A Dynamic Model of Altruistically-Motivated Transfers

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Motivation for our research agenda

- Agents are often modeled as **altruistic**, especially inside families (inter-vivos transfers, bequests, time transfers,. . .)
- Growing interest in **dynamic** altruism models in recent literature.
- **But:** Literature has limited itself to studying rather special cases of dynamic altruistic interaction. Literature
- **Our research agenda:** Want to close this gap and provide a general building-block model.

$\Rightarrow$ Can apply to interesting policy questions: Long-term care, pension reform, development aid,. . .

Our research agenda: Two papers

Tackle dynamic environments with two altruistic agents: Consumption, savings and transfers – no commitment.

1. This paper:
   - Analyzes the central tension in such models theoretically
   - Simplest-possible setting: Deterministic, no flow income.
   - Theoretical results: Point to one stable type of equilibrium.
2. Follow-up paper ("Altruistically-Motivated Transfers under Uncertainty"):
   - Adds stochastic income stream
   - Provides stable numerical algorithm for solution
   - Gives building-block model for more complex dynamic environments with altruism

Model

Setting

Markovian game in continuous time (*differential game*): Advantages

- Two infinitely-lived agents: *she (x)* and *he (x')*
- No borrowing: $k_t \geq 0$, $k'_t \geq 0$
- Safe asset: pays at rate r
- Initial endowments (k_0, k'_0), no income stream
  $\Rightarrow$ "cake eating": allows us to exploit homogeneity

Altruistic preferences

- She ranks allocations according to

$$v_0 \equiv \int_0^\infty e^{-\rho t} \left[u(c_t) + \alpha u(c'_t) \right] dt$$

- He ranks allocations according to

$$v'_0 \equiv \int_0^\infty e^{-\rho t} \left[u(c'_t) + \alpha' u(c_t) \right] dt$$

- where $\alpha, \alpha' \in [0, 1]$
- and $u(c) = \ln c$.

Formulation encompasses:

- $\alpha = \alpha' = 0$: Selfishness
- $\alpha = \alpha' = 1$: Perfect altruism
- $\alpha > 0$, $\alpha' = 0$: One-sided altruism

Benchmark: Pareto-optimal allocations

Planner with criterion $\eta v_0 + (1-\eta) v_0'$.

1. Intra-temporal optimality:

$$u_c(c_t) = \frac{(1-\eta)+\eta\alpha}{\eta + (1-\eta)\alpha'} u_c(c_t'), \qquad \text{for all } t$$

2. Inter-temporal optimality:

$$\frac{d}{dt} u_c(c_t) = (\rho - r) u_c(c_t).$$

Note: Any Pareto-optimal allocation may be implemented by choosing an appropriate division (k_0, k_0') of initial assets $K_0 = k_0 + k_0'$ and then leaving agents on their own (i.e. shutting transfers down).
$\Rightarrow$ Is an equilibrium under commitment.

Markov strategies

Payoff-relevant state: $x \equiv (k_t, k_t')$. Strategies:

- Consumption: $c(x_t) \geq 0$ and $c'(x_t) \geq 0$
- Transfer: $g(x_t) \geq 0$ and $g'(x_t) \geq 0$
  (are allowed to be mass points, i.e. *measure-valued*)

$\Rightarrow$ Law of motion for wealth:

$$\dot{k}_t = rk_t - c_t - g_t + g_t'$$
$$\dot{k}_t' = rk_t' - c_t' - g_t' + g_t$$

Best-responding: Hamilton-Jacobi-Bellman Equation

For a given strategy $\{c'(\cdot), g'(\cdot)\}$ by him, her HJB is:

$$\rho v = + \underbrace{\alpha u(c') + (rk' + y' - g' - c')v_{k'}}_{\text{his decisions}} +$$
$$+ \max_{c \geq 0} \left\{ \underbrace{u(c) + (rk + y + g' - c)v_k}_{\text{consumption-savings trade-off}} \right\} +$$
$$+ \max_{g \geq 0} \left\{ g \left[\underbrace{v_{k'} - v_k}_{\equiv \mu\text{: transfer motive}} \right] \right\}$$

- $u_c(c^*) = v_k \Rightarrow$ constant best response, crucial simplification with respect to discrete time! 2nd order
- $g^* > 0$ only if $\mu \geq 0$

Equilibrium Definition

Definition

A *Markov-Perfect Equilibrium* consists of functions $\{v, c, g\}$ and $\{v', c', g'\}$ such that

1. $\{v, c, g\}$ fulfill her HJB given $\{c', g'\}$ for all x, and
2. $\{v', c', g'\}$ fulfill his HJB him given $\{c, g\}$ for all x.

