A Method for Solving and Estimating Heterogenous Agent Macro Models (by Thomas Winberry)

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Macro Reading Group

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Outline

Motivation

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2. Different numerical algorithms have been developed to overcome this challenge (value function or policy function iteration, gold search, etc).
3. However, none of them are as general, efficient, or easy-to-apply as the standard perturbation methods (typically employed for soving RA models).
4. Thus, **is it possible to apply perturbation methods to solve heterogeneous agent models?**

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2. Two important basis:
    - Approximates the cross-sectional distribution using a parametric family. For instance, *Algan, Allais and Haan (2008)* use a globally accurate projection technique to solve Krusell-Smith (KS) model (but, this is computationally slow).
    - Usage of globally accurate and locally accurate approximations to solve for the dynamics of HA models. For example, *Reiter (2009)* employs locally accurate approximation with respect to state vector for solving KS model; however, his method relies on a fine histogram approximation, which requires many parameters to achieve aceptable accuracy.

Summary of Results

1. Based on these computation techniques, the developed method employs between 30-50 **seconds** to solve the model.
2. Besides, to illustrate the power of the method, this is used to estimate a HA model with full-information Bayesian techniques.
3. Another feature of this method is that it could be applied to a wide range of of HA model. In these slides, it will be discussed the HA model of *Khan and Thomas (2008)* (KT) and *Krussel and Smith (1998)* (KS).
4. Finally, the computational method is implemented in `Dynare`.

Benchmark Model: KT Model

Households

1. It is assumed a representative infinite-lived household, whose preferences are defined by:

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{C_t^{1-\sigma}-1}{1-\sigma}-\chi\frac{N_t^{1+\psi}}{1+\psi}\right)\right] \tag{1}$$

where C_t is consumption, N_t is labor supplied to the market. β the discount factor, σ the relative risk aversion coefficient, χ governs the disutility of labor, and $1/\psi$ is the Frish elasticity of labor supply.

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2. The representative househod is endowed with a unit of time ($N_t \in [0,1]$). The household owns all the firms in the economy. Markets are complete.

1. There exist a continuum of firms with a total unit mass, $j \in [0,1]$, which produce output y_{jt} according to:

$$y_{jt} = e^{z_t} e^{\varepsilon_{jt}} k_{jt}^{\theta} l_{jt}^{\nu}, \quad \theta + \nu < 1 \tag{2}$$

where z_t is an aggregate productivity shock, ε_{jt}, a idiosyncratic one. k_{jt} the capital input, l_{jt} is labor input. θ the elasticity of output respect to capital, and ν the elasticity of output respect to labor.

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2. **Law of motions:**
   - Aggregate shock, z_t, evolves as follows:

$$z_{t+1} = \rho_z z_t + \eta_z \omega_{t+1}^{z}, \text{ with } \omega_{t+1}^{z} \sim N(0,1) \tag{3}$$

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   - Idiosyncratic shock, ε_{jt}, evolves as follows:

$$\varepsilon_{jt+1} = \rho_\varepsilon \varepsilon_{jt} + \eta_\varepsilon \omega_{t+1}^\varepsilon, \text{ with } \omega_{t+1}^\varepsilon \sim N(0,1) \tag{4}$$

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2. After observing the two productivity shocks, **each firm hires labor** from a perfect competitive labor market, and **produces output**.
3. After production, the **firm invests** in capital for the next period. Gross investment, i_{jt}, yields:

$$k_{jt+1} = (1-\delta)k_{jt} + i_{jt} \tag{5}$$

where δ is the depreciation rate of capital (which is assumed to be homogeneous among the firms).

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3. Hence, the parameter $\mathscr{a}$ governs a region around zero investment, within which firms do not incur the fixed cost.

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3. Hence, the parameter a governs a region around zero investment, within which firms do not incur the fixed cost.
4. ξ_{jt} is uniformly distributed in the interval $[0, \underline{\xi}]$, and is i.i.d over firms and time.

1. Following KT, the Bellman equation for the firm is:

$$v(\varepsilon, k, \xi; \mathbf{s}) = \lambda(\mathbf{s}) \max_{l} \{e^{z} e^{\varepsilon} k^{\theta} l^{\nu} - w(\mathbf{s}) l\} + \max\{v^{a}(\varepsilon, k; \mathbf{s}) - \xi \lambda(\mathbf{s}) w(\mathbf{s}),\ v^{n}(\varepsilon, k; \mathbf{s})\} \quad (6)$$

where $\mathbf{s}$ is the aggregate state vector, $\lambda(\mathbf{s}) = C(\mathbf{s})^{-\sigma}$.

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where $\mathbf{s}$ is the aggregate state vector, $\lambda(\mathbf{s}) = C(\mathbf{s})^{-\sigma}$.

2. Besides:

$$v^{a}(\varepsilon, k; \mathbf{s}) = \max_{k' \in \mathbb{R}} \left[-\lambda(\mathbf{s})(k' - (1-\delta)k) + \beta \mathbb{E}[\hat{v}(\varepsilon', k'; \mathbf{s}'(z', \mathbf{s})) | \varepsilon, k; \mathbf{s}]\right] \quad (7)$$

$$v^{n}(\varepsilon, k; \mathbf{s}) = \max_{k' \in \mathbb{A}} \left[-\lambda(\mathbf{s})(k' - (1-\delta)k) + \beta \mathbb{E}[\hat{v}(\varepsilon', k'; \mathbf{s}'(z', \mathbf{s}))] | \varepsilon, k; \mathbf{s}\right] \quad (8)$$

with $\mathbb{A} = [(1-\delta-a)k,\ (1-\delta+a)k]$, $\hat{v}(\varepsilon, k; \mathbf{s}) = \mathbb{E}_{\xi}[v(\varepsilon, k, \xi; \mathbf{s})]$.

1. Let $k^a(\varepsilon, k; \mathbf{s})$ denotes the capital choice from (7), and $k^n(\varepsilon, k; \mathbf{s})$ the optimal choice from (8).

