolio decisions; the red areas mark points in the state space where the player invests in the risky asset, in blue areas the safe asset is bought. In this example, both players gamble in the relevant region.[18]

The inefficiencies explained so far are also present in an altruistic framework without stochastic labor income, see Barczyk & Kredler (2011). However, the stochastic-labor income setting here adds a different type of inefficiency which is known from (non-altruistic) Bewley models. In figure 8, we see that the equilibrium allocation is inefficient in the lower-left corner, where she is broke and he does not provide trans-

[18] Again, the figures look very similar for the two labor income states, so we only show the results for the case where her labor income is high.

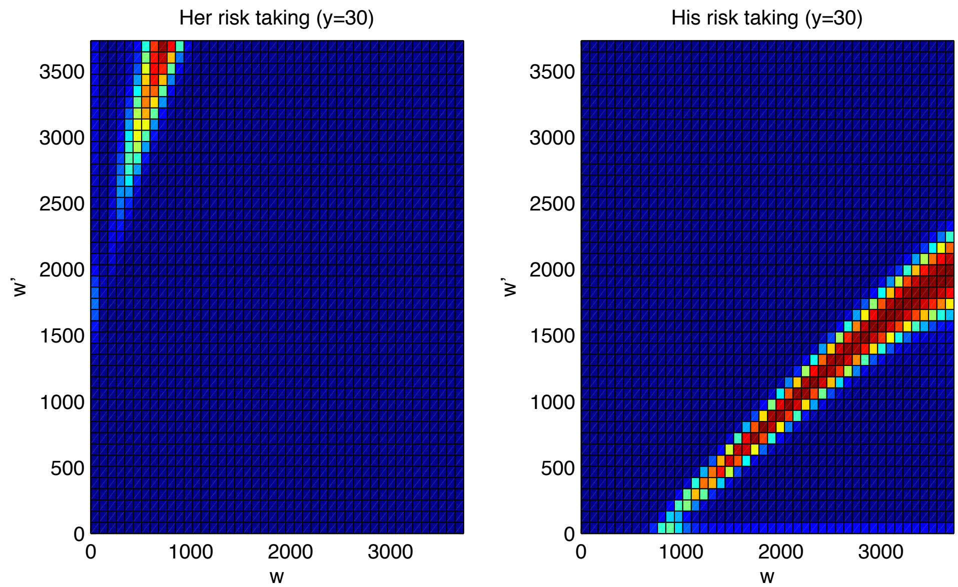


Figure 11: Risk-taking (red = risk-taker)

fers to her. At this point, she is constrained ($w = 0$ and $c = y$) in equilibrium but the Pareto planner is not ($W = w' > 0$). What would the planner do to implement Pareto gains? He would provide her with higher consumption in the constrained state but reduce her consumption in future high-labor income states, thus smoothing her consumption plan. The planner lets both agents shoulder some of her labor income risk, compensating him for taking on more risk with higher consumption levels. Finally, note that this inefficiency can also occur in settings where both players face a stochastic labor income stream. This is because the planner can condition consumption on joint assets W and joint labor income Y, which is less risky than each agent's private labor income.

For certain parameter constellations, however, this type of inefficiency does not occur. This is the case when the donor has a large degree of altruism and/or there are large differences in labor income. Then transfers flow even if the donor is not wealth-rich, and one of the transfer regions extends all the way to the point where both players are broke.

One could argue that altruistic agents, especially inside a family, should be able to write contracts that avoid under-consumption of a constrained agent. However, note that if we allowed for fully state-contingent contracts, our framework would collapse to the collective model and only efficient allocations would be chosen. Short of fully-contingent contracts, one could also imagine basic lending contracts with a fixed interest rate between altruistic agents. This possibility is clearly of interest, but lies outside the scope of the current paper.

5 Comparative statics and robustness

We now discuss comparative-statics exercises with respect to the various parameters to gain a better understanding of the workings of the model. In particular, we focus on the altruism parameters (α, α'), the utility curvature γ and the investment-risk parameter σ. In general, our algorithm performs well and is stable across the parameter space. The qualitative features of equilibrium discussed in the previous section are always preserved under imperfect altruism.

5.1 Re-parameterization of altruism

It turns that a re-parameterization for the altruism parameters is useful. Our goal is to find new altruism measures (a, a') that keep the generosity of a donor constant when other model parameters change. To find such a measure, consider a donor who has control over the recipient's consumption (as is the case in the transfer region). The donor will optimally choose transfers such that $u_c(c) = \alpha u_c(c')$. For power utility, this implies that agents' consumption levels have the ratio

$$a \equiv \alpha^{1/\gamma} = \frac{c'}{c},$$

which defines our new, more intuitive measure of altruism. We define $a' \equiv \alpha'^{1/\gamma}$ equivalently.