- Implies subgame perfection
- Technical note: We define what HJB means by limit argument ($\Delta t \rightarrow 0$) in case of non-differentiabilities and mass transfers.

Optimal decisions

Optimal savings: (Generalized) Euler equation

Consider region where $k > 0$:

$$\frac{d}{dt}u_c(c) = \underbrace{(\rho - r)u_c(c)}_{\text{standard = efficient}} + \underbrace{\left[v_{k'} - \alpha u_c(c')\right]c'_k}_{\text{altruistic-strategic distortion}} + \underbrace{\left[u_c(c) - v_{k'}\right]g'_k}_{\text{transfer-induced incentives}}$$

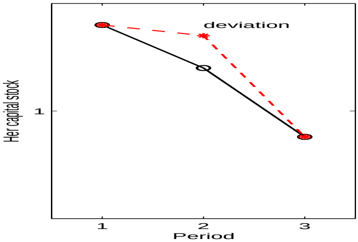
Note: 2nd and 3rd term vanish in two special cases:

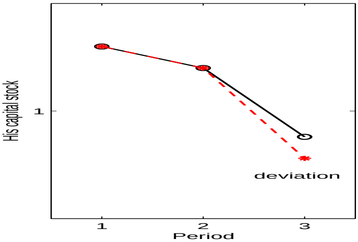
1. $\alpha = \alpha' = 0$: Have $c'_k = g'_k = 0$.
2. $\alpha = \alpha' = 1$: Have $u_c(c) = v_k = v_{k'} = u_c(c')$.

$\Rightarrow$ Expect distortions to be large for intermediate values of α and α'.

Intuition for generalized Euler equation (discrete time)

$$\frac{d}{dt}u_c(c_t) = (\rho - r)u_c(c_t) + \left[v_{k'} - \alpha u_c(c')\right]c'_k + \left[u_c(c) - v_{k'}\right]g'_k$$





	period 1	period 2	period 3
standard	$-u_c(c_1)$	$+\ \beta R u_c(c_2)$	
altruistic-strategic		$\alpha\beta u_c(c_2)\frac{\partial c'_2}{\partial k_2}$	$-\beta^2 v_{k'}(k_3, k'_3) R \frac{\partial c'_2}{\partial k_2}$
transfer-induced		$\beta u_c(c_2)\frac{\partial g'_2}{\partial k_2}$	$-\beta^2 R v_{k'}(k_3, k'_3)\frac{\partial g'_2}{\partial k_2}$

$\Rightarrow$ Typically inefficiency Inefficiencies

Equilibrium

Homogeneity: Dimension reduction

- Define the following variables: $K = k + k'$ and $P = k/K$
- Consider homogeneous (i.e. K-linear) policies, e.g. for her:

$$C(P, K) = \bar{C}(P)K, \qquad G(P, K) = \bar{G}(P)K$$

- Homogeneous policies lead to separable value functions, e.g. for her:

$$V(P, K) = \frac{1+\alpha}{\rho}\left(\frac{r}{\rho} + \ln K\right) + \bar{V}(P)$$

- Given separable V, want to play homogeneous strategies.
- Only $\bar{V}(P)$ and $\bar{V}'(P)$ to be determined.

Advantages of dimension reduction

- HJBs and Euler equations turn from PDEs into ODEs $\Rightarrow$ easier!
  Equations
- ODEs plus the number of boundary conditions tells us if to expect
  1. no equilibrium
  2. one (or a finite number of) equilibria
  3. a continuum of equilibria.

  $\Rightarrow$ an advantage of the continuous-time approach!

Solution strategy

Consider equilibria where state space $\mathcal{P} = [0, 1]$ is partitioned into $1 \leq N < \infty$ smooth regions:

- Region $i = 1, \ldots, N$: $\mathcal{P}_i = (P_{i-1}, P_i)$ where $0 = P_0 < P_1 < \cdots < P_N = 1$.
- Value functions assumed differentiable inside all $\mathcal{P}_i$
  $\Rightarrow$ Policies obey Euler equations
- *Value matching*: Can show that value functions must be continuous at *boundaries* P_i.
- **But:** Value functions may be non-differentiable at P_i (*kinks*)
  $\Rightarrow$ Discontinuities in policies

Strategy: Characterize different types of regions and then try to patch them together.

