Approximating PDEs on trinomial lattice (based on Marco Avellanada's lecture)

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The problem

We want to represent the following process on a trinomial tree:

$$dX_t = a(X_t, t)dt + s(X_t, t)dW_t$$

and solve the associated PDE with a terminal condition numerically:

$$\begin{aligned}
-\frac{\partial V}{\partial t} &= -rV + a(X_t, t)\frac{\partial V}{\partial x} + \frac{\left[s(X_t, t)\right]^2}{2}\frac{\partial^2 V}{\partial x^2} \\
V(x, T) &= f(x)
\end{aligned}$$

Setting up the tree

We will do this by creating a *lattice* (this means an equally-spaced grid) on the rectangle $[0, T] \times [\underline{x}, \overline{x}]$. We call the interval size between two grid points Δt and Δx, respectively. For our scheme, it turns out that we have to choose

$$\Delta x = \max_{0 \le t \le T, \underline{x} \le x \le \overline{x}} \{s(t, x)\}\sqrt{\Delta t} = \bar{s}\sqrt{\Delta t}$$

So if the volatitility is time-varying, we take the highest possible volatility on our grid as the benchmark $\bar{s}$.

From a given node (t_i, x_k), the process can go to the three adjacent nodes (t_{i+1}, x_{k-1}), (t_{i+1}, x_k) and (t_{i+1}, x_{k_1}) in the next period, and this happens with associated probabilities p_u ("up"), p_m ("middle") and p_d (hence the name *trinomial tree*).

We impose the following three common-sensical conditions to pin down these probabilities:

$$\begin{aligned}
p_u + p_m + p_d &= 1 && \text{(normalization)} \\
p_u\Delta x - p_d\Delta x &= a(x, t)\Delta t && \text{(drift)} \\
(\Delta x)^2(p_u + p_d) &= [s(x, t)]^2\Delta t && \text{(volatility)}
\end{aligned}$$

Note that in the last equation, we calculate the variance of the increment assuming that the mean of the increment is zero – this is not strictly correct, but it turns out that the mistake we make like this is of second order, and when the grid becomes really small, the mistake does not matter anymore. Plugging in from before, we get

$$
\begin{aligned}
p_u + p_d &= \frac{s(x,t)}{\bar{s}} \equiv k \\
p_u - p_d &= \frac{a(x,t)}{\bar{s}} \sqrt{\Delta t}
\end{aligned}
$$

Note that for the node where the maximal volatility $\bar{s}$ on the grid is realized, we have $k = 1$, and the probability of jumping to the middle point is zero. So the trinomial-tree procedure includes the *binomial-tree procedure* (with constant volatility) as a special case.

When we solve, we get the following probabilities:

$$
\begin{aligned}
p_m(x_i, t_j) &= 1 - \underbrace{\frac{s(x_i, t_j)}{\bar{s}}}_{=k(x_i, t_j)} \\
p_u(x_i, t_j) &= \frac{1}{2}\left[k(x_i, t_j) + \frac{a(x_i, t_j)}{\bar{s}} \sqrt{\Delta t}\right] \\
p_d(x_i, t_j) &= \frac{1}{2}\left[k(x_i, t_j) - \frac{a(x_i, t_j)}{\bar{s}} \sqrt{\Delta t}\right]
\end{aligned}
$$

Solving the PDE

Now, to solve the value function backward, we use the following formula:

$$
V(x_i, t_j) = e^{-r\Delta t}\Big[p_u V(x_{i+1}, t_{j+1}) + p_m V(x_i, t_{j+1}) + p_d V(x_{i-1}, t_{j+1})\Big]
$$

Note that for the nodes x_1 and x_I on the bottom and the top of the lattice we can't use this formula. At these points, we just extend the grid in a linear fashion, i.e. we use

$$
V(x_{I+1}, t) = V(x_I, t) + \Big[V(x_I, t) - V(x_{I-1}, t)\Big]
$$

and likewise on the bottom. This kills off the action in the PDE that comes from convexity (i.e. the second derivative) at these points. So we have to make sure that the upper and lower bound of the lattice are in regions where our value function has almost no convexity. [We will cover in the lecture how convexity spreads back in time from the strike and covers ever larger areas in option pricing.]

Stability condition

This scheme will work as long as the so-called *Courant-Friedrichs-Lax condition* (CFL) holds:

$$\frac{s(t,x)^2 \Delta t}{(\Delta x)^2} \leq 1 \qquad \text{(CFL)}$$

So whenever crazy things happen, check this condition! It actually imposes that we have non-negative probabilities, but the computer wouldn't notice that...