5.2 Changes in a

When thinking about changes to altruism, it is first useful to consider the extreme situations of selfishness ($a = a' = 0$) and perfect altruism ($a = a' = 1$). For both selfishness and perfect altruism, the equilibrium is described by a Bewley-type economy. For $a = a' = 0$, both agents solve their separate consumption-savings problem and transfers are zero. The agents' consumption depends solely on their own state (w, y) but is invariant in the other agent's state (w', y'). For $a = a' = 1$, a dynastic household solves a Bewley problem given the agents' joint labor income resources (see the planner's problem in section 2.3). Agents' consumption is equalized in all states and depends solely on the joint state (W, Y), but neither on the distribution of assets $P \equiv w/W$ between agents nor agents' labor income draws (y, y').

We find that the equilibrium consumption functions converge to these extremes when (a, a') approach their bounds. So consumption depends more on joint wealth when altruism is high, and more on own wealth when altruism is low. The transfer regions and over-consumption region become large when $(a, a') \to (1, 1)$; they become small and start at ever higher levels of (w, w') when $(a, a') \to (0, 0)$. In both cases, distortions to consumption-savings decisions vanish as the limit is approached; distortions and over-consumption are most pronounced for intermediate values of (a, a'). Immiseration occurs ever faster in the OC region when $(a, a') \to (1, 1)$ as consumption of the poor agent gets close to the rich agent's consumption level.

Two special cases, one-sided altruism and symmetric altruism, warrant discussion. It is important to point out that the model maintains its intricacies in these two cases.

In the case of one-sided altruism, the fundamental reason for this is that the other player's state (w', y') is always of interest for both players. A selfish agent will watch her counterpart's situation in order to gauge the likelihood of transfers, and the altruistic agent has to watch the selfish counterpart's state in order to see if transfers are needed. So dynamic strategic interactions are still present in the case of one-sided altruism. As is to be expected, in equilibrium the selfish player never gives transfers. In the region where the altruistic agent is relatively well-off, there is a transfer region and an OC region with the characteristics described in section 4.

In the case of symmetric imperfect altruism $(0 < a = a' < 1)$, the only simplification that occurs is that the equilibrium is now symmetric. Its features are as in the baseline example. Symmetric imperfect altruism is not fundamentally different from asymmetric altruism. Agents do not have the same preferences, i.e. they disagree on allocations. Preferences are only mirror-symmetric. The only case in which agents agree and the analysis simplifies is perfect altruism ($a = a' = 1$).

5.3 Changes in γ

Due to our parsimonious specification of preferences, γ plays various roles. In addition to governing risk aversion and the elasticity of inter-temporal substitution, it also impinges on transfer behavior. However, as is apparent from the definition of a, γ only affects transfer behavior in conjunction with α. It turns out that when keeping (a, a') constant, the equilibrium transfer functions are essentially invariant in γ. Distortions to consumption-savings decisions are not visible affected either. Qualitatively, the consumption functions also preserve their characteristic. The only way we find γ to affect the equilibrium is through the well-known precautionary-savings mechanism. The higher γ, the stronger the precautionary-savings motive and the larger agents' wealth under the ergodic distribution. Since more time is spent in regions with high wealth, this also means that transfers flow more often.

5.4 Changes in σ

As mentioned before, the transfers-when-constrained equilibrium does not exist when $\sigma = 0$. The reason is that in regions where joint wealth W is large, the model converges to the one studied in our previous paper and tensions arise that cannot be overcome unless there is noise. Computationally, these tensions manifest themselves in the transfer motive becoming positive on the seam between the OC and the SS region. Specifically, we find that the transfer motive increases along the seam in W and converges to the level we find in the zero-labor income model. We thus compute the simpler zero-labor income model in order to determine if the equilibrium exists for a given parameter vector.

In table 2, we report the minimal level for σ that is required so that equilibrium exists, given different levels of γ and a. For convenience, altruism is assumed to be symmetric ($a' = a$) and all other parameters are as in our baseline numerical example.[19] We find that for fixed γ, the lower bound for σ is inverse-U-shaped in a, just as altruistic-strategic distortions are (not shown here). Tensions disappear at the extremes where the model converges to the selfish or the perfectly-altruistic case and are maximal for intermediate values of altruism. When fixing a, the lower bound is

[19] When altruism is asymmetric, we find that the lower bound for σ is governed mainly by the higher altruism parameter.

