

Homework on connection between SED and PDE

(by Matthias Kredler)

1. **(The generalized Ito Rule)** Consider a diffusion X_t and another process Y_t which is calculated as a smooth function $f(\cdot)$ on X_t and time:

$$\begin{aligned}dX_t &= a(X_t)dt + s(X_t)dW_t, & X_0 \text{ given} \\ Y_t &= f(t, X_t),\end{aligned}$$

where $a(\cdot)$, $s(\cdot)$ and $f(\cdot)$ are well-behaved functions, i.e. we can differentiate them a couple of times.

- (a) Take a Taylor expansion of second order (as in the derivation for the Ito rule) for $f(t + \Delta t, X_{t+\Delta t})$ around the point (t, X_t) . Note that here we are fixing a particular realization of the Brownian Motion W_t driving the process X_t and hence a fixed realization of the process X_t itself.
- (b) Now, let's wave our hands: For very small Δt , the SDE for X_t above has its drift roughly fixed at $a(X_t)$ and its volatility roughly fixed at $s(X_t)$ over the short interval $[t, t + \Delta t]$, since we are dealing with continuous functions. So we can replace

$$\Delta X_t = a(X_t)\Delta t + s(X_t)\Delta W_t$$

in the Taylor expansion (If you're not happy with this sloppy argumentation, take Taylor expansions of $a(X_{t+h})$ around $a(X_t)$ for $0 < h \leq \Delta t$ to see that what we are missing is of second order.)

- (c) Now, use the usual rules we have derived for stochastic calculus a la Ito:

$$\begin{aligned}(\Delta W_t)^2 &\rightarrow \Delta t \\ \Delta W_t \Delta t &\rightarrow 0\end{aligned}$$

What is left of the Taylor expansion after you apply these rules? You should get the following generalized version of the Ito Rule:

$$\begin{aligned} dY_t = df(X_t) &= \frac{\partial g(t, X_t)}{\partial t} dt + \frac{\partial g(t, X_t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g(t, X_t)}{\partial x^2} (dX_t)^2 = \\ &= \frac{\partial g(t, X_t)}{\partial t} dt + \frac{\partial g(t, X_t)}{\partial x} a(X_t) dt + \\ &\quad + \frac{1}{2} \frac{\partial^2 g(t, X_t)}{\partial x^2} [s(X_t)]^2 dt + \frac{\partial g(t, X_t)}{\partial x} s(X_t) dW_t \end{aligned}$$

[Note that the Ito Rule of course also holds for Ito-processes that are *not* diffusions – however, we will very rarely deal with these processes, so I left this out here; the rule is just analogous.]

2. **(European Option)** Let us price a European call option. The underlying asset price X_t moves according to a geometric Brownian Motion:

$$dX_t = \alpha X_t dt + \sigma X_t dW_t$$

where we set $\alpha = 0.3$ and $\sigma = 0.1$; a unit of time is equal to one year. The option price at $t = 0$ is $X_0 = 100$, and it expires in one year ($T = 1$) with a strike $K = 120$. A risk-neutral agent who discounts at a constant rate r values the option as follows:

$$V(x, t) = e^{-r(T-t)} E [(X_T - K)^+ | X_t]$$

- Write down the density of the random variable X_1 .
- How could we compute an approximation to $(X_0, 0)$ using random draws from the random variable X_T ? You do not have to compute these values, describing the scheme is enough.
- What is the probability that the option ends up “in the money”, i.e. $Prob(X_1 \geq K | X_0)$?
- Now, take a grid over time (say $t = 0, 0.01, \dots, 1$) and over prices (say $x = 10, 11, \dots, 300$). Calculate the price $V(x, t)$ of the option using the closed form solution for the Black-Scholes formula (look it up on wikipedia.org) for each point on this grid.
- What would a recursive scheme to calculate $V(x, t)$ for fixed t – given we knew $V(x, t + \Delta t)$ for all values of x a step Δt into the future – look like? You don’t have to code this, just describe the scheme in words to build some intuition.

- (f) Approximate the t - and the x -derivative of this function using the following common-sensical scheme:

$$V_x(x_i, t) = \frac{V(x_i, t) - V(x_{i-1}, t)}{\Delta x}$$

$$V_{xx}(x_i) = \frac{V_x(x_{i+1}, t) - V_x(x_i, t)}{\Delta x} = \frac{V(x + \Delta x) - 2V(x) + V(x - \Delta x)}{(\Delta x)^2}$$

where $\Delta x = 1$ is the grid size. Calculate terms in the PDE that we derived in class from these approximations and see if they really sum up approximately to zero as the PDE claims.

- (g) Use the approximation scheme for second-order PDEs that is described in the file `TrinomTreePDE.pdf` to approximate the value function. How close is this to the true solution? Experiment with different bounding values for the lattice and with different grid sizes!
- (h) Look at the value function in the region where $t \rightarrow T$ and $x \sim K$. Describe what happens here. Which effect is at work? What is $\partial V(x, t)/\partial t$ at the point $(x = K, t = T)$?
3. **(Connection to first-order PDE/deterministic motion)** Suppose an individual's human capital X_t follows the following deterministic process:

$$X_t = X_0 + \alpha t, \quad X_0 \text{ given}$$

At time T , for some weird reason the model ends and the agent is rewarded with some payoff that is related to his human-capital level at this point. The value function is just the discounted value of this payoff:

$$V(x, t) = e^{-r(T-t)}g(X_T)$$

where $g(\cdot)$ is a given function.

- (a) Using a pencil and a ruler, make a graph and find a closed-form solution for the value function for all $0 \leq t \leq T$ and for all x by finding out where the agent will end up if he is at a certain point x at a given t .

- (b) Now, look at the problem from a PDE-perspective: Take the steps analogous to the ones we took in class to get a PDE for the value function. Check if the function you found in 3a indeed fulfills this PDE.
- (c) Recall that the path that X_t follows was called a “characteristic curve” for first-order PDEs. Look again at the derivation of our second-order PDE for the European Option and convince yourself that the realizations of the path for the price of the underlying ($dX_t = \alpha X_t dt + \sigma X_t dW_t$) play the role of these characteristic curves. Why do we average over a bunch of “curves” in this case?

4. Consider a general random walk with drift

$$dX_t = \alpha dt + \sigma dW_t, \quad X_0 = x$$

Let the value function be an expectation of where the process ends up at $T > 0$:

$$V(x, t) = e^{-r(T-t)} E [g(X_T) | X_t = x]$$

where $g(\cdot)$ is a given function.

- (a) Write down the solution for $V(x, t)$ as an integral using the normal density of the hitting distribution and the function $g(\cdot)$.
- (b) Derive the PDE that has to hold for $V(x, t)$; note that it has to be closely connected to the PDE we derived in class for the European Option.
- (c) Check if your solution for V from 4a fulfills the PDE in 4b.