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Firms: Optimization Problem

1. Let $k^a(\varepsilon, k; \mathbf{s})$ denotes the capital choice from (7), and $k^n(\varepsilon, k; \mathbf{s})$ the optimal choice from (8).
2. The firm will pay the fixed cost iff:

$$v^a(\varepsilon, k; \mathbf{s}) - \xi\lambda(\mathbf{s})w(\mathbf{s}) \geq v^n(\varepsilon, k; \mathbf{s}) \tag{9}$$

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3. Hence, there exists a unique value of the fixed cost ξ which makes the firm indifferent between the two options:

$$\tilde{\xi}(\varepsilon, k; \mathbf{s}) = \frac{v^a(\varepsilon, k; \mathbf{s}) - v^n(\varepsilon, k; \mathbf{s})}{\lambda(\mathbf{s})w(\mathbf{s})} \tag{10}$$

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4. To prevent values that could be outside the support of ξ, let define:

$$\hat{\xi}(\varepsilon, k; \mathbf{s}) = \min\{\max\{0, \tilde{\xi}(\varepsilon, k; \mathbf{s})\}, \underline{\xi}\} \tag{11}$$

1. Since the extensive margin decision is characterized by the cutoff (10), it is possible to compute analytically $\hat{v}(\varepsilon, k; \mathbf{s})$:

$$\hat{v}(\varepsilon, k; \mathbf{s}) = \lambda(\mathbf{s}) \max_{l} \{e^z e^{\varepsilon} k^{\theta} l^{\nu} - w(\mathbf{s}) l\} + v^a(\varepsilon, k; \mathbf{s}) \mathbb{P}(i/k \notin [-a, a])$$
$$-\lambda(\mathbf{s}) w(\mathbf{s}) \mathbb{E}[\xi \mathbf{1}(\xi \leq \hat{\xi}(\varepsilon, k; \mathbf{s}))] + v^n(\varepsilon, k; \mathbf{s})(1 - \mathbb{P}(i/k \notin [-a, a]))$$

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2. Note that:

$$\mathbb{P}(i/k \notin [-a, a]) = \int_0^{\hat{\xi}(\varepsilon, k; \mathbf{s})} dF(\xi) = \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\underline{\xi}} \quad (12)$$

$$\mathbb{E}[\xi \mathbf{1}(\xi \leq \hat{\xi}(\varepsilon, k; \mathbf{s}))] = \int_0^{\hat{\xi}(\varepsilon, k; \mathbf{s})} \xi dF(\xi) = \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})^2}{2\underline{\xi}} \quad (13)$$

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Firms: Characterization of $\hat{v}(\varepsilon, k; \mathbf{s}) = \mathbb{E}_\xi[v(\varepsilon, k, \xi; \mathbf{s})]$

1. Thus:

$$\begin{aligned}\hat{v}(\varepsilon, k; \mathbf{s}) = \lambda(\mathbf{s}) \max_l \{e^z e^\varepsilon k^\theta l^\nu - w(\mathbf{s}) l\} \\ + \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\underline{\xi}} \left(v^a(\varepsilon, k; \mathbf{s}) - \lambda(\mathbf{s}) w(\mathbf{s}) \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{2} \right) \\ + v^n(\varepsilon, k; \mathbf{s}) \left(1 - \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\underline{\xi}} \right)\end{aligned} \tag{14}$$

1. The aggregate vector **s** contains the current draw of the aggregate productivity shock, z, and the distribution (density) of firms over (ε, k)-space, $g(\varepsilon, k)$.

1. The aggregate vector $\mathbf{s}$ contains the current draw of the aggregate productivity shock, z, and the distribution (density) of firms over (ε, k)-space, $g(\varepsilon, k)$.
2. The RCE for KT model consists in a set of functions: (i) $\hat{v}$, l, k^a, k^n, $\hat{\xi}$ depending on $(\varepsilon, k; \mathbf{s})$, (ii) λ, w depending on $\mathbf{s}$, and (iii) $\mathbf{s}^{'}(z^{'}; \mathbf{s}) = \left(z^{'}; g^{'}(z, g)\right)$ such that:
    - (*Firm opt.*) Taking λ, w and $\mathbf{s}'$ as given: l, k^a, k^n, $\hat{\xi}$ solve the firm's optimization problem (6).
    - (*Household opt.*) For all $\mathbf{s}$: $\lambda(\mathbf{s}) = C(\mathbf{s})^{-\sigma}$, with

$$C(\mathbf{s}) = \mathbb{E}_{\varepsilon,k}\left[e^z e^{\varepsilon} k^{\theta} l(\varepsilon, k; \mathbf{s})^{\nu} + (1-\delta)k - k^a(\varepsilon, k; \mathbf{s})\frac{\hat{\xi}(\varepsilon,k;\mathbf{s})}{\underline{\xi}} - k^n(\varepsilon, k; \mathbf{s})\left(1 - \frac{\hat{\xi}(\varepsilon,k;\mathbf{s})}{\underline{\xi}}\right)\right]$$

$$w(\mathbf{s}) \text{ satisfies } \mathbb{E}_{\varepsilon,k}\left[l(\varepsilon, k; \mathbf{s}) + \frac{\hat{\xi}(\varepsilon,k;\mathbf{s})^2}{2\underline{\xi}}\right] = \left(\frac{w(\mathbf{s})\lambda(\mathbf{s})}{\chi}\right)^{1/\psi}$$

3

- (*Law of motion for distribution*) For all (ε', k'):

$$g'(\varepsilon', k'; z, \mathbf{s}) = \int\int\int \left(\begin{bmatrix} \mathbb{1}\left(\rho_\varepsilon \varepsilon + \eta_\varepsilon \omega'_\varepsilon = \varepsilon'\right) \times \left[\frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\underline{\xi}} \mathbb{1}\left(k^a(\varepsilon, k; z, \mathbf{s}) = k'\right) .. \right. \\ \left. \quad ... + \left(1 - \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\underline{\xi}}\right) \mathbb{1}\left(k^n(\varepsilon, k; z, \mathbf{s}) = k'\right) \right] \end{bmatrix} \times p(\omega'_\varepsilon) g(\varepsilon, k; \mathbf{s}) d\omega'_\varepsilon d\varepsilon dk \right) \tag{15}$$

where $p(.)$ is the p.d.f of idiosyncratic productivity shock.

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- (*Law of motion for aggregate shocks*)

$$z' = \rho_z z + \eta_z \omega'_z \tag{16}$$

Computational Method

1. Winberry's method involves three main steps:
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1. Winberry's method involves three main steps:
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   3. **Local accurate approximation around the stationary equilibrium.**
2. Note that steps 2 and 3 are the typical ones when a RA model is solved using perturbation techniques.
3. In step 1, the two infinite-dimensional objects to approximate are: (i) firm's value function $\hat{v}(\varepsilon, k; \mathbf{s})$, and (ii) cross-section distribution $g(\varepsilon, k)$.

Approximation of infinite-dimensional objects

Cross-section distribution

1. Following *Algan, Allais and Haan (2008)*, Winberry approximates $g(\varepsilon, k)$ using the following parametric family:

$$g(\varepsilon, k) \approx g_0 \exp\{g_1^1(\varepsilon - m_1^1) + g_1^2(\log(k) - m_1^2) \ldots \\ \ldots + \sum_{i=2}^{n_g} \sum_{j=0}^{i} g_i^j \left[(\varepsilon - m_1^1)^{i-j} (\log(k) - m_1^2)^j - m_i^j \right] \} \quad (17)$$

where n_g is the degree of approximation, $\{g_0, g_1^1, g_1^2, \langle \{g_i^j\}_{j=0}^i \rangle_{i=2}^{n_g}\}$ are parameters, and $\{m_1^1, m_1^2, \langle \{m_i^j\}_{j=0}^i \rangle_{i=2}^{n_g}\}$ are centralized moments of the distribution.