(weakly) decreasing in γ. This is due to a portfolio-variance effect that we will now explain in detail.

It turns out that there is also an upper bound on σ above which equilibrium ceases to exist for given (γ, a). We report this bound in table 3 in appendix A.4. The intuition for this upper bound is the following. Recall from the planner's problem in section 2.3 that the optimal portfolio choice of the planner is to keep half of total wealth W in each agent's account – this minimizes portfolio risk. So when agents are perfectly altruistic, they will distribute their assets in this way. However, for imperfect altruism the equilibrium dynamics are such that the poorer agent heads towards being broke. At that point, however, the portfolio risk is maximal from the family's perspective, which creates an incentive to head back to a more balanced asset distribution. When altruism gets close to perfect, value functions flatten out because agents become indifferent towards the distribution of wealth. The portfolio effect then at some point overrides the richer agent's preference for own wealth, making the transfer motive positive and thus precluding the transfers-when-constrained equilibrium. Since the portfolio effect is increasing in σ, this occurs more often for high values of σ, thus inducing an upper bound for σ for a given (γ, a)-combination. For some parameter configurations, the upper bound for σ cuts below the lower bound, so that there is no level of σ under which equilibrium exists for given (γ, a). The dashes in the table indicate when this occurs.

However, we note that the portfolio effect is not in the spirit of why we introduced shocks to assets in the first place – to overcome the underlying tensions in the setting through an element of uncertainty. There is a way to neutralize the portfolio effect, which we discuss in appendix A.4. When neutralizing the portfolio effect, we can always find σ large enough to ensure existence of equilibrium for any (γ, a). The lower bound on σ persists, but the upper bound disappears. We thus feel that the upper bound on σ is a less serious limitation for the usefulness of our model than the lower bound.

Table 2: Lower bound for σ

$a = a'$	0.1	0.3	0.5	0.7	0.9
$\gamma = 0.5$	2.5	5.3	–	–	0.9
$\gamma = 1$	1.1	3.2	–	3.1	0.9
$\gamma = 2$	0.8	1.8	2.5	2.4	0.9
$\gamma = 4$	0.4	1.0	1.3	1.6	0.9

Lowest value of σ for which the transfer motive μ stays negative throughout the state space and thus equilibrium exists (for given γ and $a = a'$). Values for ρ and r are as in the baseline example.

5.5 Other parameters

Changes to the remaining parameters in the model have the expected effects. An increase in ρ makes agents more impatient and brings consumption functions to a higher level, but leaves the qualitative features of the equilibrium unchanged. The opposite is true for an increase in r. As mentioned before, widening the gap between agents' flow labor income y and y' results in transfer regions drawing closer to the origin and eventually reaching it.

We have also computed equilibria for the case where *both* agents' labor income is stochastic. The model behaves as expected. Both agents now engage in precautionary-savings behavior. This results in a richer ergodic distribution. There is a density over the region where both players have positive wealth, there are "mass strips" on the lines where one player is broke, and finally there is a mass point where both players are broke.

6 Observable implications

In this section we discuss the observable implications of our model. While we interpret the two agents in the model as altruistically-linked households (children and parents, for example), the implications also apply for other applications of the model.

As stated in the introduction, the model is successful in generating transfer behavior that is broadly in line with the empirical evidence on inter-vivos transfers. Transfers in the model flow to liquidity-constrained individuals. This feature is consistent with findings by Cox (1990) and Cox & Jappelli (1990). Furthermore, our model predicts that transfers are increasing in the donor's wealth, increasing in the donor's labor income, and decreasing in the recipient's labor income. Again, these predictions are borne out in the data. For examples, see McGarry & Schoeni (1995), McGarry & Schoeni (1997) and Berry (2008).

A novel feature of our model is that the donor will condition transfers on the recipient's wealth and labor income in distinct ways. Our model imposes fewer restrictions on the *transfer-income derivative* than other models, but adds restrictions in that it distinguishes between transfer-*income* and transfer-*wealth* derivatives.