Regions without transfers: Characterization

Regions without transfers: $G = G' = 0$

1. Usually: **No-transfer** (NT)
   - Savings decisions distorted $\Rightarrow$ inefficiencies
   - Transitory regime: Are left eventually
2. Special case: **Self-sufficiency** (SS). Players play autarkic consumption policies: $C = \rho P, C' = \rho(1 - P)$.
   - Absorbing regime: $\dot{P} = 0$.
   - Efficient if wealth distribution sufficiently balanced: $P_0 \in [\frac{\alpha'}{1+\alpha'}, \frac{1}{1+\alpha}]$.

Transfer regions: Characterizations

1. **Mass-transfer** (MT) by one player to the other
   - Transitory: Immediately left.
2. **Flow-transfer** (FT): One player gives transfers
   - Donor's transfer schedule determines recipient's consumption/savings.
   - Transitory regime
3. **Wealth-pooling** (WP): Both players' transfer motives zero: $\mu = \mu' = 0$, or equivalently $\bar{V}_P = \bar{V}'_P = 0$.
   - Transfers indeterminate, $\dot{P}$ indeterminate $\Rightarrow$ may be absorbing.
   - Players consume as if consuming out of common stock $K = k + k'$.
   - Tragedy-of-the-commons-type inefficiencies, alleviated by altruism:

$$C_{WP} = \frac{\rho}{1+\alpha}, \qquad C'_{WP} = \frac{\rho}{1+\alpha'}.$$

Smooth equilibria?

Smooth equilibrium (consisting of one region) only in two special cases:

1. $\alpha = \alpha' = 0$: Self-sufficiency $\Rightarrow$ efficient
2. $\alpha = \alpha' = 1$: Wealth-pooling $\Rightarrow$ efficient

$\Rightarrow$ Look for non-smooth/patched equilibria for imperfect altruism.

Continuum of tragedy-of-the-commons-type equilibria

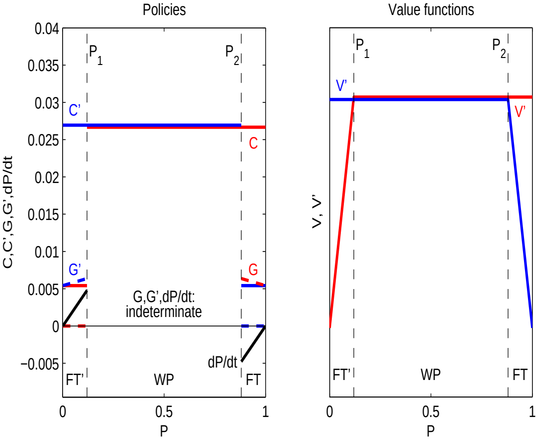
Proposition:
Iff $\alpha\alpha' > 0$ there exists a continuum of equilibria of the following type:

1. He gives transfers on $[0, P_1)$.
2. There is a wealth-pooling region $[P_1, P_2]$.
3. She gives transfers on $(P_2, 1]$.

We have $P_1 \in (0, P_{max}(\alpha, \alpha')]$ and $P_2 \in [1 - P_{max}(\alpha', \alpha), 1)$.
The economy ends up in $[P_1, P_2]$ from (almost) all initial states.

$P_{max}(\cdot)$: see extra slide Pmax

Tragedy-of-the-commons-type equilibrium ($\alpha = \alpha'$)



Why continuum of equilibria? Only **one** steady state: WP!

Transfer-when-constrained equilibrium?

Is there an equilibrium where transfers only flow at $P = 0$ and $P = 1$?

- Empirically plausible! Transfers flow often when recipient constrained.
- Insert one no-transfer (NT) region: $(0, 1)$.
- *Party Theorem*: $P = 0$, $P = 1$ are absorbing steady states. Theorem
- Problem: 2 ODEs with 4 boundary conditions $\Rightarrow$ expect 0 solutions.
- Cannot find this equilibrium numerically for any (α, α'), as expected.
- Intuition: 2 steady states, agents disagree to which one to go.

