Internal Rationality, Imperfect Market Knowledge and Asset Prices
Klaus Adam and Albert Marcet

Meng Li

Universidad Carlos III de Madrid

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Motivation

- The rational expectations hypothesis (REH): agents know how market works → unrealistic;
- Relax these assumptions and model learning about market behavior.
What They Do in the Paper

- Separate standard rationality requirements into “internal” and “external” rationality components defined in the study;
- Relax the external rationality assumption but to fully maintain internal rationality in a model;
- Present a simple asset pricing model with heterogeneous risk-neutral investors and market incompleteness;
- Consider the standard relaxation way-Bayesian rational expectations equilibrium and compare these two ways;
- Reconsider the asset pricing model with prior beliefs close to the RE beliefs to show the consistency of internal rationality with standard approaches in the adaptive learning literature.
Main Results

1. Internal rationality is not sufficient to achieve external rationality;

2. The equilibrium stock price is determined by a one-step-ahead asset pricing equation;

3. Agents need to possess a tremendous amount of additional knowledge about the market, over and above what can be derived from internal rationality alone for a BREE to emerge;

4. A model with internally rational agents whose prior beliefs are close to the RE beliefs can derive an OLS learning algorithm.
Investor: Infinitely lived, risk neutral and heterogenous in type $i \in \{1, \ldots, I\}$.
A unit mass of investors in each type endowed with $\frac{1}{I}$ units of initial stock.
A standard time-separable utility function

$$E_0^{P_i} \sum_{t=0}^{\infty} (\delta^i)^t C^i_t,$$  \hspace{2cm} (1)

where $E_0^{P_i}$ denotes the agent’s expectations in some probability space $(\Omega, S, P^i)$, where $\Omega$ is the space of realizations, $S$ the corresponding $\sigma$-algebra, and $P^i$ a subjective probability measure over $(\Omega, S)$ given to agents as a model primitive.
1. Basic asset pricing model

- The stocks $S_t^i$ owned by agents represent claims to an infinitely lived tree;
- Yields each period $D_t$ units of a perishable consumption good which are paid as dividend;
- The stock can be purchased and sold costlessly in a perfectly competitive spot market at price $P_t$.
- Budget constraint: $C_t^i + P_t S_t^i \leq (P_t + D_t) S_{t-1}^i + \xi$.
- Limit constraints on stock holdings: $0 \leq S_t^i \leq \bar{S}$, where $1 < \bar{S} < \infty$. 
Non-standard part:

- Agents view the process for \( \{P_t, D_t\} \) as external to their decision problem;
- The probability space over which they condition their choices is given by
  \[
  \Omega \equiv \Omega_P \times \Omega_D. \tag{2}
  \]
- Assume that type \( i \)'s beliefs are given by a well-defined probability measure \( \mathcal{P}^i \) over \( (\Omega, \mathcal{S}) \).
- Denote
  \[
  \Omega^t_D: \text{ the set of all possible dividend histories up to period } t; \\
  D^t \in \Omega^t_D: \text{ a typical dividend history; } \\
  \text{similar definitions for prices; }
  \]
  Then the set of all histories up to period \( t \) is given by
  \[
  \Omega^t \equiv \Omega^t_P \times \Omega^t_D \text{ and its typical element is denoted by } \omega^t \in \Omega^t. \]
1. Basic asset pricing model

**Definition**

**Internal rationality**

Agent $i$ is internally rational if she chooses the consumption and stock to maximize expected utility subject to the budget constraint, and the limit constraints, taking as given the probability measure $P^i$. 
More assumptions:

- $\mathcal{P}^i$ satisfies $E^{\mathcal{P}^i}[P_{t+1} + D_{t+1}|\omega^t] < \infty$ for all $\omega, t, i$;
- a maximum of the investor’s utility maximization problem exists.
Internal Rationality with Imperfect Market Knowledge

2. Optimality conditions

**FOC:**

\[ P_t < \delta^i E_t^P (P_{t+1} + D_{t+1}) \text{ and } S_t^i = \bar{S} \]

\[ P_t = \delta^i E_t^P (P_{t+1} + D_{t+1}) \text{ and } S_t^i \in [0, \bar{S}] \]

\[ P_t > \delta^i E_t^P (P_{t+1} + D_{t+1}) \text{ and } S_t^i = 0 \]

Notice that the optimality conditions are of the one-step-ahead form.
3. Standard belief formulation: a singularity

Standard belief formulation imposes additional restrictions on beliefs:

- Agents formulate probability beliefs only over the reduced state space \( \Omega_D \);
- Agents’ choices are contingent on the history of dividends only.