In order to compute transfer-income derivatives, we consider an augmented version of our baseline numerical example in which both agents face draws from an labor income process with the same support (y_1, y_2). We compute the change in transfers Δg following a downward jump in the donor's labor income paired with an upward jump in the recipient's labor income (conditional on transfers still being positive after the change to labor incomes). Dividing Δg by $\Delta y = y_2 - y_1$ gives us the analog of the transfer-income derivative (TID). We find that the TID may be below or above unity, depending on the hazard rates of labor income changes for agents. This softens the restrictions from the one- and two-period models considered so far in the literature, which impose that the TID equal unity.

The intuition is that in situations where a donor gives transfers, he essentially dictates the allocation, taking into account his wealth and future expected labor income of both agents. If the hazard rates for both labor income processes are the same, then an increase in his labor income paired with a decrease of her labor income of the same size leaves the dynasty's life-time wealth unchanged, and the donor implements the same consumption policies as before. In this situation, we find TIDs indistinguishable from unity. When considering asymmetric hazard rates, we are able to generate TIDs that differ from unity (in both directions). This is because changing both labor incomes by the same amount has different implications on the dynasty's permanent labor income. To see this, consider the situation where the agent with the more persistent labor income process draws a bad shock while the agent with a less persistent process draws a good shock. This is definitely bad news for expected total family labor income. The effect we find is similar to the effect in McGarry (2006), except that her model yields transfer-income derivatives strictly lower than unity.

For wealth changes, the picture in our model is fundamentally different and we think we are able to point to a new way of testing the restrictions imposed by altruism models in the data. Due to the consumption discontinuity, transfer-wealth derivatives are lower than unity if altruism is imperfect. In section A.5 of the appendix, we derive the following approximation to the transfer-wealth derivative (TWD):

$$TWD(x) = \frac{1}{1 + \frac{c_{lim}(x) - c(x)}{g'(x)}},$$

where $x = (0, w'; y, y')$ is a point in the state space where he gives transfers and $c_{lim}(x) \equiv \lim_{w \to 0} c(w', w; y, y')$ is her limit consumption before going broke. We see that the larger the recipient's party (i.e. the drop consumption upon going broke), the lower the transfer-wealth derivative. The intuition for this may be gleaned from the following example: say he is rich and provides transfers of 10,000\$ per year to her. She has flow labor income of 10,000\$ per year, so her consumption is 20,000\$. The standard altruism model would then predict that if we transfer 10,000\$ from him to her in the beginning of the year, he would choose transfers equal to zero and both would consume as before. We see that transfers react one-to-one to a redistribution of wealth. In our model, however, she will party once she has received 10,000\$ since she knows that he will provide transfers later in the year once she becomes broke. If the party is such that she spends at a rate of 30,000\$ per year, her wealth will decline at a rate of 20,000\$ per year so that she becomes broke after half a year. He will then provide another 5,000\$ of transfers over the second half of the year. As can be seen in this example, the transfer derivative is one half: observed transfers decrease by 5,000\$ following a wealth redistribution of 10,000\$.

The fundamental difference between labor income and wealth in our model suggests caution when testing the transfer-derivative restriction. The model says that it is crucial to differentiate between flow labor income and the stock of wealth. This is not easy to do using real-world data (think of labor income in the form of a large one-time bonus, e.g.). Furthermore, while it allows these derivatives to be different from unity it still has predictions on their sign and size.

Second, we turn to the model's implications on consumption behavior. It predicts that consumption levels should, in general, depend on relatives' assets and labor income. In families where the wealth distribution is very biased and even if we do not observe transfers (this would occur while the family is inside the OC region), poor households' consumption should strongly depend on the rich households' resources and only weakly on their own resources. This interdependence should not be observed for families with a balanced asset distribution.

There is a large literature that is concerned not with *levels* but with *changes* in consumption over time, and tests the implications of the Euler equations on longitudinal data. Our model adds terms to the standard (selfish) Euler equation. It predicts that marginal utility should grow fastest when a family is inside the OC region. Empirically, this would become evident in consumption growth being lowest for families with a biased (but not degenerate) wealth distribution. Drops in consumption should be sharpest when one household begins to receive transfers from another. Consumption growth should be higher in regions where one household receives transfers or where the family's wealth distribution is balanced. Indeed, consumption growth should be the same in these regions as for otherwise similar households that lack family ties to other households.

Finally, an extension of our model which allows for endogenous portfolio choice makes the following prediction. Households should exhibit the riskiest investment behavior when their wealth is low relative to that of the rest of the family, even when they are not yet dependent on transfers from them (on the seam between SS and OC region, see figure 11).

