Homework: The Kolmogorov Forward Equation

(by Matthias Kredler)

1. **(Building intuition for the forward equation)** Write down the forward equation for the following cases and formulate in words which effect the single terms on the right-hand side of the forward equation capture. If you have problems with the intuition, look at the approximation of the process on a trinomial tree!

   (a) $a(x) = \alpha x$, $s(x) = 0$

   (b) $a(x) = 0$, $s(x) = \sigma$

   (c) Only consider an interval $[c, d]$ with $0 < c < d$. Let $n(0, x) = \beta x$, $a(x) = 0$ and $\tilde{s}(x) = 1 + \sigma x$ on this interval. What happens to the density on a small increment of time Δt if $\beta > 0$? What if $\beta < 0$?

   (d) Again, consider an interval $[c, d]$ with $c > 0$. This time, let $n(0, x) = const. > 0$, $a(x) = 0$ and $\tilde{s}(x) = 1 + \sigma x + \omega x^2$. Why doesn't anything happen when $\omega = 0$? What happens if $\omega > 0$ or $\omega < 0$?

2. **(Invariant distribution of the Ornstein-Uhlenbeck process)** Consider the folllowing AR(1)-like process in continuous time, which is called *Ornstein-Uhlenbeck process*:

   $$dX_t = -\mu X_t dt + \sigma dW_t$$

   We guess that the invariant distribution is normal with mean zero and variance ω^2, which is to be found by the method of undetermined coefficients. Write down the forward equation for the process, then plug in the derivatives from the normal distribution we have guessed[1], find the correct ω^2 and show that the guess was corrrect. Compare your solution to the invariant distribution of the equivalent discrete-time process – are the solutions similar?

3. *(Dirac's delta)* Consider a Brownian Motion with variance σ that is killed at rate ν. The process is kept stationary by inserting the killed particles at $x = 0$ continuously.

[1] Note that the time derivative is zero – an essential property of an invariant distribution!

(a) For $x \neq 0$, the heat equation holds, adjusted for killing:

$$f_t = \frac{\sigma^2}{2} f_{xx} - \nu f$$

Setting $f_t = 0$ (stationarity), show that for the region $(0, \infty)$, the stationary density has to be of the form

$$f(x) = C \exp\left(-\underbrace{\frac{\sqrt{2\nu}}{\sigma}}_{\equiv \alpha} x \right)$$

(b) Using a symmetry argument, show that we must have

$$C = \frac{\sqrt{\nu}}{\sqrt{2}\sigma}$$

(c) Suppose we didn't inject the particles at just $x = 0$, but uniformly over a small interval $[-d, d]$. Then the forward equation in this region would be

$$f_t = \frac{\sigma^2}{2} f_{xx} - \nu f + \frac{\nu}{2d}$$

If f_x has to be the same from both sides for the border points d and $-d$, what does this mean for f_{xx} when we let $d \to 0$?

(d) When going all the way to the limit, we see that we get a jump in the first derivative – this is a "mass point" in the second derivative. Note that the PDE is satisfied even at $x = 0$ in the weak sense since

$$\frac{\sigma^2}{2} f_{xx} + \delta\nu = -\frac{\sigma^2}{2}\delta 2C\alpha - \delta\nu = 0$$

Why can we just forget about the term $-\nu f$?

4. Take derivative of the closed-form solution for heat-equation to prove that PDE holds

5. Find expected payoff for $X_t = K$ (strike) for $t \to T$ (expiration); connection to smooth pasting

6. Trick: Differentiate heat equation to get behavior of derivatives