Macro II (M.A.E., UC3M)
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Problem Set 5
Due: 28 February 2020
You are encouraged to work in groups; however, every student has to hand in his/her own version of the solution.

1. **Bellman equation for consumption-savings problem.** Consider the standard consumption-savings problem: A consumer with initial assets $a_0 \geq 0$ receives a constant endowment stream $w > 0$ each period and can save at a fixed gross interest rate R, so the budget constraint is

$$c_t + \frac{a_{t+1}}{R} \leq a_t + w \qquad \text{for } t = 0, 1, \ldots$$

Also, there is a no-borrowing constraint: $a_{t+1} \geq 0$ for $t = 0, 1, \ldots$. Let us also assume that there is an exogenous upper bound on assets $\bar{a} > 0$. The consumer orders consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u(\cdot)$ is a continuous, strictly increasing, strictly concave function with the property that $u(0) = 0$. You may use theorems from class to answer the following questions.

   (a) State this problem as a dynamic-programming problem: Give the state, the control variable, the feasible-set correspondence, and the return function.

   (b) Write the Bellman equation.

   (c) Is there a unique function V satisfying the Bellman equation? Explain.

   (d) Does the solution $V(\cdot)$ too the Bellman equation give us the value function from the underlying sequence problem? Explain.

   (e) Show that the value function is increasing.

   (f) Show that the value function is concave and that the optimal policy function is continuous.

   (g) How would you construct the sequence $\{a_{t+1}\}_{t=0}^{\infty}$ for the optimal plan given a_0?

   (h) Derive the Euler equation from the Bellman equation.

2. **Coding the stochastic growth model.** Modify your program for the deterministic growth model from previous problem sets to accommodate stochastic productivity (i.e. again capital is chosen on a discrete grid). Let the production function be given by $y_t = A_t k_t^{\alpha}$, where total factor productivity A_t follows a two-state Markov chain: $A_t \in \{1-a, 1+a\} = \{0.95, 1.05\}$, and $\text{Prob}(A_{t+1} = A_t) = p = 0.8$.

(a) Base your code on the previous exercise on the deterministic growth model. Say exactly at which point(s) you had to modify your code.

(b) Plot the value function and comment briefly on it. Does it share the properties with the deterministic model?

(c) How do the results (value function and optimal policy) change in the parameter a and p? Comment.

(d) In which sense is there a "steady state" now? How could you find it (you do not have to carry this out in a program, just describe the method)?

3. **Firm exit.** Time is infinite. A firm is producing a good with technology

$$y_t = A_t k_t^\alpha,$$

where $\alpha \in (0, 1)$. The price of y equals one in all periods, and the firm can rent capital at a fixed rate r each period. The firm's productivity evolves according to

$$A_{t+1} = A_t(1 + z_{t+1}),$$

where $z_t \in \{z_1, \ldots, z_n\}$ is an i.i.d. shock process. The timing is as follows: After observing z_t, the company can decide if to stay in the market or not. If it leaves the market, the payoff is zero in the current and in all subsequent periods. If it stays, it has to pay a fixed cost $C > 0$ in order to operate in this period; it can then choose k_t and sell y_t. The firm maximizes expected profits, discounted at a factor β.

(a) What is the optimal choice for k_t once the firm has decided to stay in the market?

(b) Find an expression for $V_s(\cdot)$, the value of staying in the market is. What are the arguments of the function $V_s(\cdot)$?

(c) State the problem in dynamic-programming form: Give the state variable(s), control(s), the law of motion for the state, the feasible set and the return function.

(d) Write the Bellman equation for the firm.