7 Conclusion

In this paper we have studied a fully-dynamic framework of altruistically-motivated transfers under uncertainty. A major strength of the framework is that it is consistent with stylized facts on inter-vivos transfers. Another strength of the model is that it delivers precise predictions on the timing of transfers. Thus, an immediate application would be to study the timing of inter-vivos transfers, remittances or development aid.

For future research the model is suitable as a building block for dynamic models of the (extended) family.[20] Obvious applications are the welfare implications of gov-

[20]For an introduction to the Markov-chain approximation method for continuous-time problems see our computational online-appendix. There we also demonstrate how to compute the equilibrium

ernment policies such as pensions, health insurance, welfare programs or the estate tax. For all of these issues, the interaction of government-provided transfers with transfers inside the family is likely to be important.

Some applications will find it necessary to introduce non-monetary transfers (in-kind, time, co-residence) into our model. In a follow-up project, we plan to extend the current framework using both labor-supply and time-transfer decisions in order to study the macroeconomic consequences of government long-term-care policies. While government-provided care potentially crowds out family-provided care, it could provide households with insurance.

Finally, one can view our framework as a theory of partial insurance. The crucial difference between this theory and others is that impoverished agents receive transfers even if they have never given to the donor in the past and are unable to reciprocate in the future. An example of this might be altruistic behavior towards people with terminal diseases or disabilities.

of the current model, and how to extend the current algorithm to overlapping-generations settings, finite-horizon settings, time or age dependence of value functions, etc. We provide a code toolbox, which can be used to replicate all the results presented in the paper, and for use in future research.

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A Appendix

A.1 His equations

For the convenience of the reader, this section keeps track of the equations for him, which are mirror-symmetric copies of the respective equations for her. The planner's

Euler equation for him is already implied by (7) when we invoke intra-temporal optimality from (6):

$$\mathcal{A}u_c(c') = (\rho - r)u_c(c').$$

His HJB is given by

$$\begin{aligned}\rho v' =& \alpha' u(c) + (rw + y - g - c)v'_w + (rw' + y' + g)v'_{k'} + \\ &+ \xi[v'(\cdot, \tilde{y}) - v'(\cdot, y)] + \xi[v'(\cdot, \tilde{y}') - v'(\cdot, y')] + \tfrac{\sigma^2}{2}(w^2 v'_{ww} + w'^2 v'_{w'w'})\} + \\ &+ \max_{g' \geq 0}\{g'[v'_w - v'_{w'}]\} + \max_{c' \geq 0}\{u(c') - c' v'_{k'}\}.\end{aligned}$$

The associated Euler equation for him is

$$\mathcal{A}u_c(c') = (\rho - r)u_c(c') + \big[v'_w - \alpha' u_c(c)]c_{w'} + \big[u_c(c') - v'_w]g_{w'}.$$

The expressions for his optimal policies when she is broke are straightforward mirror-symmetric copies of her optimal policies, which are given in sections 3.2 and A.3.

A.2 Optimal portfolio weights in planner's problem

We will show here that the planner optimally chooses to hold one-half of wealth in each player's account. This allocation minimizes the variance of a portfolio in the two risky assets and is thus obviously what the risk-averse planner should do. Let P be the fraction of W that goes to her asset, and $1 - P$ be the one allocated to his asset. Then, the law of motion for W is given by

$$dW_t = (rW_t + Y_t - c_t - c'_t)dt + P_t W_t \sigma dB_t + (1 - P_t)W_t \sigma dB'_t.$$

The variance is given by

$$E[(dW_t)^2] = P_t^2 W_t^2 \sigma^2 dt + (1 - P_t)^2 W_t^2 \sigma^2 dt,$$

which is minimized for $P^* = 1/2$. Substituting P^* back into the law of motion yields

$$dW_t = (rW_t + Y_t - c_t - c'_t)dt + W_t \underbrace{\frac{\sigma}{\sqrt{2}}}_{\equiv \sigma_W} \underbrace{\frac{dB_t + dB'_t}{\sqrt{2}}}_{\equiv dB_{W,t}}.$$

$B_{W,t}$ as defined here is standard Brownian motion since it has unit variance, has normal increments and is serially uncorrelated. This gives us the law of motion stated in the planner's problem (5).

A.3 Transfers decision when both players are broke

This appendix treats the transfer decision when both players are broke, i.e. now $w = 0$ in addition to $w' = 0$. We will consider the situation where her flow labor income is higher than his: $y > y'$. Then we only have to consider the possibility of her giving transfers to him (see the end of this section for a proof of this statement).

