1 Euler equations under uncertainty

1.1 General setting and notation

Time is discrete and infinite. Let s_t for $t = 1, 2, \ldots$ denote a sequence of shocks to an economy. s_t is drawn from a finite set and the the process may have an arbitrary probability distribution over time. We will denote histories of shocks up to t as follows:

$$s^t \equiv (s_1, s_2, \ldots, s_t)$$

As an example, take a process s_t that can only take two values $\{\underline{s}, \bar{s}\}$ and consider the specific history

$$s^3 = (\bar{s}, \underline{s}, \bar{s}).$$

Often, we will want to refer to a sub-history of a history s^t. For example, to refer to the history of shocks in s^t up to time $t-1$, we write $s^t_{\rightarrow t-1}$. In our example, we would have

$$s^3_{\rightarrow 2} = (\bar{s}, \underline{s}).$$

To refer to single shocks s_k in a given history s^t we use subindeces. For example, to read off the last shock in a history s^t we write s^t_t. In our example, this would be

$$s^3_3 = \bar{s}.$$

The probability that a history s^t occurs is denoted by $\pi_t(s^t)$. We may view $\{\pi_t(\cdot)\}_{t=1}^{\infty}$ as a sequence of functions mapping from the set $\{s^t\}$ of possible histories at t to $\mathbf{R}$. The sequence of probability functions $\{\pi_t(\cdot)\}_{t=1}^{\infty}$ fulfills the following (obvious) consistency requirements:

$$\sum_{s^t} \pi_t(s^t) = 1 \qquad \text{for all } t$$

$$\sum_{s^{t+1}: s^{t+1}_{\rightarrow t} = s^t} \pi_{t+1}(s^{t+1}) = \pi_t(s^t) \qquad \text{for all } s^t, \text{ for all } t$$

The first requirement says that unconditional probabilities of all histories at must sum up to one at any t; the second says that the sum of probabilities following a particular node in the event tree must equal the probability of reaching that node. Conditional probabilities are given by

$$Prob(s_{t+1} = s' | s^t) = \frac{\pi_{t+1}((s^t, s'))}{\pi_t(s^t)}.$$

The policy function at time t (i.e. choices made by agents at time t) is conditioned on information at time t. Formally, they are functions defined on the set of histories at that point: $g_t : \{s^t\} \rightarrow \mathbf{R}$. This requirement ensures that “agents cannot see into the future”.[1] (Would a model be a reasonable description of an

[1] In measure-theoretic terms, one would say that these functions are *measurable* with respect to the filtration F_t created by the shock history s^t.

agent in an uncertain environment if policy functions at t were defined on a set $\{s^{t+k}\}$ for $k > 0$? Why/Why not?)

The objective of our agent is to choose a sequence of policy functions $\{g_t(\cdot)\}_{t=0}^{\infty}$ in order to maximize

$$\max_{\{g_t(\cdot)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) F_t(s^t, \{g_\tau(s^t_{\to\tau})\}_{\tau=0}^{t}),$$

where $F_t(\cdot)$ is the *return function* at t, which we allow to depend on the history of the shock and all decisions taken along this history up to t.

1.2 Example: The stochastic neo-classical growth model

Consider a neo-classical-growth economy with production function

$$y_t = z_t F(k_t),$$

where z_t is an i.i.d. productivity shock that takes a low value $\underline{z} > 0$ with probability 0.5 and a high value $\bar{z} > \underline{z}$ with probability 0.5 each period.

So the probability functions $\pi_t(\cdot)$ have the following properties:

$$\pi_t(z^t) = 0.5^t \qquad \text{for all } z^t,\ t$$
$$Prob(z_{t+1} = \bar{z}|z^t) = Prob(z_{t+1} = \bar{z}) = 0.5 \qquad \text{for all } z^t,\ t$$

The capital stock for period $t+1$ is chosen in period t, so it is a policy function at t and we write $k_{t+1}(z^t)$. Also consumption at t is decided at t, so write $c_t(z^t)$. The feasibility constraint for the agent at node z^t is

$$k_{t+1}(z^t) \leq z_t F(k_t(z^t_{\to t-1})) + (1-\delta)k_t(z^t_{\to t-1}) - c_t(z^t).$$

Since this constraint will always hold with equality, we can write our usual criterion as[2]

$$U = \sum_{t=1}^{\infty} \beta^t \sum_{z^t} \pi_t(z^t) u\Big(\underbrace{z_t F(k_t(z^t_{\to t-1})) + (1-\delta)k_t(z^t_{\to t-1})}_{\equiv f(k_t(z^t_{\to t-1}), z_{t+1})} - k_{t+1}(z^t)\Big)$$

1.3 Euler equations for growth model

In an event-tree figure, we can easily see that the choice $k_{t+1}(z^t)$ affects utility at the node z^t and at the *two* subsequent nodes z^{t+1} that follow up. So the

[2]Note that we could also write a Lagrangian keeping the feasibility constraint as an inequality constraint (or as an equality constraint, arguing as above). The two approaches are equivalent and yield the same results; your choice between the choice should be guided by which makes your calculations easier.

first-order condition with respect to $k_{t+1}(z^t)$ is

$$\begin{aligned}\frac{\partial U}{\partial k_{t+1}(z^t)} =& -\beta^t \pi_t(z^t) u'\big(c_t(z^t)\big) + \\ &+ \sum_{z^{t+1}:z^{t+1}_{\to t}=z^t} \beta^{t+1}\pi_{t+1}(z^{t+1}) u'\big(c_{t+1}(z^{t+1})\big) f_k\big(k_{t+1}(z^{t+1}_{\to t}), z_{t+1}\big) \\ =& \;\; 0\end{aligned}$$

Dividing by $\beta^t \pi_t(z^t)$ we obtain

$$u'\big(c_t(s^t)\big) = \beta \sum_{z^{t+1}:z^{t+1}_{\to t}=z^t} \frac{\pi_{t+1}(z^{t+1})}{\pi_t(z^t)} u'\big(c_{t+1}(z^{t+1})\big) f_k\big(k_{t+1}(z^{t+1}_{\to t}), z_{t+1}\big),$$

where we recognize the conditional probabilities in the fractions π_{t+1}/π_t. Of course we could simplify these to 0.5 due to the i.i.d. assumption, but we will stay with the more general notation because this carries over to other settings. Now we can bring the Euler equation into its typical form, which is

$$\begin{aligned}u'\big(c_t(s^t)\big) &= \beta E\Big[u'\big(c_{t+1}(z^{t+1})\big) f_k\big(k_{t+1}(z^{t+1}_{\to t}), z_{t+1}\big)\Big| z^t\Big] = \\ &= \beta E_t\Big[u'\big(c_{t+1}(z^{t+1})\big) f_k\big(k_{t+1}(z^{t+1}_{\to t}), z_{t+1}\big)\Big]\end{aligned}$$

where the second line is just different notation but means exactly the same as the first line.[3] In papers you will mostly see the following short-hand notation:

$$u'(c_t) = \beta E_t\Big[u'(c_{t+1}) f_k(k_{t+1}, z_{t+1})\Big],$$

where the dependence of c_{t+1} and k_{t+1} on histories is understood.

The intuition for this Euler equation is straighforward: The marginal utility loss from investing one more unit at node z^t must equal the marginal discounted expected gain, which is the marginal increase in productivity times marginal utility of consumption at the respective nodes at $t+1$.

[3]Note that this is just the definition of conditional expectations: $E_t[h(z^{t+k})] \equiv E[h(z^{t+k})|z^t]$ for any function $h(\cdot)$, $k > 0$ and some stochastic process z_t.