Macro II (UC3M, M.A.E. Econ)
Professor: Matthias Kredler
Problem Set 6
Due: 6 March 2020
You are encouraged to work in groups; however, every student has to hand in his/her own version of the solution.

1. **American Option.** Consider an agent who owns an option that allows her to buy an asset at a fixed price $\bar{p} > 0$ (the *strike*) in periods $t = 0, 1, \ldots, T$. The agent is not genuinely interested in the asset, only in making a profit on re-selling it; there is no discounting (i.e. $\beta = 1$), so the payoff is $p_t - \bar{p}$ if exercising the option at t (when the price takes value p_t) and 0 if the option is never exercised. We assume that the price p_t is an i.i.d. process drawn from a cumulative distribution function (cdf) $F(p)$ with support $[0, \infty)$.

   (a) Find the value function $V_T(\cdot)$ at T and characterize the optimal policy $g_T(\cdot)$.

   (b) Find the value function $V_{T-1}(\cdot)$ at $T-1$, taking as given $V_T(\cdot)$ and write it as much as you can in terms of primitives, i.e. $\bar{p}$ and $F(p)$. Characterize the optimal policy $g_{T+1}(\cdot)$.

   (c) Write the Bellman equation for general $0 \leq t < T$. Show that $V_t(p) \geq V_{t+1}(p)$ for all t and for all fixed p and characterize in which way the optimal policy $g_t(\cdot)$ changes as we go back in time.

2. **Exploiting homogeneity to reduce the state.** Consider an entrepreneur with revenue $A_t^{1-\alpha} n_t^{\alpha}$, where A_t is the entrepreneur's ability and n_t is the number of workers he hires. Ability follows the process

$$A_{t+1} = A_t(1 + \epsilon_{t+1}), \qquad \text{where } \epsilon_t \sim \text{i.i.d. with } \mathbb{E}[\epsilon] = \bar{\epsilon}.$$

   The entrepreneur's only costs are to hire workers at wage w_t every period, where w_t follows a first-order Markov process that is independent of the process for A_t. The entrepreneur discounts future profits with factor $\beta \in (0, 1)$. We further assume that $\beta(1 + \bar{\epsilon}) < 1$.

   (a) Show that the entrepreneur's profit function $\pi(A, w)$ is linear in ability A.

   (b) Write down the Bellman equation for the entrepreneur's value function $V(A, w)$.

   (c) Now, guess that the value function is of the form $V(A, w) = AU(w)$ for some function $U(\cdot)$. Derive the functional equation that U has to fulfill. Why is $AU(w)$ a solution to the Bellman equation for V?

3. **Investment in a risky asset.** Time is discrete and infinite. Consider a household that starts out with wealth $a_0 > 0$ at $t = 0$. In each period, the

household splits its wealth into consumption and savings. All savings go into a single asset, which is risky. It pays a gross return $R_t \in \{\bar{R}_1, \dots, \bar{R}_n\}$ each period, where $\bar{R}_i > 0$ for all i. There is no other source of income for the household than savings. Denote the probability of observing a history $R^t = \{R_0, \dots, R_t\}$ by $\pi_t(R^t)$ and assume that $\pi_t(R^t) > 0$ for all $R^t \in \mathcal{R}_t$ (where $\mathcal{R}_t$ is the set of all possible histories at t) and for all $t = 0, 1, \dots$. No borrowing is allowed. The household has logarithmic utility from consumption and maximizes expected utility, where future periods are discounted at factor $\beta \in (0, 1)$.

(a) Sketch an event tree. Write down the household's budget constraint at a typical node of the tree, stating exactly the dependency of the different variables on histories (the *measurability* requirements).

(b) Write down the household's time-0 problem in sequential form.

(c) Show that the no-borrowing constraint will never bind (i.e. optimal savings are always positive)

(d) State the Lagrangean and derive the household's Euler Equation.

(e) Briefly interpret the Euler Equation.

4. **Recursive competitive equilibrium with taxation.** Consider an infinite-horizon economy with a continuum of households indexed by $i \in [0, 1]$. Household i's budget constraint is

$$c_t^{(i)} + k_{t+1}^{(i)} = (\leq) A(1 - \tau_t) k_t^{(i)},$$

i.e. each household has a linear technology in the capital it owns, $k_t^{(i)}$. Output from this technology, $Ak_t^{(i)}$, is taxed at a rate τ_t. The household can use the net revenue from production for consumption, $c_t^{(i)}$, or investment, $k_{t+1}^{(i)}$. Capital depreciates fully after use. Government spending $G_t > 0$ follows a first-order Markov process. The government balances its budget in each period, i.e. it sets τ_t at each t to satisfy

$$\tau_t A K_t = G_t,$$

where K_t is the aggregate capital stock.[1] Households have rational expectations about future tax rates, and their preferences are given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln c_t^{(i)}.$$

(a) Social planner:

i. Write down the resource constraint at time t of a social planner who chooses aggregate variables $\{C_t, K_{t+1}\}_{t=0}^{\infty}$ and takes the stochastic sequence $\{G_t\}_{t=0}^{\infty}$ as given.

[1] We assume that the initial K_0 is large enough such that it is always feasible for the government to pay for G_t in equilibrium.

ii. Write down the Bellman equation for this social planner.

iii. From the Bellman equation, find the Euler equation for the planner's problem (assume that the usual regularity conditions hold).

(b) Recursive competitive equilibrium (RCE):[2]

i. Write down the household's Bellman equation.

ii. Define the RCE.

iii. Find the household's Euler equation in RCE.

(c) Compare the Euler equations from the planner's problem and the household's problem to discuss if capital accumulation is efficient in RCE, or if it is below or above the efficient level.

[2]Hint: The state variables for the recursive competitive equilibrium (i.e. the variables that determine prices) are the same as in the planner's problem above.