Macro II (M.E.A., UC3M)
Professor: Matthias Kredler
Problem Set 3
Due: 14 Feb 2020
You are encouraged to work in groups; however, every student has to hand in his/her own version of the solution.

1. **Value-function iteration in Matlab. You may hand in solutions as a group for this exercise; the due date for this problem is 21 Feb, i.e. you have one wee more than for the other problems.** For this exercise, go back to the *Matlab* code you wrote in Problem Set 1.

   (a) Modify the code to iterate on the following operator T:

   $$V_{n+1}(k) = TV_n(k) = \max_{k' \in K} \{ln(k^\alpha + (1-\delta)k - k') + \beta V_n(k')\}$$

   where K is the set of grid points for capital in your code. Make the number of iterations n arbitrary. On paper, write down in matrix form how you operationalized T in the code. At how many points did you have to change the code with respect to the exercise in finite-time dynamic programming?

   (b) Now, measure the distance between V_{n+1} and V_n by the supremum-norm and store this distance for each n. Run your loop until $||V_{n+1} - V_n|| < \epsilon = 10^{-6}$ (Tip: Have a look at the `while`-loop.). If your code runs too long, relax the convergence criterion or lower the number of grid points. Do the distances shrink as fast as the Banach fixed-point theorem predicts? What is the economic meaning of the limit point you found?

2. **Contractive linear mappings.**

   (a) Consider a firm which makes a deterministic profit π each period and which discounts at a rate $\beta = 1/(1+r)$. Write an operator T, $T : X \to X$, that tells us the value of the firm V_t at time t given that we know the firm's value V_{t+1} one period ahead. Using properties of this operator, show that the value of the firm must be

   $$V = \sum_{t=0}^{\infty} \beta^t \pi = \frac{\pi}{1-\beta}.$$

   (b) Now, suppose that there is finite number of states of the world $\{s_i\}_{i=1}^n$ each period in which the firm makes profits $\pi_1 < \pi_2 < \cdots < \pi_n$, respectively. The transition between those states is stochastic and follows a Markov process, i.e. the current state is the only determinant for the probability distribution over states in the next period:

   $$\text{Prob}(s_{t+1} = s_i | s^t = (s_1, \ldots, s_j)) = \text{Prob}(s_{t+1} = s_i | s_t = s_j) = p_{ij}.$$

We collect the *transition probabilities* p_{ij} in an $n \times n$-matrix $P = [p_{ij}]$ (the *transition matrix*). Write an operator $T : X \to X$ that maps the vector $V_{t+1} = [V_{t+1}(s_1), \ldots, V_{t+1}(s_n)]$ for the value of the firm in the different states of the world at $t+1$ into a vector V_t. Using the properties of this operator, show that there is unique vector V for the value of the firm and say how this vector can be calculated.

(c) Now, suppose that there is a continuum of states $s_t \in [0,1]$ and an associated profit function $\pi : [0,1] \to \mathbb{R}$, which is a continuous, strictly increasing function in s (so we have "ordered" states by their payoffs). Consider the following operator:

$$V_t(s) = TV_{t+1}(s) = \pi(s) + \beta \int_0^1 P(s,r)V_{t+1}(r)\mathrm{d}r$$

where $P(s,r)$ is a continuous function on the unit square and we have $\int_0^1 P(s,r)\mathrm{d}r = 1$ for all $s \in [0,1]$. The "matrix row" $f_s(r) \equiv P(s,r)$ gives us the density function of $r = s_{t+1}$ given that we are in state s today. Again, show that there is a unique function $V(s)$ for the value of the firm and say how we can calculate (or approximate) it.

Note: The function $P(\cdot,\cdot)$ is something like a matrix for mapping one function to another; in functional analysis, such a function is called a *kernel*. Note that any such kernel P gives us a linear operator (try additiveness and scalar multiplication). "Inverting" the kernel P (or rather the "matrix" $I + \beta P$, where I is the identity) to find the solution to the functional equation directly is a classical problem in functional analysis. Keywords: Fredholm alternatives, Fredholm determinant. Some good books: Peter D. Lax: "Functional Analysis", a cheaper one: Riesz & Sz.-Nagy: "Functional Analysis")

3. **Contraction mapping in savings model.** Consider a consumer in infinite time with assets a. He has no labor income and faces a no-borrowing constraint. Let $A = \{a \in \mathbb{R} : a \geq 0\}$ be the state space, and let $C(A)$ be the space of bounded and continuous functions on A. Consider the following operator T:

$$(TV)(a) = \max_{a' \in [0,Ra]} \{u(a - a'/R) + \beta V(a')\},$$

where $\beta \in (0,1)$, $R > 1$ and where $u : \mathbb{R} \to \mathbb{R}$ is a continuous, bounded, and increasing function. You may use theorems from class to show your results.

(a) Show that T maps $C(A)$ to itself.

(b) Show that T has a unique fixed point.

4. **Counterexamples to maximum theorem and its corollary.** Consider the problem

$$h(x) = \sup_{y \in \Gamma(x)} f(x, y),$$
$$G(x) = \{y \in Y : f(x, y) = h(x)\}.$$

where $x \in X \subset \mathbb{R}^n$ and $y \in Y \subset \mathbb{R}^m$. $f : X \times Y \to \mathbb{R}$ is some function and $\Gamma : X \rightrightarrows Y$ is a non-empty correspondence.

(a) Find examples (i.e. define X, Y, f, and Γ and solve for h and G) in which the following occur. In each case, state which assumption(s) of Berge's Maximum Theorem and/or the corollary we saw in class is/are violated

   i. The value function $h(\cdot)$ is discontinuous.
   ii. The policy correspondence $G(x)$ is empty for some $x \in X$.
   iii. The policy correspondence $G(x)$ is multi-valued for some $x \in X$.
   iv. The policy correspondence $G(\cdot)$ is single-valued for all x, but discontinuous at some $x \in X$.

(b) Let all assumptions of the Maximum Theorem hold. Find an example where $G(x)$ is not lower-hemi-continuous for some $x \in X$. Which assumption of the corollary to the Maximum Theorem fails in your example?

5. **Consumption-savings problem with habit formation**. Consider an agent whose current utility depends not only on current consumption c_t, but also on past consumption; her criterion is

$$\sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1}),$$

where $\beta \in (0, 1)$ and $c_{-1} > 0$ is given. Also, the agent is endowed with initial assets $a_0 > 0$ and receives a fixed income $y > 0$ each period. Assume that the agent can save at an exogenously-given gross interest rate $R > 1$ and that there is an exogenous borrowing limit $\bar{a}$.

(a) Bring this problem into dynamic-programming form: What are the state, the control(s), the feasible-set correspondence and the return function?

(b) Write down the Bellman equation.

(c) Find the Euler equation and interpret it briefly.