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2. When $n_g = 2$, the approximate function lies in the family of multivariate Normal distributions.

1. The parameter vector $\mathbf{g} = \left\{ g_0 \quad \dots \quad g_{n_g}^{n_g} \right\}$, and the moment vector $\mathbf{m} = \left\{ m_1^1 \quad \dots \quad m_{n_g}^{n_g} \right\}$ have to be consistent with each other. That is, moments should be implied by the parameters. Thus:

$$
\begin{aligned}
m_1^1 &= \int \int \varepsilon g(\varepsilon, k) d\varepsilon dk \\
m_1^2 &= \int \int \log(k) g(\varepsilon, k) d\varepsilon dk \\
m_i^j &= \int \int (\varepsilon - m_1^1)^{i-j} (\log(k) - m_1^2)^j g(\varepsilon, k) d\varepsilon dk \\
&\quad \text{for } i = 1, \dots, n_g, \; j = 0, \dots, i
\end{aligned} \tag{18}
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1. The parameter vector $\mathbf{g} = \begin{Bmatrix} g_0 & \dots & g_{n_g}^{n_g} \end{Bmatrix}$, and the moment vector $\mathbf{m} = \begin{Bmatrix} m_1^1 & \dots & m_{n_g}^{n_g} \end{Bmatrix}$ have to be consistent with each other. That is, moments should be implied by the parameters. Thus:

$$\begin{aligned} m_1^1 &= \int \int \varepsilon g(\varepsilon, k) d\varepsilon dk \\ m_1^2 &= \int \int \log(k) g(\varepsilon, k) d\varepsilon dk \\ m_i^j &= \int \int (\varepsilon - m_1^1)^{i-j} (\log(k) - m_1^2)^j g(\varepsilon, k) d\varepsilon dk \\ &\text{for } i = 1, \dots, n_g, \; j = 0, \dots, i \end{aligned} \tag{18}$$

2. Plugging in the approximate functional form (17) in system (18) results in a non-linear system on **m** and **g**. But, *Algan, Allais and Haan (2008)* develop a robust method for solving **g** given a vector **m**. Thus, **m** completely characterizes the approximated density.

1. Using the law of motions 3, 4 and 15, and the approximated density $g(\varepsilon, k; \mathbf{m})$, one can get:

$$m_1^{1\prime}(z, \mathbf{m}) = \int\int\int (\rho_\varepsilon \varepsilon + \omega_\varepsilon') p(\omega_\varepsilon') g(\varepsilon, k; \mathbf{m}) d\omega_\varepsilon' d\varepsilon dk$$

$$m_1^{2\prime}(z, \mathbf{m}) = \int\int\int \log(k') p(\omega_\varepsilon') g(\varepsilon, k; \mathbf{m}) d\omega_\varepsilon' d\varepsilon dk \qquad (19)$$

$$m_i^{j\prime}(z, \mathbf{m}) = \int\int\int (\rho_\varepsilon \varepsilon + \omega_\varepsilon' - m_1^{1\prime})^{i-j} \boldsymbol{\kappa}_j p(\omega_\varepsilon') g(\varepsilon, k; \mathbf{m}) d\omega_\varepsilon' d\varepsilon dk$$

where

$$\log(k') = \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\underline{\xi}} \log(k^a(\varepsilon, k; z, \mathbf{m})) + \left(1 - \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\underline{\xi}}\right) \log(k^n(\varepsilon, k; z, \mathbf{m}))$$

$$\boldsymbol{\kappa}_j = \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\underline{\xi}} \left[\log(k^a(\varepsilon, k; z, \mathbf{m})) - m_1^{2\prime}\right]^j + \left(1 - \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\underline{\xi}}\right) \left[\log(k^n(\varepsilon, k; z, \mathbf{m})) - m_1^{2\prime}\right]^j$$

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2. Besides, one can iterate system (19) in order to find the steady-state values of moment vector, $\mathbf{m}^\star$.

Approximation of infinite-dimensional objects

Cross-section distribution

1. System (19) represents a mapping from the current aggregate state $(z, \mathbf{m})$ into the next period moments $\mathbf{m}'(z, \mathbf{m})$.
2. Besides, one can iterate system (19) in order to find the steady-state values of moment vector, $\mathbf{m}^{\star}$.
3. Even though, theoretically, there is no guarantee for the convergence of a non-linear system like (19), Winberry shows that, in practice, convergence happens.

1. Winberry approximates firm's value function with respect to individual states (ε, k) using orthogonal polynomials. Hence:

$$\hat{v}(\varepsilon, k; z, \mathbf{m}) \approx \sum_{i=1}^{n_\varepsilon} \sum_{j=1}^{n_k} \vartheta_{ij}(z, \mathbf{m}) T_i(\varepsilon) T_j(k) \tag{20}$$

where n_ε and n_k are the degree of approximation of individual states ε and k, respectively. $T_i(\varepsilon)$ and $T_j(k)$ are Chebyshev polynomials, and $\vartheta_{ij}(z, \mathbf{m})$ are coefficients on those polynomials.

Approximation of infinite-dimensional objects

Firm Value Function

1. With this approximation, we can obtain a numerical approximation of Bellman equation (6) using collocation. Let define a set of grid points $\{\langle \varepsilon_i \rangle_{i=1}^{n_\varepsilon}, \langle k_j \rangle_{j=1}^{n_k}\}$.

1. With this approximation, we can obtain a numerical approximation of Bellman equation (6) using collocation. Let define a set of grid points $\{\langle\varepsilon_i\rangle_{i=1}^{n_\varepsilon}, \langle k_j\rangle_{j=1}^{n_k}\}$.
2. First, equation (7) it is approximated:

$$v^a(\varepsilon_i, k_j; z, \mathbf{m}) \approx -\lambda(z, \mathbf{m})(k^a(\varepsilon_i, k_j; z, \mathbf{m}) - (1-\delta)k_j)... \\ + \beta\, \mathbb{E}_{z'|z}\left[\mathbb{E}_{\omega'_\varepsilon}\left(\hat{v}(\rho_\varepsilon \varepsilon + \eta_\varepsilon \omega'_\varepsilon, k^a(\varepsilon_i, k_j; z, \mathbf{m}); z', \mathbf{m}'(z, \mathbf{m}))\right)\right] \tag{21}$$

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3. Now, equation (8):

$$v^n(\varepsilon_i, k_j; z, \mathbf{m}) \approx -\lambda(z, \mathbf{m})(k^n(\varepsilon_i, k_j; z, \mathbf{m}) - (1-\delta)k_j)\ldots \\ + \beta\, \mathbb{E}_{z'|z}\left[\mathbb{E}_{\omega'_\varepsilon}\left(\hat{v}(\rho_\varepsilon \varepsilon + \eta_\varepsilon \omega'_\varepsilon, k^n(\varepsilon_i, k_j; z, \mathbf{m}); z', \mathbf{m}'(z, \mathbf{m}))\right)\right] \tag{22}$$

