Macro II (UC3M, MA/PhD Econ)
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Problem Set 7
Due: 14 April 2011
You are encouraged to work in groups; however, every student has to hand in his/her own version of the solution.

1. **Principle of optimality.** Consider a saver with linear utility; her criterion to order streams of consumption is

$$\sum_{t=0}^{\infty} \beta^t \alpha c_t$$

where $c_t \geq 0$ for all t and $\beta \in (0,1)$. She can save at rate $R = 1/\beta$ and has initial assets a_0. There is a no-borrowing constraint, i.e. we impose $a_t \geq 0$ for all t

   (a) Consider the case where the saver "doesn't care": $\alpha = 0$.
      i. Bring the problem into dynamic-programming form.
      ii. Write the Bellman equation.
      iii. Show that $V(a) = a$ is a solution to the Bellman equation.
      iv. Is $V(a) = a$ the true value function?
      v. Explain why our theorems linking the sequence problem and the functional equation hold/fail.

   (b) Now, consider the case where the saver *does* care and set $\alpha = 1$.
      i. Write the Bellman equation.
      ii. Find the true value function $V^*(a)$ and show that it satisfies the Bellman equation.
      iii. Characterize the set of optimal policies for this problem. Compare your finding to the theorems in the principle of optimality which concern optimal policies.

2. **Increasingness of value function in finite-horizon dynamic programming.** Consider the problem

$$\begin{aligned} \max_{\{x_{t+1}\}_{t=0}^{T}} & \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) \\ \text{s.t. } & x_{t+1} \in \Gamma(x_t) \qquad \text{for } t = 0, \ldots, T \\ & x_0 \text{ given} \end{aligned}$$

Assume that $F(\cdot, x')$ is a strictly increasing function and that Γ is an increasing correspondence, i.e. $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$ for any x, x'. Show that all value functions $\{V_t(\cdot)\}_{t=0}^{T}$ are strictly increasing.

3. **Concavity of value function.** Consider the framework from class: We want to show that the value function is strictly concave under assumptions A1, A2, A5 and A6.

   (a) Prove that the set $S' = \{f \in C(X) : f \text{ is weakly concave}\}$ is a closed subset of $C(X)$. (Hint: Work directly with the definition of weak concavity).

   (b) Prove that $T(S') \subseteq S'' \subset S'$, where S'' further restricts S' to *strictly* concave functions.

   (c) Apply our results from class to show the desired result.

4. **Vintage capital.** Consider the neo-classical growth model, but assume that the age of the capital stock (its *vintage*) matters.[1] The consumption good is produced according to a production function $y_t = F(k_{t,0}, k_{t,1}, \ldots, k_{t,n})$, where $k_{t,n}$ is capital at t of vintage $t - n$, i.e. capital that is of age i at t.[2] F is continuous, strictly increasing in all of its arguments and strictly concave. It also satisfies the Inada-conditions $\lim_{x_i \to 0} \partial F(x_0, \ldots, x_n)/\partial x_i = \infty$ for $i = 0, \ldots, n$. New capital is formed according to

$$c_t + k_{t+1,0} \leq y_t,$$

   where $c_t \geq 0$ is consumption at t. There is an initial vector $(k_{0,0}, \ldots, k_{0,n})$ given. The agent ranks sequences of consumption by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

   where $\beta \in (0,1)$, $u' > 0$, $u'' < 0$ and $\lim_{c \to 0} u'(c) = \infty$.

   (a) Write down the sequence problem for a social planner in this economy.

   (b) Find the first-order condition for investment from the sequence problem. Interpret briefly.

   (c) Show that the optimal investment will always be interior, i.e. there will never be a corner solution for $k_{t+1,0}$.

   (d) Bring the problem into dynamic-programming form: Find the state, control(s), feasible-set correspondence and return function.

   (e) Write down the Bellman equation.

   (f) Use the theorems from class to characterize the value function and optimal-policy correspondence as far as possible.

[1] If a large part of the capital stock of an economy was formed at a certain time, say after a war, then it is likely that a lot of this capital will have to be replaced at the same time (in a so-called *replacement echo*). Note that the assumptions of geometric depreciation rule out this kind of behavior in the standard growth model.

[2] So capital depreciates fully at age n. We assume zero depreciation before age n here. However, note that we could easily incorporate depreciation by modifying the function F accordingly.