Macro II (M.E.A., UC3M)
Professor: Matthias Kredler
Problem Set 4
Due: 21 Feb 2020
Recall that also the computation problem from Problem Set 2 is due on 21 Feb. You are encouraged to work in groups; however, every student has to hand in his/her own version of the solution.

1. **Functional equation for policy function.** We have seen that under mild conditions, there is a unique optimal policy function $g(k)$ in the deterministic neo-classical growth model.

   (a) Find the functional equation that g must fulfill (hint: stare at the Euler equation). What is the unknown in this equation?

   (b) Suppose you have a candidate policy function $g_n(k)$ and you want to find a next candidate $g_{n+1}(k)$ in the style of what we did for V_n in class. Write down how you would find g_{n+1}, and define an operator S such that $g_{n+1} = Sg_n$. (You do not have to show that continuity of g_n is preserved, nor that there always exists a unique optimal policy.) Which solution algorithm does this procedure suggest?

   Note: This algorithm indeed works well for growth and savings problems and is preferred to value-function iteration by most applied researchers.

   (c) Now, consider a general deterministic dynamic-programming environment with state $x_t \in \mathbb{R}$, return function $F(x_t, x_{t+1})$ and feasible set $x_{t+1} \in \Gamma(x_t)$. Assume that the problem is such that the optimal policy is always interior, i.e. there is a unique optimal choice $x^*_{t+1} = g(x^*_t)$ that lies in the interior of $\Gamma(x^*_t)$ for any x^*_t. Write down the functional equation that the policy function g has to fulfill, and find an operator S whose fixed point is given by g. (Again, you do not have to show that continuity is preserved, nor that there is necessarily a unique optimal policy, nor that there is a unique fixed point to S.)

2. **Bellman Equation of a firm with finite state space.** Consider a firm that operates in periods $t = 0, 1, \dots$ and discounts future profits at the gross interest rate $R > 1$. The firm's only choice in each period is how many products to offer, i.e. to choose a number $n \in \mathcal{N} \equiv \{0, 1, \dots N\}$, where $0 < N < \infty$ is given. The firm's Bellman equation is

$$V(n) = \max_{n' \in \mathcal{N}} \left\{ F(n, n') + R^{-1} V(n') \right\},$$

   where $F : \mathcal{N} \times \mathcal{N} \to [0, 1]$ is a given payoff function from having n products today and n' products tomorrow and $V : \mathcal{N} \to \mathbb{R}$ is the value function. Show that there is a unique value function V that solves the firm's Bellman equation.

3. **Putty-clay investment.** Consider an infinite-horizon economy in which output is produced according to $y_t = F(k_t)$, where $F(\cdot)$ satisfies the usual properties: $F' > 0$, $F'' < 0$, $F(0) = 0$, $\lim_{k\to\infty} F'(k) = 0$. The representative agent ranks sequences of consumption by

$$\sum_{t=0}^{\infty} \beta^t \sqrt{c_t}.$$

Investment is irreversible: One unit of y_t may be converted (i.e. *invested*) into one unit of k_{t+1}, but capital k_t cannot be converted back into consumption goods y_t again (so consumption c_t cannot exceed production y_t in any given period, or in other words gross investment cannot be negative). A fraction $\delta \in (0, 1)$ of the capital stock depreciates each period.

(a) Bring the planner's problem of this economy into dynamic-programming form: Say what the state, control(s), the feasible-set correspondence and the return function are.

(b) Write down the Bellman equation.

(c) Does the Bellman Equation have a unique solution? Why/why not?

(d) Assume that the planner chooses investment at $t+1$ above the lower bound, i.e. assume there is an interior solution at $t+1$. Under this assumption, derive the Euler equation for investment. Be careful to argue if/which corner solutions can occur for investment at t.[1]

4. **Investment with adjustment costs.** Consider a firm that produces a good y according to technology $y_t = k_t$ for $t = 0, 1, \ldots, T$, where k is capital. The firm rents capital on competitive markets each period at a constant rental rate $q > 0$. The firm faces a downward-sloping demand function: The price is determined by $p_t = p(y_t)$ in all periods, where $p(y) > 0$ and $p'(y) < 0$ for all $y \geq 0$. The firm discounts future profits at the market interest rate $R > 1$. It incurs an adjustment cost $\frac{c}{2}(k_{t+1} - k_t)^2$, $c > 0$, when changing the capital stock in all but the last period (i.e. for $t = 0, 1, \ldots, T-1$)[2]. The firm starts out with initial capital stock $k_0 = 0$.

(a) Write the firm's problem as a sequence problem.

(b) Bring this problem into dynamic-programming form: What is the state, the control(s), the feasible-set correspondence, and the return function?

[1] Note: Differentiability of the value function in infinite-horizon dynamic programming is not guaranteed when optimal choices occur at boundaries. See Rincón-Zapatero (your maths teacher!) & Santos in JET, 2009, for results that establish differentiability of the value function and thus generalize the Envelope Theorem in the case of non-interior choices; they discuss putty-clay investment as an application.

[2] This may be interpreted as going through costly changes in the production process, bureaucratic requirements for new plants etc.

(c) Write the value function for T and the Bellman equation for a generic $t = 0, \ldots, T-1$.

(d) Find the optimality condition for investment (i.e. the Euler equation) and interpret it in at most two sentences (you may assume that the optimal path $\{k_t\}_{t=0}^{T}$ is increasing for the interpretation).

(e) Find the steady-state capital stock for the firm, i.e. a number $\bar{k}$ from which the firm would not not want to deviate ever if the firm started with $k_0 = \bar{k}$. What property does this level $\bar{k}$ have when comparing it to a static problem?