1. Thus, the numerical approximation of equation (14) is:

$$\hat{v}(\varepsilon_i, k_j; z, \mathbf{m}) = \lambda(z, \mathbf{m}) \max_{l} \{e^z e^{\varepsilon_i} k_j^{\theta} l^{\nu} - w(z, \mathbf{m}) l\} \ldots$$
$$+ \frac{\hat{\xi}(\varepsilon_i, k_j; z, \mathbf{m})}{\underline{\xi}} \left(\tilde{v}^a(\varepsilon_i, k_j; z, \mathbf{m}) - \lambda(z, \mathbf{m}) w(z, \mathbf{m}) \frac{\hat{\xi}(\varepsilon_i, k_j; z, \mathbf{m})}{2} \right)$$
$$+ \tilde{v}^n(\varepsilon_i, k_j; z, \mathbf{m}) \left(1 - \frac{\hat{\xi}(\varepsilon_i, k_j; z, \mathbf{m})}{\underline{\xi}} \right) \tag{23}$$

where $\tilde{v}^a(.)$ and $\tilde{v}^n(.)$ are the right-hand side of equations (21, 22), respectively.

1. The last step of first stage is to approximate the equilibrium conditions. Winberry shows that these approximated eq. conditions may be written as a system of $2n_\varepsilon n_k + n_g + 2 + n_g + 1$ equations.

Approximation of infinite-dimensional objects

Approximated Equilibrium Conditions

1. The last step of first stage is to approximate the equilibrium conditions. Winberry shows that these approximated eq. conditions may be written as a system of $2n_\varepsilon n_k + n_g + 2 + n_g + 1$ equations.
2. Let $\{\tau_i^\varepsilon, \omega_i^\varepsilon\}_{i=1}^{m_\varepsilon}$ denote the weights and nodes of the one-dimensional Gauss-Hermite quadrature used to approximate: $\mathbb{E}_{\omega'_\varepsilon}\left(\hat{v}(\rho_\varepsilon \varepsilon + \eta_\varepsilon \omega'_\varepsilon, k^q(\varepsilon_i, k_j; z, \mathbf{m}); z', \mathbf{m}'(z, \mathbf{m}))\right)$ for $q = a, n$.

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2. Let $\{\tau_i^\varepsilon, \omega_i^\varepsilon\}_{i=1}^{m_\varepsilon}$ denote the weights and nodes of the one-dimensional Gauss-Hermite quadrature used to approximate: $\mathbb{E}_{\omega'_\varepsilon}\left(\hat{\vartheta}(\rho_\varepsilon \varepsilon + \eta_\varepsilon \omega'_\varepsilon, k^q(\varepsilon_i, k_j; z, \mathbf{m}); z', \mathbf{m}'(z, \mathbf{m}))\right)$ for $q = a, n$.
3. Hence:

$$\mathbb{E}_{\omega'_\varepsilon}\left(\hat{\vartheta}(\rho_\varepsilon \varepsilon + \eta_\varepsilon \omega'_\varepsilon, k^q(\varepsilon_i, k_j; z, \mathbf{m}); z', \mathbf{m}'(z, \mathbf{m}))\right) = \ldots$$
$$\ldots \sum_{o=1}^{m_\varepsilon} \tau_o^\varepsilon \sum_{p'=1}^{n_\varepsilon} \sum_{r'=1}^{n_k} \vartheta'_{p'r'} T_{p'}(\rho_\varepsilon \varepsilon_i + \eta_\varepsilon \omega_o^\varepsilon) T_{r'}(k^q(\varepsilon_i, k_j)) \tag{24}$$

where $\vartheta'_{p'r'} = \vartheta_{p'r'}(z', \mathbf{m}'(z, \mathbf{m}))$.

1. Thus, equation (23) can be written as follows:

$$\mathbb{E}\left[\sum_{p=1}^{n_\varepsilon}\sum_{r=1}^{n_k}\vartheta_{pr}T_p(\varepsilon_i)T_r(k_j)-\lambda(e^z e^{\varepsilon_i}k_j^\theta l(\varepsilon_i,k_j)^\nu - w\,l(\varepsilon_i,k_j)+(1-\delta)k_j)\ldots\right.$$

$$+\frac{\hat{\xi}(\varepsilon_i,k_j)}{\underline{\xi}}(\lambda(k^a(\varepsilon_i,k_j)-w\frac{\hat{\xi}(\varepsilon_i,k_j)}{2}))+\left(1-\frac{\hat{\xi}(\varepsilon_i,k_j)}{\underline{\xi}}\right)\lambda k^n(\varepsilon_i,k_j)\ldots$$

$$-\beta\frac{\hat{\xi}(\varepsilon_i,k_j)}{\underline{\xi}}\sum_{o=1}^{m_\varepsilon}\tau_o^\varepsilon\sum_{p'=1}^{n_\varepsilon}\sum_{r'=1}^{n_k}\vartheta'_{p'r'}T_{p'}(\rho_\varepsilon\varepsilon_i+\eta_\varepsilon\omega_o^\varepsilon)T_{r'}(k^a(\varepsilon_i,k_j))\ldots$$

$$\left.-\beta\left(1-\frac{\hat{\xi}(\varepsilon_i,k_j)}{\underline{\xi}}\right)\sum_{o=1}^{m_\varepsilon}\tau_o^\varepsilon\sum_{p'=1}^{n_\varepsilon}\sum_{r'=1}^{n_k}\vartheta'_{p'r'}T_{p'}(\rho_\varepsilon\varepsilon_i+\eta_\varepsilon\omega_o^\varepsilon)T_{r'}(k^n(\varepsilon_i,k_j))\right]=0 \tag{25}$$

for the collocation nodes $i=1,\ldots,n_\varepsilon$ and $j=1,\ldots,n_k$.

