Homework on stochastic control

(by Matthias Kredler)

1. **(The Merton consumption-investment problem)** Consider an agent who can either consume his assets or invest them in a risky asset. The value of the risky asset follows

$$dX_t = \alpha X_t dt + \sigma X_t dB_t.$$

The consumer chooses a consumption rate c_t, i.e. per unit of time (say a year) he would take out a fraction c_t from his wealth and consume it. Over a short time interval Δt his assets Y would evolve as

$$\Delta Y_t = \underbrace{-c_t \Delta t Y_t}_{\text{remove for consumption}} + \underbrace{(1 - c_t \Delta t) Y_t}_{\text{left for saving}} \underbrace{(\alpha \Delta t + \sigma \Delta B_t)}_{\text{stochastic return}}$$

   (a) Take the limit $\Delta t \to 0$ to derive

$$dY_t = (\alpha - c_t) Y_t dt + \sigma Y_t dB_t.$$

   (b) The consumer ranks consumption streams according to

$$E_0 \int_0^\infty e^{-\beta t} \frac{1}{1+\gamma} (c_t Y_t)^{1+\gamma} \, dt$$

   Write down the HJB that the value function $v(Y)$ must fulfill.

   (c) Try the guess $v(Y) = \Phi Y^{1+\gamma}$: Plug it into the value function, back out the undetermined coefficient Φ and show that it works.[1] What is the optimal policy? Is this surprising when you think of the name for this utility function?

2. **(Merton II/practice in continuous-time modelling)** Now, suppose that the consumer has two assets at his disposal: One is safe and the other one is risky:

$$dZ_t = r Z_t dt$$
$$dX_t = \alpha X_t dt + \sigma X_t dB_t$$

The consumer puts a fraction x_t of his wealth into the risky asset, a fraction z_t into the safe asset and consumes the rest.

[1] As in discrete time, it is very rare that there is a closed-form solution for the value function – this is one of the very few cases were a closed-form solution has been found.

(a) Write down the law of motion for his wealth over a short time period Δt, assuming that the consumed part must be taken out of the portolio at the beginning of this period.

(b) Take limits to get the law of motion for wealth in continuous time.

3. **(Ayagari in continuous time)** Consider an infinitely-lived agent choosing a consumption rate c in a small open economy facing an exogenous interest r and a stochastically varying i.i.d. wage with mean w and variance σ^2; his wealth y follows the process

$$dY_t = (rY_t - C_t)dt + w + \sigma dB_t$$

The consumer ranks consumption streams according to

$$E_0 \int_0^\infty e^{-\beta t} \frac{1}{1+\gamma} C_t^{1+\gamma} dt$$

(a) Why is this the correct form to write the problem in continuous time? Set up the corresponding problem in discrete time. Then make the time increments smaller and take limits to get the above equation.

(b) Derive the Hamilton-Jacobi-Bellman equation that characterizes the value function $v(Y)$.

(c) Given the optimal rule from the HJB, write down the forward equation that tells us how the density $n(t, Y)$ describing the distribution of agents over the Y-grid evolves over time. From this, you should get a 2nd-order ordinary differential equation (ODE) that the stationary density $\bar{n}(Y)$ must fulfill.

(d) Write a program for a trinomial tree that iterates back on the value function for some initial guess. Either approximate the derivatives as Christian told us in his talk (up-wind differencing for the first derivative), or do as if the agent faced the simulated problem altogether, i.e. let him choose the p_u- and p_d-values, inferring the appropriate consumption from the first and second moment of ΔY_t resulting from p_u and p_d.

(e) With the optimal policy in hand, you can now map any start density $n(0, Y)$ forward in time, and it should converge to the stationary density $\bar{n}(Y)$. Do this by applying the procedure described in the notes on the forward equation.

4. **(Neoclassical growth/two state variables)** Consider the following economy: The investment technology is standard and subject to depreciation, production has decreasing returns and is subject to a mean-reverting productivity shock:

$$
\begin{aligned}
dK_t &= (-\delta K_t + I_t)dt \\
C_t &= A\exp(Z_t)K_t^\alpha - I_t \\
dZ_t &= -\mu Z_t dt + \sigma dB_t
\end{aligned}
$$

We want to find the value function $v(K_t, Z_t)$ for a planner in this economy who maximizes

$$
E_0 \int_0^\infty e^{-\beta t} \frac{1}{1+\gamma} C_t^{1+\gamma} dt
$$

(a) Show that the HJB is

$$
\begin{aligned}
0 = &- \beta v(K,Z) - \mu Z \frac{\partial v(K,Z)}{\partial Z} + \frac{\sigma^2}{2} \frac{\partial^2 v(K,Z)}{\partial Z^2} + \\
&+ \max_I \left\{ \frac{1}{1+\gamma} [A\exp(Z)K^\alpha - I]^{1+\gamma} + (\underbrace{I - \delta K}_{=\dot{K}}) \frac{\partial v(K,Z)}{\partial K} \right\}.
\end{aligned}
$$

(b) I think that the following approximation strategy should work well: Create a grid on K and Z. For the movements on the Z-grid, jump up, down or stay with the probabilities p_u, p_m and p_d prescribed by the trinomial tree in one dimension. To get $\partial v/\partial K$, difference up-wind: For values of I that yield $\dot{K} > 0$, take the first difference in v one grid point above; analogously, take the downward difference for $\dot{K} < 0$. If none of these yields an interior solution, choose $\dot{K} = 0$. My hunch is that when you start with a concave guess for the value function, this concavity should be preserved and we should never have the case that both the upward and the downward problem yield an interior solution.

(c) Compare your solution to the one for a discrete problem with the same parameters and a Markov chain for Z that has the same variance and first-lag autocovariance as our process for Z. Tell me how the two methods compare in terms of computation time and reliability (I have never done this, so I'm genuinely interested!).