Her problem is to maximize $H(c, g)$ in (10) subject to the additional constraint

$$c + g \leq y, \tag{15}$$

which says that she cannot spend more than her current labor income.

Obviously, if the unconstrained maximizers (c_0, g_{unc}) from section 3.2 fulfill this constraint, then they are also the solution to the constrained problem. Also the intuition from the unconstrained case applies.

From now on we will be concerned with the case where the constraint binds, which means that the unconstrained maximizers (c_0, g_{unc}) fall *outside* the feasible set defined by (15) and the non-negativity constraints $c \geq 0$ and $g \geq 0$. We will now show that the optimal choice must fulfill her budget constraint with equality and that transfers do not go into savings. Formally, we want to show that the constrained maximizers (c_{constr}, g_{constr}) lie on the line segment D defined by

$$D = \{(c, g) : \ c + g = y, g \leq c'_0 - y'\}.$$

We will first show why $g \leq c'_0 - y'$ must hold. If we consider a pair (c, g) such that $g > c'_0 - y'$, a (feasible) decrease in g always increases H since $\partial H/\partial g < 0$ in this region (recall that he is saving the marginal transfer unit, which she dislikes). Thus we must have $g_{constr} \leq c'_0 - y'$.

Second, to see why $c + g = y$ must hold with equality, note that the objective function H is strictly concave in (c, g) in the region where $g \leq c'_0 - y'$ by strict concavity of u. So any interior pair $\{(\tilde{c}, \tilde{g}) : \tilde{c} + \tilde{g} < 0, \tilde{g} \leq c'_0 - y'\}$ must be dominated by any convex combination of $(\tilde{c}, \tilde{g})$ and the unconstrained maximum (c_0, g_{unc}) since $H(\tilde{c}, \tilde{g}) \leq H(c_0, g_{unc})$ and H is strictly concave (note that $c_0 \geq 0$, $g_{unc} \geq 0$ and $g_{unc} \leq c_0' - y'$ by definition of g_{unc}). Since $(\tilde{c}, \tilde{g})$ was interior, there thus must always be possible improvements and $(\tilde{c}, \tilde{g})$ cannot be optimal.

We have now established that the constrained maximizers must lie on the line segment D.

Since in the constrained case the budget constraint is always fulfilled with equality (i.e. $c+g = y$) and g is always such that all transfers are consumed, we can reduce the problem of maximizing (10) subject to (15) to a simpler auxiliary problem in one choice variable by writing

$$\max_{g \in [0, c_0' - y']} \quad \big\{u(y-g) + \alpha u(y'+g)\big\}. \tag{16}$$

Here, we recognize a static altruism problem subject to the additional constraint $g \leq c_0' - y'$ (which comes from the fact that she never wants to give transfers that flow into savings); we see that apart from this upper bound on transfers the problem is now independent of the dynamic aspects of the game, i.e. the derivatives of the value functions v_w etc.

Note that the function to be maximized, $\tilde{u}(g) \equiv u(y-g) + \alpha u(y'+g)$, is strictly concave (again by strict concavity of u) and the maximand is chosen from a closed interval. It will thus be possible to characterize the solution by just checking if the unconstrained maximizer of $\tilde{u}(g)$ falls into the constrained set. We define