NT-NT-NT structure

Introduce shock

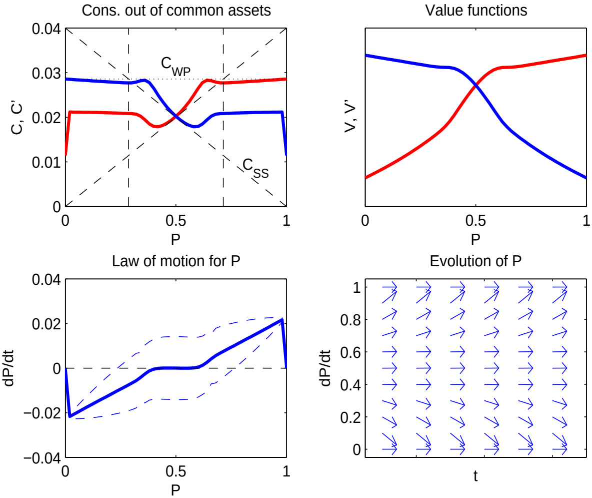
- Introduce (uncorrelated) shocks:

$$dk_t = (r - c_t - g_t + g_t')dt + \sigma k_t dB_t$$
$$dk_t' = (r - c_t' - g_t' + g_t)dt + \sigma k_t' dB_t'$$

where $E[dB_t dB_t'] = 0$.

- Interpretation: Asset is risky now
- Get 2 **second**-order ODEs with 4 boundary conditions: Exactly identified, expect unique solution!
- Numerically: Obtain stable equilibrium.
- Intuition: Chance decides to which of 2 steady states to go ⇒ resolves tension.

Transfer-when-constrained equilibrium (with shock)



(Dynamic) Samaritan's Dilemma

- **Party theorem**: Recipient's consumption path jumps downward when going broke More
  - Intuition: Recipient is not fully taking into account cost of transfers to donor
  - Gross violation of efficiency
  - Equivalent to (well-known) *Samaritan's Dilemma* in discrete time: Aid can create perverse incentives
- **Dynamic Samaritan's Dilemma:** Inefficiencies feed back to long before reaching zero wealth (new)
  - Intuition: Like tragedy of the commons – consume out of a joint stock (donor's assets)
  - Also donor's savings decision is distorted (new)
  - Inefficiencies before party are of lower order than party

Which equilibrium should we select?

The **tragedy-of-the-commons-type equilibrium** is not stable with respect to:

1. introduction of noise
2. finite horizon
3. introduction of in-kind transfers

The **transfer-when-constrained equilibrium** is

1. stable with respect to all of the above
2. indeed the unique equilibrium obtained as limit of sequence of finite games (with noise)
3. empirically more plausible

$\Rightarrow$ **We think: Future research should focus on the transfer-when-constrained equilibrium.**

Further result: The *prodigal-son dilemma*

Proposition:
There cannot be a mass transfer from him to her at $P = 0$ (neither by her to him at $P = 1$) unless $\alpha = \alpha' = 1$.

- Intuition as in prodigal-son parable:
  1. Father cannot commit to not-give transfers in future
  2. Son knows this and parties (the *Samaritan's dilemma*)
  3. Father should foresee this and not pay out bequest early.

In general, for imperfect altruism we show:
There cannot be equilibria where the rich player lifts the poor one out of poverty and both are self-sufficient ever after.
$\Rightarrow$ No-commitment is crucial for this result!

Conclusions

Contributions

Solved simplest-possible dynamic model of altruism.

1. **Euler equations:** Understand savings incentives and distortions.
2. Found **new type of equilibrium** in altruism models: Tragedy-of-the-commons-type. New feature: Under-consumption.
3. New points in transfers-when-constrained equilibrium:
   - Inefficiencies feed back long in time
   - Also donor's savings decision distorted
   - Party distortion ($\simeq$ Samaritan's dilemma) is inefficiency of different order than distortions before.
4. Stability analysis: **Focus on transfers-when-constrained equilibrium** (need shock)
5. Clear-cut distinction between **prodigal-son dilemma** and Samaritan's dilemma.
6. Technically: Deal with non-smoothness and mass-point policies in differential game.

Research agenda: Outlook

Follow-up paper:
"Altruistically-motivated transfers under uncertainty"

- Introduces **stochastic income stream**.
- Focuses on transfers-when-constrained equilibrium.
- Gives numerically-stable **algorithm** to calculate equilibrium.
- Solves **extensions** of basic model:
  1. OLG structure
  2. Finite-horizon setting
  3. Portfolio decision: Poor agent "gambles for resurrection".