1. The optimal labor demand is given by the FOC:

$$\nu e^z e^\varepsilon k^\theta l^{\nu-1} - w(\mathbf{s}) = 0$$

Its numerical version, for the grid $\{\varepsilon_i, k_j\}$, is:

$$l(\varepsilon_i, k_j) = \left(\frac{\nu e^z e^{\varepsilon_i} k_j^\theta}{w}\right)^{\frac{1}{1-\nu}} \tag{26}$$

1. The optimal capital decision in case the firm assumes the fix cost is given by the FOC:

$$-\lambda(\mathbf{s}) + \beta \frac{\partial}{\partial k'} \mathbb{E}[\hat{v}(\varepsilon^{'}, k^{'}; \mathbf{s}^{'}(z^{'}, \mathbf{s}))|\varepsilon, k; \mathbf{s}] = 0$$

Its numerical version, for the grid $\{\varepsilon_i, k_j\}$, is:

$$\mathbb{E}\left[\lambda - \beta \sum_{o=1}^{m_\varepsilon} \tau_o^\varepsilon \sum_{p'=1}^{n_\varepsilon} \sum_{r'=1}^{n_k} \vartheta^{'}_{p' r'} T_{p'}(\rho_\varepsilon \varepsilon_i + \eta_\varepsilon \omega_o^\varepsilon) T'_{r'}(k^a(\varepsilon_i, k_j))\right] = 0 \tag{27}$$

where $T'_{r'}(k^a(\varepsilon_i, k_j)) = \frac{\partial}{\partial k^a} T_{r'}(k^a(\varepsilon_i, k_j))$

1. The optimal capital decision in case the firm does not assume the fix cost is:

$$k^n(\varepsilon_i, k_j) = \begin{cases} (1-\delta+a)k_j, & \text{if } k^a(\varepsilon_i, k_j) > (1-\delta+a)k_j \\ k^a(\varepsilon_i, k_j), & \text{if } k^a(\varepsilon_i, k_j) \in [(1-\delta-a)k_j, (1-\delta+a)k_j] \\ (1-\delta-a)k_j, & \text{if } k^a(\varepsilon_i, k_j) < (1-\delta-a)k_j \end{cases} \tag{28}$$

1. The numerical version of optimal threshold $\tilde{\xi}(\varepsilon_i, k_j; z, \mathbf{m})$ is:

$$\tilde{\xi}(\varepsilon_i, k_j) = \frac{1}{w\lambda}\Big[-\lambda(k^a(\varepsilon_i, k_j) - k^n(\varepsilon_i, k_j))...$$

$$+\beta \sum_{o=1}^{m_\varepsilon} \tau_o^\varepsilon \sum_{p'=1}^{n_\varepsilon} \sum_{r'=1}^{n_k} \vartheta'_{p'r'} T_{p'}(\rho_\varepsilon \varepsilon_i + \eta_\varepsilon \omega_o^\varepsilon)\left(T_{r'}(k^a(\varepsilon_i, k_j)) - T_{r'}(k^n(\varepsilon_i, k_j))\right)\Big] \tag{29}$$

1. The numerical version of optimal threshold $\tilde{\xi}(\varepsilon_i, k_j; z, \mathbf{m})$ is:

$$\tilde{\xi}(\varepsilon_i, k_j) = \frac{1}{\omega\lambda}\Bigg[-\lambda(k^a(\varepsilon_i, k_j) - k^n(\varepsilon_i, k_j))\ldots$$
$$+\beta \sum_{o=1}^{m_\varepsilon} \tau_o^\varepsilon \sum_{p'=1}^{n_\varepsilon} \sum_{r'=1}^{n_k} \vartheta'_{p'r'} T_{p'}(\rho_\varepsilon \varepsilon_i + \eta_\varepsilon \omega_o^\varepsilon)\left(T_{r'}(k^a(\varepsilon_i, k_j)) - T_{r'}(k^n(\varepsilon_i, k_j))\right)\Bigg] \tag{29}$$

2. And, the numerical version of the bounded threshold is defined:

$$\hat{\xi}(\varepsilon_i, k_j) = \min\{\max\{0, \tilde{\xi}(\varepsilon_i, k_j)\}, \underline{\xi}\} \tag{30}$$

1. Now, let $\{\tau_i^g, \langle \varepsilon_i, k_i \rangle\}_{i=1}^{m_g}$ denote the weights and nodes of the two-dimensional Gauss-Legendre quadrature used to approximate the integral with respect to the distribution.

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2. Then, household optimization condition may be written as follows:

$$\lambda - \left(\sum_{h=1}^{m_g} \tau_h^g \left[e^z e^{\varepsilon_h} k_h^\theta l(\varepsilon_h, k_h) + (1-\delta) k_h \ \ldots \right.\right.$$

$$\left.\left. - \frac{\hat{\xi}(\varepsilon_h, k_h)}{\underline{\xi}} k^a(\varepsilon_h, k_h) - \left(1 - \frac{\hat{\xi}(\varepsilon_h, k_h)}{\underline{\xi}}\right) k^n(\varepsilon_h, k_h) \right] g(\varepsilon_h, k_h | \mathbf{m}) \right)^{-\sigma} = 0 \tag{31}$$

1. And, the numerical version of optimal labor supply choice is:

$$\left(\frac{w\lambda}{\chi}\right)^{\frac{1}{\psi}} - \sum_{h=1}^{m_g} \tau_h^g \left[l(\varepsilon_h, k_h) + \frac{\hat{\xi}(\varepsilon_h, k_h)^2}{2\underline{\xi}} \right] g(\varepsilon_h, k_h|\mathbf{m}) = 0 \quad (32)$$

with

$$g(\varepsilon_i, k_j|\mathbf{m}) = g_0 \exp\{g_1^1(\varepsilon - m_1^1) + g_1^2(\log(k) - m_1^2)...$$
$$... + \sum_{i_\varepsilon=2}^{n_g} \sum_{j_k=0}^{i_\varepsilon} g_{i_\varepsilon}^{j_k} \left[(\varepsilon_i - m_1^1)^{i_\varepsilon - j_k} (\log(k_j) - m_1^2)^{j_k} - m_{i_\varepsilon}^{j_k} \right] \}$$

1. The approximated law of motion for the distribution, based on equation (19), will be:

$$
\begin{aligned}
0 &= m_1^{1\prime} - \sum_{h=1}^{m_g} \tau_h^g \sum_{o=1}^{m_\varepsilon} \tau_o^\varepsilon (\rho_\varepsilon \varepsilon_h + \eta_\varepsilon \omega_o^\varepsilon) g(\varepsilon_h, k_h|\mathbf{m}) \\
0 &= m_1^{2\prime} - \sum_{h=1}^{m_g} \tau_h^g \sum_{o=1}^{m_\varepsilon} \tau_o^\varepsilon \left[\frac{\hat{\xi}(\varepsilon_h, k_h)}{\underline{\xi}} \log(k^a(\varepsilon_h, k_h)) + \left(1 - \frac{\hat{\xi}(\varepsilon_h, k_h)}{\underline{\xi}}\right) \log(k^n(\varepsilon_h, k_h)) \right] g(\varepsilon_h, k_h|\mathbf{m}) \qquad (33) \\
0 &= m_i^{j\prime} - \sum_{h=1}^{m_g} \tau_h^g \sum_{o=1}^{m_\varepsilon} \tau_o^\varepsilon \left[(\rho_\varepsilon \varepsilon_h + \omega_o^\varepsilon - m_1^{1\prime})^{i-j} \tilde{\boldsymbol{\kappa}}_j \right] g(\varepsilon_h, k_h|\mathbf{m})
\end{aligned}
$$

where

$$
\tilde{\boldsymbol{\kappa}}_j = \frac{\hat{\xi}(\varepsilon_h, k_h)}{\underline{\xi}} \left[\log(k^a(\varepsilon_h, k_h)) - m_1^{2\prime} \right]^j + \left(1 - \frac{\hat{\xi}(\varepsilon_h, k_h)}{\underline{\xi}}\right) \left[\log(k^n(\varepsilon_h, k_h)) - m_1^{2\prime} \right]^j
$$