Agents are then endowed with the knowledge that each realization \( D^t \in \Omega_D^t \) is associated with a given level of the stock price \( P_t \). However, knowledge of this singularity is not a consequence of agents’ ability to maximize their utility or to behave rationally given their subjective beliefs.
Let $(\Omega_D, S_D, P_D)$ be a probability space with $\Omega_D$ denoting the space of dividend histories and $P_D$ the ‘objective’ probability measure for dividends. Let $\omega_D \in \Omega_D$ denote a typical infinite history of dividends.

**Definition**

**IREE**

An Internally Rational Expectations Equilibrium (IREE) consists of a sequence of equilibrium price functions $\{P_t\}_{t=0}^{\infty}$ where $P_t: \Omega_D^t \to \mathbb{R}_+$ for each $t$, contingent choices $\{C_t^i, S_t^i\}_{t=0}^{\infty}$ and probability beliefs $P^i$ for each agent $i$, such that

1. all agents $i = 1, \ldots, I$ are internally rational, and
2. when agents evaluate $\{C_t^i, S_t^i\}$ at equilibrium prices, markets clear for all $t$ and all $\omega_D \in \Omega_D$ almost surely in $P_D$. 
The previous first order conditions imply that the asset is held by the agent type with the most optimistic beliefs about the discounted expected price and dividend in the next period.

Equilibrium prices thus satisfy:

\[ P_t = \max_{i \in I} \left[ \delta^i E_t^P (P_{t+1} + D_{t+1}) \right]. \]
5. Is internal rationality sufficient to derive a singularity in beliefs?

Let $m_t : \Omega_D^t \rightarrow \{1, \ldots, I\}$ denote the marginal agent pricing the asset in period $t$ in equilibrium:

$$m_t = \arg\max_{i \in I} [\delta^i E^P_t (P_{t+1} + D_{t+1})].$$

The equilibrium price can be rewritten as

$$P_t = \delta^{m_t} E^{P_{m_t}}_t (P_{t+1} + D_{t+1}).$$
Internal Rationality with Imperfect Market Knowledge

5. Is internal rationality sufficient to derive a singularity in beliefs?

The answer in general is NO.

▶ Suppose that agents know that the equilibrium price satisfies above equation each period and that this is common knowledge.

▶ Then $P_t = \delta^{m_t} E_t^{P_{mt}} (P_{t+1} + D_{t+1})$.

▶ Iterate forward $T$ times to find

$$P_t = \delta^{m_t} E_t^{P_{mt}} (D_{t+1}) + \delta^{m_t} E_t^{P_{mt}} (\delta^{m_{t+1}} E_{t+1}^{P_{mt+1}} D_{t+2}) + \delta^{m_t} E_t^{P_{mt}} (\delta^{m_{t+1}} E_{t+1}^{P_{mt+1}} (\delta^{m_{t+2}} E_{t+2}^{P_{mt+2}} D_{t+3})) + \cdots$$

$$+ \delta^{m_t} E_t^{P_{mt}} (\delta^{m_{t+1}} E_{t+1}^{P_{mt+1}} (\cdots \delta^{m_{t+T}} E_{t+T}^{P_{mt+T}} (P_{t+T+1} + D_{t+T+1})))$$

.$$
5. Is internal rationality sufficient to derive a singularity in beliefs?

> Since agent $i$ is not marginal in all periods and
> since agent $i$ can rationally believe other agents to hold rather different beliefs,
> own beliefs about dividends fail to restrict the beliefs agent $i$ can entertain about prices.

Summary: knowledge of the existence of a degeneracy falls short of informing agents about its exact location.
Conclusions

Limited knowledge about the market gives rise to learning from market outcomes, so that expectations regarding the future market outcomes become an important determinant of the current market outcome.