Macro II (UC3M, M.A.E.)
Professor: Matthias Kredler
Problem Set 7
Due: 25 March 2020, 23:59h, by Aula Global
You are encouraged to work in groups; however, every student has to hand in his/her own version of the solution.

1. **Setting up budget constraints in continuous time.** Consider a saver in a continuous-time environment who can save at constant interest rate r, receives a time-varying, deterministic wage w_t, and chooses a consumption rate c_t.

   (a) Set up the budget constraint over a short time interval $[t, t + \Delta t]$ given assets a_t. As opposed to what we did in class, assume that consumption spending and wage income are taken out of the asset stock at the end of the time interval (i.e. at $t + \Delta t$) and not in the beginning (at t).

   (b) From this, derive the flow budget constraint in differential form, i.e. find an expression for $\dot{a}_t = \mathrm{d}a/\mathrm{d}t$.

   (c) Explain why this is the same flow budget constraint as the one derived in class.

2. **Growth model in continuous time: HJB and EE.** Consider the neoclassical growth model in continuous time. Capital k_t produces $f(k_t)\Delta t$ units of consumption goods over a short interval $[t, t + \Delta t]$, and $\delta k_t \Delta t$ units of capital depreciate over this period. The planner chooses a consumption function $c_t : [0, T] \to \mathbb{R}^+$ to maximize

$$\int_0^T e^{-\rho t} u(c_t)\mathrm{d}t + e^{-\rho T} H(k_T),$$

   where $\rho > 0$ and $u(\cdot), f(\cdot)$ are functions with the usual properties. The terminal payoff $H(\cdot)$ is such that $H' \geq 0$. The initial capital stock $k_0 > 0$ is given.

   (a) Show that the law of motion for capital is given by

$$\dot{k}_t = f(k_t) - \delta k_t - c_t.$$

   Which variables are stocks and which are flows?

   (b) Write an equation analogous to the familiar $Y = C + I + G$ (spending side of GDP) for this economy. What is investment in this economy?

   (c) Set up Bellman's principle for the value function V.

   (d) From Bellman's principle, show that the Hamilton-Jacobi-Bellman equation (HJB) for the planner is

$$-V_t(k,t) + \rho V(k,t) = \max_{c \geq 0} \Big\{ u(c) + \big[f(k) - \delta k - c\big] V_k(k,t) \Big\},$$

   where subscripts of V denote partial derivatives.

   (e) Find the first-order condition (FOC) for consumption and interpret it briefly.

   (f) Derive the Euler equation and write it in terms of marginal utility $u'(c)$. Interpret briefly.

3. **Continuous-time firm investment in quality**. Consider a firm in continuous time with an infinite horizon which maximizes discounted profits (using the market interest rate $r > 0$).The firm sells one product, whose quality we denote by $q_t \in \mathbb{R}$. The firm faces a time-invariant demand function $y(q_t, p_t)$, which is increasing in q_t and decreasing in the price p_t. $y(q_t, p_t)$ denotes the number of units sold at t when the price is p_t and the quality is q_t. Each unit of the good is produced at cost $c > 0$. The firm can set the price p_t freely each t. As for quality q_t, when the firm makes an investment $x_t \Delta t$ over a short interval $[t, t + \Delta t)$, then the quality of its good increases by $g(x_t)\Delta t$ over that interval. We assume that g is an increasing concave function that satisfies $\lim_{x\to 0} g'(x) = \infty$ and $\lim_{x\to\infty} g'(x) = 0$.

   (a) Derive a differential equation for the law of motion for quality q_t.
   (b) State Bellman's principle for the firm.
   (c) Derive the Hamilton-Jacobi-Bellman equation from Bellman's principle.
   (d) Derive the first-order condition for x and interpret it briefly.

4. **Discrete career ladder.** Time t is infinite and continuous. Consider a worker who at each t can be in one out of finitely many positions, $p_t \in \{1, 2, \ldots, n\}$. The worker starts with $p_0 = 1$. The worker gets paid wage $w_t = \omega p_t$, where $\omega > 0$ is a parameter. At each t, the worker decides how much effort $h_t \geq 0$ to exert in order to get promoted. Given effort $h \geq 0$, a promotion happens with the following probability over a short time interval Δt:
$$\mathbb{P}(p_{t+\Delta t} = p_t + 1 | p_t) = h\Delta t.$$
With probability $(1 - h\Delta t)$, the worker stays in the current position, p_t. The worker can never be demoted. The worker maximizes expected discounted wages and faces a flow cost $\frac{1}{2}\gamma h^2$ for promotion effort, where $\gamma > 0$ is a parameter. To be precise, the worker maximizes
$$\mathbb{E}_0 \int_0^\infty e^{-rt} \left[\omega p_t - \tfrac{1}{2}\gamma h_t^2\right] \mathrm{d}t.$$

   (a) Find the value of having reached career position n.
   (b) Write Bellman's principle for the worker's value function when in a position $p < n$.
   (c) Derive the Hamilton-Jacobi-Bellman equation equation (for $p < n$).
   (d) Using the first-order condition, find the optimal effort choice and interpret briefly.
   (e) Describe how you would solve for the value functions and effort choices (no explicit solution required, only describe an algorithm).