1. And, the approximated moment consistency system, based on equation (18), will be:

$$
\begin{aligned}
0 &= m_1^1 - \sum_{h=1}^{m_g} \tau_h^g \varepsilon_h g(\varepsilon_h, k_h|\mathbf{m}) \\
0 &= m_1^2 - \sum_{h=1}^{m_g} \tau_h^g \log(k_h) g(\varepsilon_h, k_h|\mathbf{m}) \\
0 &= m_i^j - \sum_{h=1}^{m_g} \tau_h^g \left[(\varepsilon_h - m_1^1)^{i-j} \left(\log(k_h) - m_1^2\right)^j \right] g(\varepsilon_h, k_h|\mathbf{m}) \\
& \text{for } i = 2, ..., n_g; \; j = 0, ..., i
\end{aligned} \tag{34}
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0 &= m_i^j - \sum_{h=1}^{m_g} \tau_h^g \left[(\varepsilon_h - m_1^1)^{i-j} \left(\log(k_h) - m_1^2\right)^j \right] g(\varepsilon_h, k_h|\mathbf{m}) \\
& \text{for } i = 2, ..., n_g;\ j = 0, ..., i
\end{aligned}
\tag{34}
$$

2. Finally, the law of motion for the aggregate productivity shock is:

$$
\mathbb{E}[z' - \rho_z z] = 0 \tag{35}
$$

1. Therefore, it is straightforward to see that equations (23), (27), (31), (32), (33), (34) and (35) defines a system of $2n_\varepsilon n_k + n_g + 2 + n_g + 1$ equations.

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2. All these equation can be defined as a mapping $f(\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x}; \eta)$ such that:

$$\mathbb{E}[f(\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x}; \eta)] = 0 \tag{36}$$

where $\mathbf{y} = \left\{\vartheta, \mathbf{k}^a, \mathbf{g}, \lambda, w\right\}$ is the control variables vector and $\mathbf{x} = \left\{z, \mathbf{m}\right\}$, the state variables vector. η represents the perturbation parameter, and $\mathbf{k}^a$ denotes the target capital stock along the collocation grid.

Computation of Stationary Equilibrium

1. It can be noted that equation (36) has the canonical form studied by *Schmitt-Grohe and Uribe (2004)*.

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3. In principle, the above non-linear system can be solved using some root-finding algorithm. However, in practice, such system is quite large that numerical solvers fail to converge usually.
4. Winberry's method for solving the stationary equilibrium is similar to the developed by *Hopenhayn and Rogerson (1993)*.

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   1. Guess a value for equilibrium wage, w^0.

Computation of Stationary Equilibrium

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   4. Finally, aggregate individual firms' labor demand:

$$L_D = \int \left(l(\varepsilon, k) + \frac{\hat{\xi}(\varepsilon, k)^2}{2\underline{\xi}} \right) g(\varepsilon, k) d\varepsilon dk$$

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$$L_D = \int \left(l(\varepsilon, k) + \frac{\hat{\xi}(\varepsilon, k)^2}{2\underline{\xi}} \right) g(\varepsilon, k) d\varepsilon dk$$

2. Labor supply can be computed from: $L_S = \left(\frac{w^0 \lambda^0}{\chi} \right)^{1/\psi}$ where λ^0 may be computed using $C^0 = Y^0 - I^0$

Local accurate approximation around the stationary equilibrium

1. Once steady state vector, $(\mathbf{y}^{\star}, \mathbf{x}^{\star})$, is computed, it may be applied a Taylor expansion around such value in order to compute the aggregate Dynamics.

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$$\mathbf{y} = \mathscr{g}(\mathbf{x}; \eta)$$
$$\mathbf{x}' = \mathscr{h}(\mathbf{x}; \eta) + \eta \times \phi \omega_z'$$

where $\phi = (1,\ \mathbf{0}_{n_g \times 1})'$.

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$$\mathbf{y} = g(\mathbf{x}; \eta)$$
$$\mathbf{x}' = h(\mathbf{x}; \eta) + \eta \times \phi \omega_z'$$

where $\phi = (1,\ \mathbf{0}_{n_g \times 1})'$.

3. Then, a first order Taylor approximation around steady-stat yields:

$$\begin{aligned} g(\mathbf{x}; 1) &\approx g_{\mathbf{x}}(\mathbf{x}^\star; 0)(\mathbf{x} - \mathbf{x}^\star) + g_\eta(\mathbf{x}^\star; 0) \\ h(\mathbf{x}; 1) &\approx h_{\mathbf{x}}(\mathbf{x}^\star; 0)(\mathbf{x} - \mathbf{x}^\star) + h_\eta(\mathbf{x}^\star; 0)(1 - 0) + \phi \omega_z' \end{aligned} \tag{37}$$

Results KT model

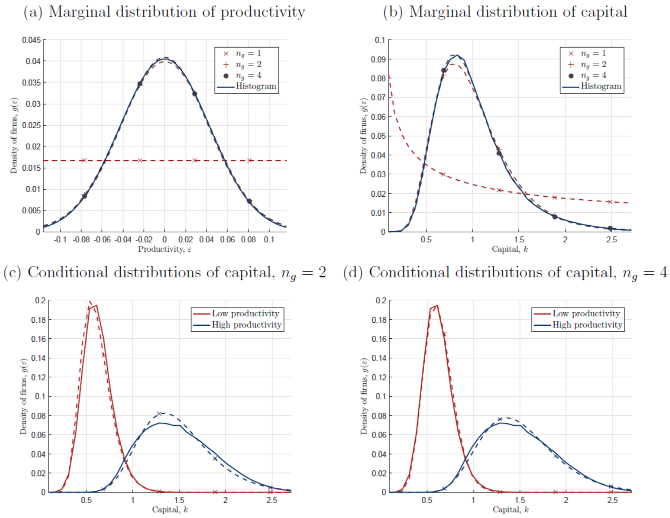

Figure 1: Stationary Distribution of Firms (Winberry (2018))

Results KT model

Table 1: Aggregate Variable in Steady State (Winberry (2018))

Variable	$n_g = 1$	$n_g = 2$	$n_g = 3$	$n_g = 4$	$n_g = 6$	**Histogram**
Output	0.509	0.498	0.499	0.499	0.499	0.498
Consumption	0.555	0.413	0.413	0.413	0.413	0.413
Capital	1.200	1.007	1.007	1.008	1.007	1.007
Wage	0.985	0.958	0.958	0.958	0.958	0.958
Marg. Utility	1.803	2.424	2.422	2.422	2.422	2.422

