Macro II (M.E.A., UC3M)
Professor: Matthias Kredler
Problem Set 2
Due: 7 February 2020

Recall that also the computation problem from Problem Set 1 is due on 7 Feb. You are encouraged to work in groups; however, every student has to hand in his/her own version of the solution.

1. **Neo-classical human-capital accumulation.** Consider the problem of an agent maximizing the present value of his/her lifetime earnings over T periods by dividing his/her 1 unit of time between market work and human capital investment. Let h_t denote the human capital stock of the agent at age t and let $1 - l_t$ be his/her market work. Let $h_t(1 - l_t)w$ be the agent's income at time t, where w is the rental rate of human capital (or efficiency wage rate) that the agent takes as given. The amount of time the agent does not spend for market work, i.e. l_t, is used to produce human capital according to $f(h_t, l_t) = (h_t l_t)^\alpha$ with $\alpha \in (0, 1)$. Human capital depreciates at rate δ. Hence, the problem of the agent is

$$\max_{\{l_t, h_{t+1}\}_1^T} \sum_{t=1}^{T} \frac{w h_t (1 - l_t)}{(1 + r)^{t-1}}$$

subject to

$$l_t \in [0, 1], \quad h_{t+1} = f(h_t, l_t) + h_t(1 - \delta), \quad h_1 > 0 \text{ given,}$$

where r is the real interest rate.

(a) State this finite-horizon problem as a dynamic-programming problem; say what the state, the control(s), the return function and the feasibility correspondence are. Write down the Bellman Equation(s).

(b) From the Bellman Equation, find the Euler equations for this problem. Restrict attention to the case in which all optimal choices are interior.[1]

(c) Interpret the Euler equations in a couple of sentences.

2. **Hall's random-walk hypothesis.** Consider a consumer who lives for two periods $t = 1, 2$ and faces stochastic income draws y_1, y_2 (which may be dependent random variables). The consumer seeks to maximize

$$u(c_1) + \beta \mathbb{E}_1 \big[u(c_2)\big]$$

[1]Note: Differentiability of the value function in Dynamic Programming (especially in the infinite-horizon case) is not guaranteed when optimal choices occur at boundaries; see Rincón-Zapatero (your maths teacher!) & Santos in JET, 2009, for results that establish differentiability of the value function and thus generalize the Envelope Theorem in the case of non-interior choices. Other contexts in which differentiability (and thus the Envelope Theorem) may fail: Presence of discrete choices, two-player games. See the Working Paper "A General and Intuitive Envelope Theorem" by Andrew Clausen and Carlo Strub for a nice discussion and examples.

where $\beta \in (0,1)$, $u'(c) > 0$, $u''(c) < 0$ for all c and $\lim_{c \to 0} u'(c) = \infty$. The consumer has access to saving and borrowing at interest rate $R > 1$; she has to leave period 2 with non-negative assets.

(a) Find the state and the value function for $t = 2$.

(b) Find the state for $t = 1$. Write the value function $V_1(\cdot)$ at $t = 1$ recursively. i.e. using $V_2(\cdot)$.

(c) Find the Euler equation for the consumer.

(d) Assume that $R\beta = 1$.[2] Simplify the Euler equation and interpret it.

(e) Now, assume additionally that $u(c) = -\frac{1}{2}(\bar{c} - c)^2$; the resulting equation is called *Hall's random-walk hypothesis*[3] – why?

(f) Hall showed that the random-walk hypothesis holds for models with any number of periods (not only $T = 2$). To test it, he ran the following regressions on data:

$$c_t = \beta_0 + \beta_1 c_{t-1} + \beta_2 c_{t-2} + \beta_3 c_{t-3} + \beta_4 c_{t-4} + \epsilon_t$$

Which coefficients do we expect under the random-walk hypothesis? Which test could be used to reject the random-walk hypothesis?

(g) Another test Hall ran is

$$c_t = \beta_0 + \beta_1 c_{t-1} + \beta_2 y_{t-1} + \epsilon_t$$

Which coefficients should we expect in this regression under the random-walk hypothesis and how could we test it?

(h) Hall could not reject the random-walk hypothesis in 1978. The later literature, however, *could* find predictability in $\Delta c_t = c_{t+1} - c_t$: Predictable changes in income were found to lead to predictable changes in consumption, in which *borrowing constraints* play a crucial role. Explain how in the two-period model above the presence of a borrowing constraint ($a_2 \geq 0$) can lead to a violation of the Euler equation and hence the random-walk hypothesis.

3. **Transversality condition in consumption-savings problem.** Consider the deterministic consumption-savings problem without an endowment stream. The budget constraint is

$$a_{t+1} \leq R(a_t - c_t)$$

Given $a_0 > 0$, the consumer wants to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $0 < \beta < 1$, $u' > 0$, $u'' < 0$ and $\lim_{c \to 0} u'(c) = \infty$ and the consumer cannot borrow, i.e. $a_{t+1} \geq 0$ for all t.

[2] Note that this is a prediction that arises from many macroeconomic models in which R is endogenized.

[3] It was stated by Robert Hall in 1978.

(a) State the Euler equations and transversality condition for this problem (Will we always have interior solutions?).

(b) Repeat the proof from class for sufficiency of these conditions for this specific problem; state exactly at which point you use each assumption.

(c) Now, assume the special case $u(c) = c^{1-\gamma}/(1-\gamma)$.

   i. Show that a constant savings rate, i.e. setting $c_t = (1-s)a_t$ for all t, solves the Euler equations.

   ii. Under which conditions on R and β is the transversality condition fulfilled? At which rates do a_t, c_t, $u'(c_t)$ and $u(c_t)$ grow?

   iii. For the case where the transversality condition is not fulfilled, show that a (unique) maximizing sequence does not exist.