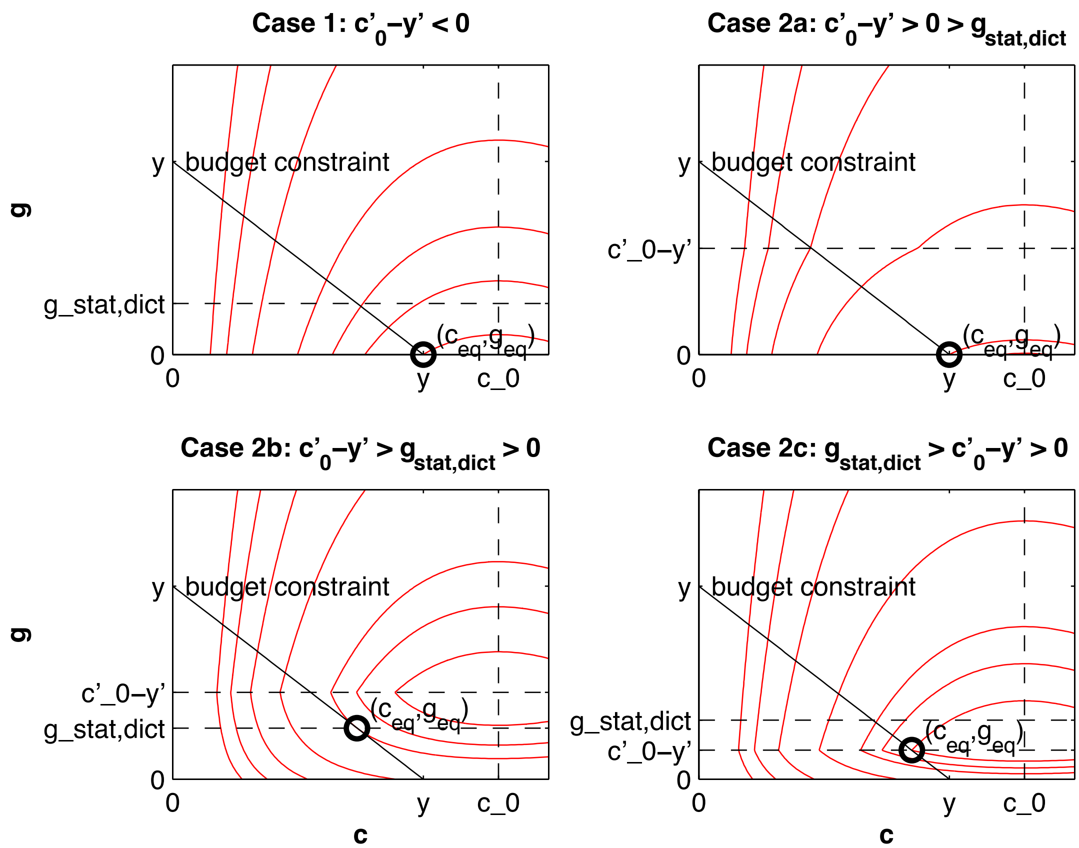
$$g_{stat,dict} = \arg \max_{-\infty < \tilde{g} < \infty} \left\{u(y-\tilde{g}) + \alpha u(y'+\tilde{g})\right\},$$

where the subscript "stat,dict" suggests that this is the transfer that she would choose in a static problem if she could also force negative transfer upon him. For CES utility, we obtain $g_{stat,dict} = (\alpha^{1/\gamma} y - y')/(1 + \alpha^{1/\gamma})$.

The following cases can arise in the case where her constraint binds. Figure 12 depicts the level lines of her Hamiltonian as curves, the constraint as a line and the optimal choice as a circle.

1. $c_0' < y'$: He saves even when transfers are zero, so she should set the transfer to zero since $v_{w'} < v_w$. The transfer margin is dominated by the consumption

Figure 12: Her transfer decision when $w = w' = 0$



margin.

2. $c'_0 \geq y'$: The following sub-cases can arise:

   (a) $g_{stat,dict} \leq 0$: She sets $g_{constr} = 0$ since the criterion in (16) is decreasing in g for all $g \in D$. The transfer margin is dominated by the consumption margin.

   (b) $0 < g_{stat,dict} < c'_0 - y'$: There is an interior solution and she sets $g_{constr} = g_{stat,dict}$; the unconstrained maximizer for the problem (16) falls into the feasible set. Here, she sets the transfer margin equal to the consumption margin and the allocation is the one predicted by the static altruism model.

(c) $g_{stat,dict} > c_0' - y'$: A corner solution occurs in the auxiliary problem (16). She increases transfers until reaching his satiation point where he would start to save the marginal transfer. She stops the transfer at this point and sets $g_{constr} = c_0' - y'$. Her transfer margin is *lower* than the consumption margin when considering a marginal *increase* in transfers at the optimum, but the transfer margin is *larger* than the consumption margin when considering a marginal *decrease* in the transfer at the optimum. This is what the kinks in the level lines of the Hamiltonian indicate.

It is important to recall that we were always operating under the assumption that the unconstrained maximizer (c_0, g_{unc}) does not fall into feasible set, so she sets savings to zero. This means that her consumption margin is larger than her savings margin in all cases.

We can summarize the solution for all (constrained) cases by

$$g_{constr} = \max\left\{0, \min\{g_{stat,dict}, c_0' - y'\}\right\}.$$

Since the budget constraint always holds with equality when she is constrained, her consumption is always given by $c_{constr} = y - g_{constr}$. His consumption is $c'_{constr} = y' + g_{constr}$ (recall that she never gives transfers that would go into his savings, so we need not enforce the upper bound c_0').