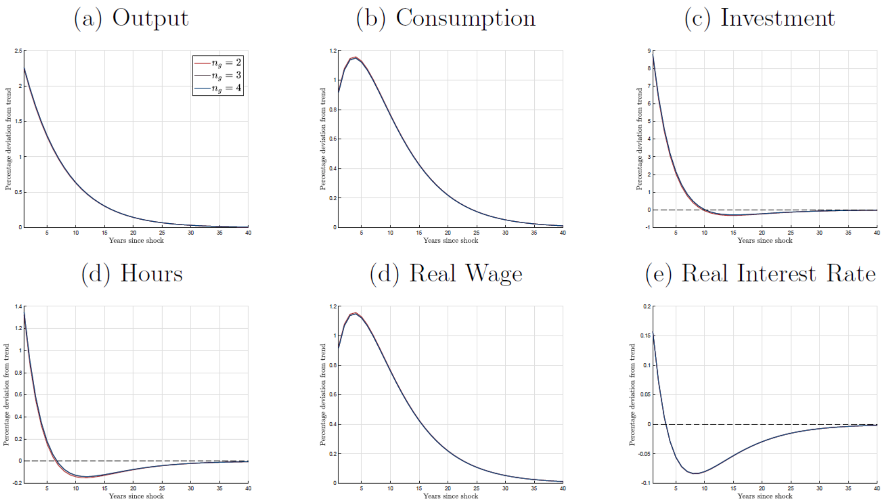

Figure 2: Impulse Responses of Aggregates: First Order Perturbation (Winberry (2018))

Continuum of households $i \in [0,1]$, with preferences over consumption c_{it}:

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_{it}^{1-\sigma} - 1}{1-\sigma}\right] \tag{38}$$

1. Each household supplies ε_{it} efficiency units of labor inelastically, and this follows a 2-state Markov process $\varepsilon_{it} \in \{0,1\}$ with transition probabilities $\pi(\varepsilon'|\varepsilon)$.
2. When $\varepsilon_{it} = 1$, households receive $w_t(1-\tau)$; when $\varepsilon_{it} = 0$, households receive unemployment benefit bw_t (financed with tax labor).
3. $a_{it+1} \geq \underline{a}$ (borrowing constraint).

Representative firm, produces output Y_t according to:

$$Y_t = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} \tag{39}$$

with z_t aggregate prod. shock.

Aggregate TFP evolces according to:

$$z_{t+1} = \rho_z z_t + \sigma_z \omega_{t+1} \tag{40}$$

The aggregate state for this economy $\boldsymbol{s} = (z, g)$. The RCE is a list of functions $a'(\varepsilon, a; \boldsymbol{s})$, $r(\boldsymbol{s})$, $w(\boldsymbol{s})$, $g'(\boldsymbol{s})$ such that:

1. **HH:** taking $r(\boldsymbol{s})$, $w(\boldsymbol{s})$ and $g'(\boldsymbol{s})$ as given, $a'(.)$ satisfies:

$$c(\varepsilon, a; \boldsymbol{s})^{-\sigma} \geq \beta \mathbb{E}[(1 + r(\boldsymbol{s}'))c(\varepsilon', a'; \boldsymbol{s}')^{-\sigma} | \varepsilon, z, g]$$

   with equality if $a'(\varepsilon, a; \boldsymbol{s}) > \underline{a}$.

2. **Firms and Market Clearing:**

$$r(\boldsymbol{s}) = \alpha e^z K^{\alpha-1} L^{1-\alpha} - \delta$$
$$w(\boldsymbol{s}) = (1-\alpha) e^z K^{\alpha} L^{-\alpha}$$
$$K = \sum_{\varepsilon} \int a dg(\varepsilon, a)$$

3. **Evolution of distribution**

$$g'(\varepsilon, \mathscr{A}_a) = \sum_{\tilde{\varepsilon}} \pi(\varepsilon | \tilde{\varepsilon}) \int \mathbb{1}\{a'(\tilde{\varepsilon}, a; \boldsymbol{s}) \in \mathscr{A}_a\} g(\tilde{\varepsilon}, da)$$

Mass at Constraint:

$$\hat{m}_{\varepsilon t+1} = \frac{1}{\pi(\varepsilon)} \left[\sum_{\tilde{\varepsilon}} (1 - \hat{m}_{\tilde{\varepsilon} t}) \pi(\varepsilon|\tilde{\varepsilon}) \pi(\tilde{\varepsilon}) \int \mathbb{1}\{a'_t(\tilde{\varepsilon}, a) = \underline{a}\} g_{\tilde{\varepsilon} t}(a) da \ldots \right.$$

$$\left. \ldots + \sum_{\tilde{\varepsilon}} \hat{m}_{\tilde{\varepsilon} t} \pi(\varepsilon|\tilde{\varepsilon}) \pi(\tilde{\varepsilon}) \mathbb{1}\{a'_t(\tilde{\varepsilon}, \underline{a}) = \underline{a}\} \right]$$

Distribution away from Constraint:

$$g_{\varepsilon t}(a) \approx g^0_{\varepsilon t} \exp \left\{ g^1_{\varepsilon t}(a - m^1_{\varepsilon t}) + \sum_{i=2}^{n_g} g^i_{\varepsilon t} \left[(a - m^1_{\varepsilon t})^i - m^i_{\varepsilon t} \right] \right\}$$

Relation between $\mathbf{g}_{\varepsilon t}$ and $\mathbf{m}_{\varepsilon t}$:

$$m^1_{\varepsilon t} = \int a g_{\varepsilon t}(a) da$$

$$m^i_{\varepsilon t} = \int (a - m^1_{\varepsilon t})^i g_{\varepsilon t}(a) da, \quad i = 2, ..., n_g$$

Evolution of moments:

$$m^1_{\varepsilon t+1} = \frac{1}{\pi(\varepsilon)} \left[\sum_{\tilde{\varepsilon}} (1 - \hat{m}_{\tilde{\varepsilon} t}) \pi(\varepsilon|\tilde{\varepsilon}) \pi(\tilde{\varepsilon}) \int a'_t(\tilde{\varepsilon}, a) g_{\tilde{\varepsilon} t}(a) da + \sum_{\tilde{\varepsilon}} \hat{m}_{\tilde{\varepsilon} t} \pi(\varepsilon|\tilde{\varepsilon}) \pi(\tilde{\varepsilon}) a'_t(\tilde{\varepsilon}, \underline{a}) \right]$$

$$m^i_{\varepsilon t+1} = \frac{1}{\pi(\varepsilon)} \left[\sum_{\tilde{\varepsilon}} (1 - \hat{m}_{\tilde{\varepsilon} t}) \pi(\varepsilon|\tilde{\varepsilon}) \pi(\tilde{\varepsilon}) \int (a'_t(\tilde{\varepsilon}, a) - m^1_{\varepsilon t+1})^i g_{\tilde{\varepsilon} t}(a) da + \sum_{\tilde{\varepsilon}} \hat{m}_{\tilde{\varepsilon} t} \pi(\varepsilon|\tilde{\varepsilon}) \pi(\tilde{\varepsilon}) (a'_t(\tilde{\varepsilon}, \underline{a}) - m^1_{\varepsilon t+1})^i \right]$$

