Definition of the state in dynamic programming


Example: Growth with productivity shocks

Consider a standard growth model in which output is produced according to $y_t = A_t k_t^\alpha$. $A_t$ is a productivity shock that follows a first-order Markov process with conditional density $f(A_{t+1}|A_t)$, $k_t$ is capital at $t$. The investment technology is standard; the planner has to choose tomorrow’s capital (the control) given a feasible-set correspondence

$$k_{t+1} \in \Gamma(k_t, A_t) \equiv [0, A_t k_t^\alpha + (1 - \delta)k_t].$$

The period-$t$ return to the planner is

$$F(k_t, A_t; k_{t+1}) = \ln (A_t k_t^\alpha + (1 - \delta)k_t - k_{t+1}).$$

The planner discounts the future at factor $\beta \in (0, 1)$. Thus the Bellman equation is given by

$$V(k, A) = \max_{k' \in \Gamma(k, A)} \left\{ F(k, A; k') + \beta \mathbb{E}[V(k', A')|A] \right\}.$$

General formulation

Consider now a more general stochastic environment with infinite horizon. There is a shock sequence $z_t \in \mathbb{R}^{N_z}$ which follows a first-order Markov process. In our example, $A_t \triangleq z_t$ (meaning: “$A$ takes the role of $z$”) and $N_z = 1$. Suppose that we are given a vector $y_t \in \mathbb{R}^{N_y}$ which informs us about the feasible set for a control $u_t \in \mathbb{R}^{N_u}$ and the return in period $t$. Specifically, we are given a feasibility correspondence, $\Gamma$, and a return function, $F$:

$$u_t \in \Gamma(y_t, z_t), \quad \Gamma : \mathbb{R}^{N_y+N_z} \Rightarrow \mathbb{R}^{N_u},$$

$$F(y_t, z_t, u_t), \quad F : \mathbb{R}^{N_y+N_z+N_u} \rightarrow \mathbb{R}.$$ 

In the example, $k_t \triangleq y_t$ and $k_{t+1} \triangleq u_t$. Finally, suppose that we are given a law of motion for $y$, i.e. a function $h$ that tells us which value $y$ takes tomorrow:

$$y_{t+1} = h(y_t, z_t, u_t, z_{t+1}), \quad h : \mathbb{R}^{N_y+N_z+N_u+N_z} \rightarrow \mathbb{R}^{N_y}.$$ 

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In the example, the function $h$ is of trivial form: $k_{t+1}$ is both the control at $t$ and (part of) the state tomorrow, thus $y_{t+1} = u_t$. Given this environment, we can always write down a valid functional equation for a value function $V(\cdot)$ that reads:

$$V(z, y) = \max_{u \in \Gamma(z, y)} \left\{ F(z, y; u) + \beta \int V(z', h(y, z, u, z')) f(z'|z) dz' \right\}$$

However, we will see this formulation may be wasteful. It may be that we can condense the state of the economy into a vector $x$ of lower dimensionality than $(z, y)$. This turns out to be of huge value for both analytical and computational purposes.

**Definition:** The state of the economy is the smallest set of variables, a vector $x \in \mathbb{R}^{N_x}$, $N_x \leq N_z + N_y$, that allows us to determine all of the following:

- the feasible set, i.e. $\bar{\Gamma}(x) = \Gamma(z, y)$ for some correspondence $\bar{\Gamma} : \mathbb{R}^{N_z} \rightarrow \mathbb{R}^{N_u}$,
- the return given a control $u$, i.e. $\bar{F}(x, u) = F(z, y, u)$, for some function $\bar{F} : \mathbb{R}^{N_x+N_u} \rightarrow \mathbb{R}$,
- the law of motion for $y$ given $u$ and $z'$, i.e. $y' = \bar{h}(x, u, z')$ for some function $\bar{h} : \mathbb{R}^{N_x+N_u+N_z} \rightarrow \mathbb{R}^{N_z}$,
- and the conditional expectation in the Bellman equation, i.e. $\bar{f}(z'|x) = f(z'|z)$ for some function $\bar{f}$.

The Bellman equation using the state $x$ is then

$$\bar{V}(x) = \max_{u \in G(x)} \left\{ \bar{F}(x, u) + \beta \int \bar{V}(\bar{h}(x, u, z')) f(z'|x) dz' \right\}.$$ 

**Back to the example: i.i.d. shocks and the cash-on-hand trick**

We now go back to our example. It turns out that for a specific case it is possible to return the dimensionality of the state space to one. This case is the one where the shock $A_t$ is i.i.d. As the new state, let us define cash-on-hand as $x_t = A_t k_t^\alpha + (1 - \delta) k_t$. The other objects are

$$\bar{\Gamma}(x) = [0, x],$$
$$\bar{F}(x, k') = \ln(x - k'),$$
$$\bar{h}(x, k', A') = A'(k')^\alpha + (1 - \delta) k'.$$
The Bellman equation is

\[
\tilde{V}(x) = \max_{k' \in \tilde{G}(x)} \left\{ \tilde{F}(x, k') + \beta \int \tilde{V}(\tilde{h}(x, k', A')) f(A') dA' \right\}.
\]

Since the conditional density of \( A' \) equal the unconditional density by the i.i.d. assumption, the expectation in the Bellman equation is correct. We thus see that cash-on-hand gives us full information about the economic environment at \( t \) and that the state can be condensed in this case.