1 Homework on the Ito integral

(by Matthias Kredler)

1. For the following stochastic differential equations, we want to get approximations by finite approximations as we talked about in class:

$$\begin{aligned} dX_t &= \alpha dt + \sigma dW_t & X_0 &= 0 \\ dX_t &= \alpha X_t dt + \sigma X_t dW_t & X_0 &= 1 \\ dX_t &= W_t dW_t & X_0 &= 1 \\ dX_t &= \sqrt{|W_t|} dW_t & X_0 &= 1 \end{aligned}$$

   Use the parameter values $\alpha = 0.02$ and $\sigma = 0.01$. Generate draws from Brownian Motion for the time interval $t \in [0, 100]$ on a grid down to size 10^{-3}. Take a grid of mesh size $\Delta t_j = 10^j$ for $j = -3, -2, \ldots 1$ and create the increments of Brownian Motion $\Delta_j W_{t_i} = W_{t_i + \Delta t_j} - W_{t_i}$ starting with $W_0 = 0$. Then, approximate $X_t^{(j)}$ by the following procedure (note that j is for the j-th approximation scheme, and i indexes the i'th grid point on the time grid for the respective approximation):

$$X_{t_{i+1}}^{(j)} = a(X_{t_i}, W_{t_i})\Delta t_j + s(X_{t_i}, W_{t_i})\Delta_j W_{t_i}$$

   where you have to substitute in the functions $a(\cdot)$ and $s(\cdot)$ from the respective case above. If you need the value X_{t_i} to calculate these functions, take the value that you have obtained "on the way" in the respective approximation of grid size 10^j, i.e. take $X_{t_i}^{(j)}$. Also, do the above sum for the Δt-term and the ΔW_t-term separately and plot them to see how much each of them "contributes" to the process.

2. Next, we want to get a better intuition for Ito's Lemma by taking step-wise Taylor approximations of the following functions of stochastic processes (each line defines one example: first a function $Y_t = f(X_t)$ and then a stochastic process X_t to plug into the function):

$$\begin{aligned} Y_t &= \exp(X_t) & X_t &= \left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t \\ Y_t &= \left(X_t\right)^2 & X_t &= W_t \\ Y_t &= \sqrt{|X_t|} & X_t &= W_t \end{aligned}$$

Again, take a grid of mesh size 10^j for $j = -3, -2, \ldots 1$. Note that we can just calculate Y_t as a function of t and W_t for each grid point t_i now. At each of these grid points, take a second-order Taylor expansion around t_i and W_{t_i} to approximate $f(t_i + \Delta t_j, W_{t_i} + \Delta_j W_{t_i})$. Also, please keep the term in $(\Delta_j W_{t_i})^3$ from the third-order expansion (but drop all other third-order terms, and also everything of fourth order and higher).

At each of the grid points, calculate the six terms separately. Also, calculate the cumulative sums of each term over time to see the "contribution" of each term to the value of X_t. To check the famous Ito-term, also calculate the following term at each grid point t_i:

$$\frac{1}{2} \frac{\partial^2 f(X_t)}{\partial X_t^2} \Delta t$$

Check if indeed – as Ito's Lemma claims – we have cumulatively

$$\frac{1}{2} \sum_i \frac{\partial^2 f(X_{t_i})}{\partial X_{t_i}^2} \Delta W_{t_i} \sim \frac{1}{2} \sum_i \frac{\partial^2 f(X_{t_i})}{\partial X_{t_i}^2} \Delta t$$

Also, check if the terms that fall out in the derivation of Ito's Lemma are really very small numerically (even when summing them up over the entire grid) as we let the grid size become small. For each of the terms that drop out, state of which order the term is mathematically (or its standard deviation conditional on information at time t_i, for stochastic terms).

3. Finally, find the solutions to the first three SDEs in problem 1 (recall that this means writing a function $X_t = g(t, W_t)$ that does not depend on the shock history but *only* on the current state of the Brownian Motion W_t). Then, compare the numeric solutions to the explicit solutions – here you should be able to use some stuff you have calculated in problem 2 (make sure to throw in the same sample paths for W_t in both problems!). Also, compare the "contributions" of each term in both cases. What can you conclude? [I don't know if there is an explicit solution to the square-root example – maybe you can find one?]