Provides **building block** for larger models:

- Barczyk (2010): Quantitative study of Ricardian Equivalence
- Can study crowding-out of family transfers by government policies. Want to do: Pension systems, long-term-care insurance.

Extra slides

Extra: Literature

- Becker (1974): Static model of altruism
- Unitary/Collective model (Chiappori, 1988): Circumvents strategic interactions by assuming commitment
- Laitner (1988): Two-period OLG with two-sided altruism; players overlap for only one period
- Nishiyama (2002): Four-period OLG with one-sided altruism; players overlap for two periods, consumption residually determined.
- Two-period models (no commitment):
  - Cox (1987)
  - Lindbeck & Weibull (1988)
  - Altig & Davis (1991)
  - Altonji, Hayashi & Kotlikoff (1997)
  - Fernandes (2005, 2008)
- Special assumptions: Fuster et al. (2007), Kaplan (2010)

back Facts on transfers

Extra: Facts on inter-vivos transfers

1. Quantitatively important: Larger than bequests in cross section (Cox & Raines, 1985)
2. Flow frequently: About 40% of parents gave to children in previous 12 months (Soldo & Hill, 1995; Berry, 2008)
3. Transfers occur . . .
   - Within families
   - When recipient is liquidity-constrained
   - From wealthy family members. . .
   - . . . to disadvantaged family members: Unemployed, low assets, low income

back

Extra: Advantages of Markov concept

- The usual ones:
  1. Smaller set of equilibria than for fully-contingent strategy space
  2. Predictions on data conditional on state only, not entire history
  3. Takes seriously the notion that bygones are bygones
- Macro-related:
  1. Fits well into dynamic-programming models in macro
  2. Exploit optimal-control techniques
- Forgiveness implicit in Markov assumption may be realistic, especially for altruistic agents

Extra: Advantages of continuous time

- Can neglect second-order effects
- Avoid curse of dimensionality in computation (see Doraszelski & Judd, 2008)
- Timing protocol doesn't matter (as much)
- Can study timing of transfers differentially
- Have ODEs and boundary conditions: Easier to determine number of equilibria.

back

Extra: First-order condition

Her FOC:

$$u_c(c) = v_k$$

Value of saving at $t + \Delta t$:

$$v_k(k_{t+\Delta t}, k'_{t+\Delta t}) = v_k(k_t, k'_t) + \underbrace{v_{kk'}(k_t, k'_t)\left[rk'_t - c'_t \dots\right]\Delta t}_{\text{of second order}} + \dots$$

- His (contemporaneous) consumption choice c' has no effect on her marginal value of saving
- Interactions with other policies also 2nd-order
- Constant best-response function for c given c'
- Simplification with respect to discrete time
- **But:** c' enters v_k over time, of course! See this in (generalized) Euler equation...

back

Extra: Inefficiencies in Δt-equilibrium

$$H(c,c') = u(c)\Delta t + \alpha u(c')\Delta t + [y + rk - c]\Delta t v_k + [y' + rk' - c']\Delta t v_{k'}$$
$$H'(c,c') = \alpha' u(c)\Delta t + u(c')\Delta t + [y + rk - c]\Delta t v'_k + [y' + rk' - c']\Delta t v'_{k'}$$

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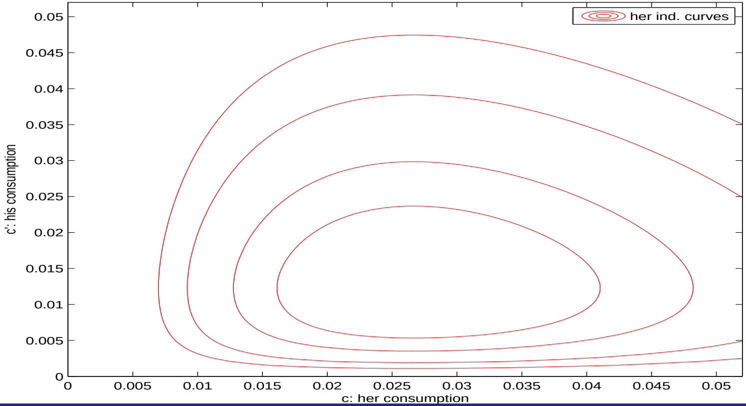