Finally, we can also conclude the following: When both players are bankrupt, then only the player with the higher flow labor income (the "labor income-rich" player) can possibly give transfer. Note that $g_{stat,dict}$ is negative for the labor income-poor player, so it is a dominant strategy for him to set transfers to zero. For this to hold, we technically need that altruism parameters are lower than one – if α was larger than one, then she might want to give to him although she has lower flow labor income than he does.

A.4 Upper bound for σ

Table 3 reports the upper bound on σ, which is due to the portfolio effect discussed in section 5.4. The bound is decreasing in a: Value functions become flatter as altruism increases, and it becomes easier for the portfolio effect to override agent's preference for own assets, causing transfer motives to turn positive. The bound is for the most part decreasing in γ: The more risk-averse agents become, the stronger the portfolio effect.

Table 3: Upper bound for σ

$a = a'$	0.1	0.3	0.5	0.7	0.9
$\gamma = 0.5$	10.6	7.6	–	–	1.2
$\gamma = 1$	25.0	8.8	–	3.5	1.3
$\gamma = 2$	5.8	4.9	4.3	3.1	1.2
$\gamma = 4$	3.4	3.2	3.0	2.6	1.3

Highest value of σ for which the transfer motive μ stays negative throughout the state space and thus equilibrium exists (given γ and $a = a'$). ρ and r are as in the baseline example.

We can think of two ways to neutralize the portfolio effect while maintaining the smoothing force of shocks.

The first is to assume that only the asset distribution $P = w/(w + w')$ between agents is subject to disturbances, but total family assets $W = w+w'$ are unaffected by shocks. Technically, this approach is paramount to finding a viscosity (i.e. smooth) solution to the HJBs in the cake-eating setting studied by Barczyk & Kredler (2011). Indeed, we find that there is no upper bound on σ when we follow this approach. The reason we did not choose this modeling approach in the first place is that it lacks a micro-foundation: There are very few real-world examples of such shocks.

A second, more micro-founded, way of shutting down the portfolio effect would be to make shocks proportional to consumption c and not assets w. This would also be in line with some of the examples we gave to motivate our shocks to assets (e.g. expenditure shocks, such as repair of consumer durables, or health shocks). To see why this approach is likely to work, observe that for perfectly-altruistic agents consumption functions are invariant in P. Thus expenditure risk would be invariant in P, canceling the portfolio effect. The problem with this approach is that it is unclear how such consumption shocks should be handled when an agent is constrained: Suppose both agents are broke and are hit by a negative expenditure shock they cannot pay for (this may always happen since Brownian-motion shocks are unbounded). It is not straightforward to handle this situation, so we chose not to take this avenue.

A.5 Derivation of the transfer-wealth derivative

Suppose that the econometrician measures the cumulative transfers over a time interval Δt (say a month or a year). At a given point $x_0 = (0, w_0'; y_0, y_0')$ in the state space, he gives transfers $g'(x_0)\Delta t > 0$ to her over this time interval (to a first order). We will now study how the observed transfers change when the donor's initial wealth is decreased by Δw, while the recipient's initial wealth is increased by Δw. We assume that Δw is small enough so that at some transfers still flow over Δt when starting the economy from the new endowment point $(\Delta w, w_0' - \Delta w; y_0, y_0')$ (this being for the case that there are no changes to labor income).

Denote by $c_{lim}(x_0) = \lim_{w \to 0} c'(w, w_0'; y_0, y_0')$ her consumption when given a small amount of assets. The law of motion for her wealth at the new endowment point $(\Delta w, w_0' - \Delta w; y_0, y_0')$ is given by $\dot{w}' = y_0' - c_{lim}(x_0)$, to a first approximation. This means that it will take a time interval $\tilde{\Delta t} \equiv \Delta w / [c_{lim}(x_0) - y_0']$ until she runs out of wealth. She then starts to receive a transfer $g'(x_0)$, so that the total transfer received over Δt is given by $g'(x_0)(\Delta t - \tilde{\Delta t})$. Note that the events that his or her labor income change are of lower order and may be dropped to a first approximation.

The transfer-wealth derivative at x_0 is then defined the change in transfers (observed over Δt) divided by the change in wealth Δw:

$$TWD(x_0) \equiv \frac{(\Delta t - \tilde{\Delta t}) g'(x_0) - g'(x_0)\Delta t}{\Delta w} = \frac{-g'(x_0)}{c_{lim}(x_0) - y_0} = \frac{-1}{1 + \frac{c_{lim}(x_0) - c(x_0)}{g'(x_0)}},$$

where the last step uses the fact that $c(x_0) = y_0 + g'(x_0)$.