Integrals are solved using Gauss-Legendre quadrature, with nodes $\{a_j\}_{j=1}^{m_g}$ and weights $\{\tau_j\}_{j=1}^{m_g}$. So:

$$\int a'(\tilde{\varepsilon}, a) g_{\tilde{\varepsilon}t}(a) da = \sum_{j=1}^{m_g} \tau_j a'(\tilde{\varepsilon}, a_j) g_{\tilde{\varepsilon}t}(a_j)$$

$$\int (a'(\tilde{\varepsilon}, a) - m^1_{\varepsilon t+1})^i g_{\tilde{\varepsilon}t}(a) da = \sum_{j=1}^{m_g} \tau_j [a'(\tilde{\varepsilon}, a_j) - m^1_{\varepsilon t+1}]^i g_{\tilde{\varepsilon}t}(a_j)$$

With the same nodes and weights, **aggregate capital** is computed:

$$K_t = \sum_{\varepsilon} \pi(\varepsilon) \sum_{j=1}^{m_g} \tau_j a_j g_{\varepsilon t}(a_j)$$

Calling $\psi_t(\varepsilon, a) = \mathbb{E}_t[\beta(1 + r_{t+1})(c_{t+1}(\varepsilon', a'_t(\varepsilon, a)))^{-\sigma}]$, then:

$$a'_t(\varepsilon, a) = \max\{\underline{a}, w_t((1-\tau)\varepsilon + b(1-\varepsilon)) + (1+r_t)a - \psi_t(\varepsilon, a)^{-1/\sigma}\}$$

$$c_t(\varepsilon, a) = w_t((1-\tau)\varepsilon + b(1-\varepsilon)) + (1+r_t)a - a'_t(\varepsilon, a)$$

$\psi_t(\varepsilon, a)$ is approximated by:

$$\hat{\psi}_t \approx \exp\left\{\sum_{i=1}^{n_\psi} \theta_{\varepsilon i t} T_i(\xi(a))\right\}$$

thus:

$$\hat{a}'_t(\varepsilon, a) = \max\{\underline{a}, w_t((1-\tau)\varepsilon + b(1-\varepsilon)) + (1+r_t)a - \hat{\psi}_t(\varepsilon, a)^{-1/\sigma}\}$$

$$\hat{c}_t(\varepsilon, a) = w_t((1-\tau)\varepsilon + b(1-\varepsilon)) + (1+r_t)a - \hat{a}'_t(\varepsilon, a)$$

when borrowing constraint is not binding:

$$\exp\left\{\sum_{i=1}^{n_\psi} \theta_{\varepsilon i t} T_i(\xi(a))\right\} = \mathbb{E}_t\left[\beta(1+r_{t+1})\sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon)\hat{c}_t(\varepsilon', \hat{a}'_t(\varepsilon, a_j))^{-\sigma}\right]$$

The approximated equilibrium can be written in the form:

$$\mathbb{E}[f(\mathbf{y}, \mathbf{y}', \mathbf{x}, \mathbf{x}'; \eta)] = 0$$

Then, the steady-state can be found by:

$$f(\mathbf{y}^\star, \mathbf{y}^\star, \mathbf{x}^\star, \mathbf{x}^\star; \eta) = 0$$

Algorithm:

1. Compute factor prices $r(K)$ and $w(K)$
2. Solve for the parameters of conditional expectation $\boldsymbol{\theta}$.
3. Using implied decision rules, solve for invariant $\boldsymbol{m}$ and $\boldsymbol{g}$.
4. Update aggregate capital $K' = \sum_\varepsilon \pi(\varepsilon) \sum_{j=1}^{m_g} \tau_j a'(\varepsilon, a_j) g_\varepsilon(a_j)$.
5. Solve for a zero of $K' - K$.

Results KT model

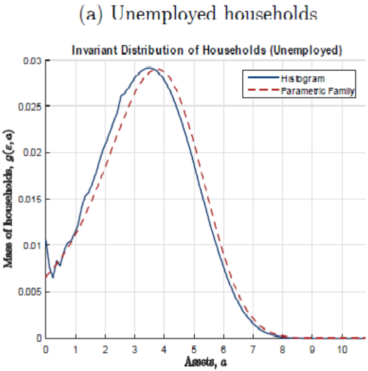

Figure 3: Stationary Distribution of Households (Winberry (2016))

Results KT model

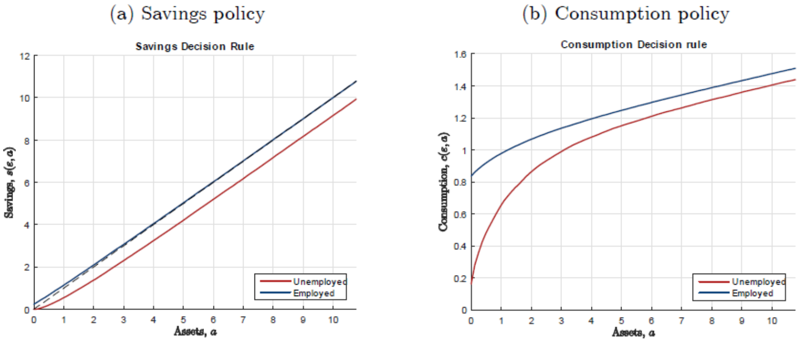

Figure 4: Decision Rules in Steady State (Winberry (2016))

Results KT model

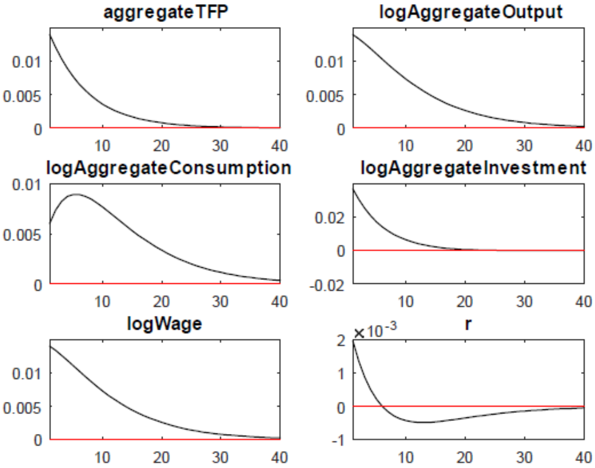

Figure 5: Impulse Response to Aggregate TFP shock (Winberry (2016))