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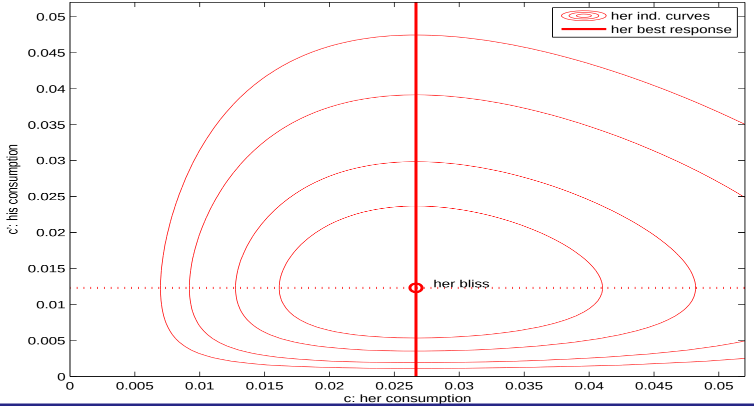

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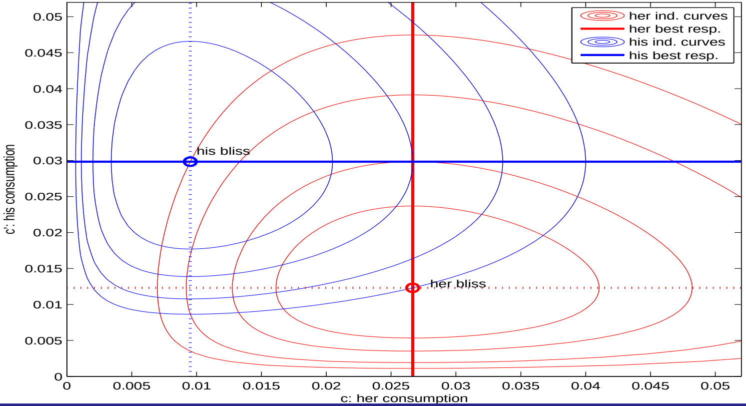

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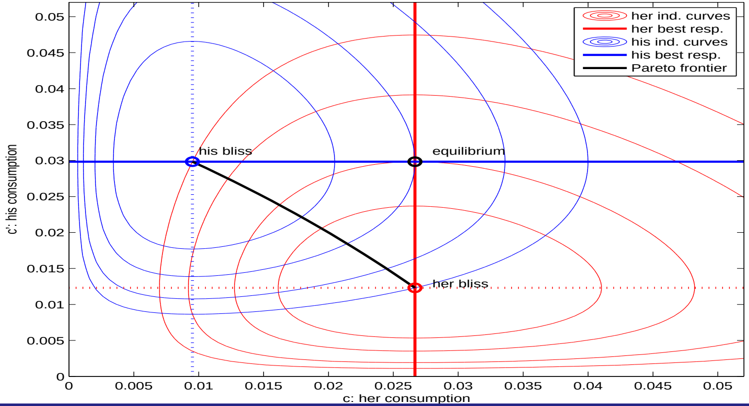


HJB and Euler equation in P-K-space

- Her HJB:

$$\rho V = \alpha \ln C' - C' \frac{1+\alpha}{\rho} + PC' V_P + G' V_P + \\ + \max_{C \geq 0} \left\{ \ln C - C \frac{1+\alpha}{\rho} - C(1-P) V_P \right\} + \max_{G \geq 0} \{-G V_P\}.$$

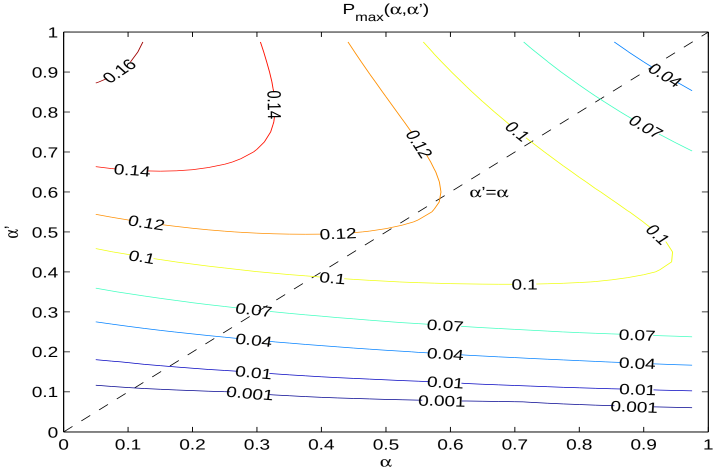
- Her Euler equation:

$$\frac{d}{dt} V_P(t) = \left[PC' - (1-P)C - G + G' \right] V_{PP} = \\ = \left[\rho - C - C' \right] V_P + \left[\frac{1}{C} - \frac{\alpha}{C'} - V_P \right] C'_P - G'_P V_P$$

back

Extra: P_{max}

P_{max} solves an ugly implicit equation, see paper appendix 10. back



Extra: Formal statement of Party Theorem

Proposition (Party Theorem):
If $\alpha' > 0$ (+weak technical assumptions), then any equilibrium where a NT-region borders $P = 0$ has the following properties:

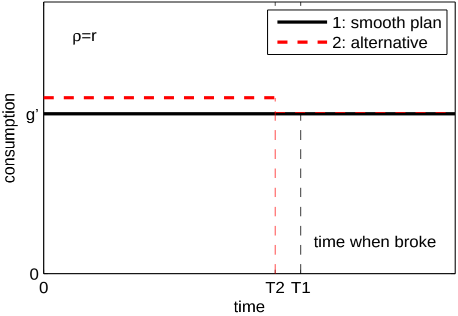
1. $\dot{P}_{lim} < 0$ and $\dot{P}_0 = 0$: Her being broke is an absorbing state.
2. $C(0) = \frac{\alpha'}{1+\alpha'}$: When she is broke, his preferred allocation is played.
3. $C_{lim} = \exp\left(\frac{1-\alpha\alpha'}{1+\alpha'}\right) C(0)$: (*Party*) On reaching $P = 0$, her consumption path has a downward jump unless $\alpha = \alpha' = 1$.
4. $V_P'(0) > 0$: He strictly prefers her being broke to her returning to be unconstrained.

back

Extra: Why non-smooth consumption plan?

- No altruism
- Means-tested benefit: Get $g' > 0$ conditional on $k = 0$
- Only initial endowment: $k_0 > 0$, $y = 0$
- $\rho = r$: Choose $c_t = const$ for $t < T$ (T: time when broke)

Would you prefer plan 1 or plan 2?



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Extra: Party incentives

Recipient's problem with Δk assets left:

$$\max_{c} \left\{ u(c)\Delta t + \alpha u(c')\Delta t + e^{-\rho \Delta t} V(0, k' + \dot{k}'\Delta t) \right\}$$

$$\Delta t = \frac{\Delta k}{c - y - r\Delta k}$$

First-order condition for c:

$$\underbrace{u_c(c)\Delta t}_{1} + \underbrace{\frac{d\Delta t}{dc}}_{2} \left[\underbrace{u(c) + \alpha u(c') - \rho V(0, k')}_{A} + \underbrace{\dot{k}' V_{k'}(0, k')}_{B} \right] = 0$$

1. $u(c) \uparrow$: Positive (more utility now)
2. $\Delta t \downarrow$: Negative (go broke sooner)
   - A Lose party flow utility, replaced by zero-assets flow value
   - B Get poorer donor (if $\dot{k}' > 0$, i.e. he saves).

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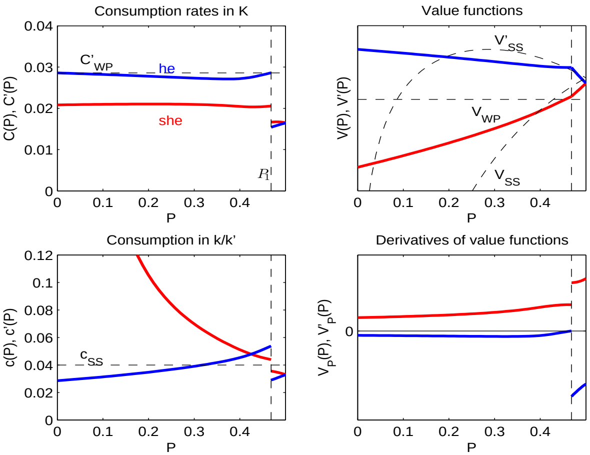
Extra: NT-NT-NT equilibrium? No!



No equilibrium policies exist at kink P_1! back