4. If you're still in the mood and your code is running reasonably fast, you can run your code for a large number of paths for W_t and then take

the mean squared error of the approximation procedures. How fast are they getting smaller when the grid is getting finer?

5. Prove that the cross term in the Taylor expansion of $f(\bar{t}+\Delta t, W_{\bar{t}+\Delta t}) - f(\bar{t}, W_{\bar{t}})$, which is

$$\lim_{N\to\infty} \sum_i \left[\left.\frac{\partial f(t,W)}{\partial t}\right|_{t=s(i),W=W_{s(i)}} \left.\frac{\partial f(t,W)}{\partial W}\right|_{t=s(i),W=W_{s(i)}} (\Delta s)(\Delta W_{s_i}) \right] = 0$$

when we take ever finer choppings $S_N = \{s_1, \ldots, s_{10^N}\}$ into tenths, hundredths, etc. of a small interval $[\bar{t}, \bar{t}+\delta]$. As in class, argue that we can make δ so small that the derivatives don't change over our grid.

6. Apply the Ito rule to the following processes:

$$\begin{aligned} Y_t &= \frac{W_t^2 - t}{2} \\ Y_t &= \exp(\alpha t) \\ Y_t &= \exp\left[(\alpha - \sigma^2/2)t + \sigma W_t\right] \\ Y_t &= (\alpha t + \sigma W_t)^3 \end{aligned}$$

For the first three processes, you should see that they solve certain stochastic differential equations (SDE). Think about what the terms in the Ito rule and the SDE mean (i.e. how you would approximate them in Matlab), what the Ito Rule exactly claims in this case, what solves what and how we found the solution.

7. **(Ito's rule applied to a value function)** Suppose that an agent posesses one asset with price X_t. He cannot make any decision but keeping his one unit of this asset. Let his value at time t be given by a function $V(X_t)$ that is twice differentiable. X_t follows a diffusion (i.e. an Ito process whose drift and volatility only depend on the level of the process itself):

$$dX_t = a(X_t)dt + s(X_t)dW_t$$

where $a(\cdot)$ and $s(\cdot)$ are well-behaved functions. Give an approximation using the Ito-rule for what will happen to the value function when his asset is at a certain price $X_t = \bar{x}$ over a small increment of time Δt for different paths of W_t, i.e approximate $V(X_{t+\Delta t}) - V(X_t)$. How does this increment behave on expectation? Which properties of the value function matter? Give an intuition for each term.

8. **(Intuition for *smooth pasting*)** Suppose that an agent has a stock whose price behaves like a Brownian Motion:

$$dX_t = \sigma dW_t$$

The value for the agent at a certain stock price is

$$V(X_t) = aX_t + b(X_t - \bar{x})^+$$
$$(y)^+ \equiv \max\{0, y\}$$

Suppose that at time $\bar{t}$, the stock's price is exactly on the kink of the function, i.e. $X_{\bar{t}} = \bar{x}$. Since the function is not differentiable at this point, we cannot use the Ito Rule. Use the fact that $X_{\bar{t}+\Delta t} - X_{\bar{t}}$ is normally distributed to calculate the expected increment of the value function over a small time horizon Δt, i.e. calculate

$$A(\Delta t) \equiv E[V(X_{\bar{t}+\Delta t} - V(X_{\bar{t}})].$$

Use the fact that

$$\int_0^\infty \psi(x) x dx = k$$

for some positive constant k, where $\psi(\cdot)$ is the density of the standard normal distribution function.

(a) What happens to $A(\Delta t)$ as $\Delta t \to 0$?

(b) Now, interpret $V(X_t)$ is the value of taking an action at the respective price level X_t, and that this value was not discounted but invariant over time. Assume that waiting costs the agent a large amount of money, say $C\Delta t$ over a time horizon of Δt, C being very large. Prove that the agent would still choose to wait for a bit to exercise. Is this still true if we discount? And if we include an arbitrary drift αdt in the process for X_t?