Certainty equivalence principle in stochastic differential games: an inverse problem approach*

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Abstract

This paper aims to characterize a class of stochastic differential games which satisfy the certainty equivalence principle, beyond the cases with quadratic, linear or logarithmic value functions. We focus on scalar games with linear dynamics in the players' strategies and with separable payoff functionals. Our results are based on the resolution of an inverse problem that determines strictly concave utility functions of the players so that the game satisfies the certainty equivalence principle. Besides establishing necessary and sufficient conditions, the results obtained in this paper are also a tool for discovering new closed-form solutions, as we show in two specific applications: in a generalization of a dynamic advertising model, and in a game of noncooperative exploitation of a productive asset.

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1 Introduction

Many problems in economics, management and operations research are naturally modeled as differential games. When the decisions of the agents are affected by uncertain events, it is usual to model the movement of the state variable as subject to stochastic disturbances. The theory of stochastic differential games provides suitable tools to analyze the interaction of the decision agents in this framework. Standard references that study differential games are Mehlmann,[1] Başar and Olsder[2] or Dockner et al.[3] Long[4] and Jorgensen and Zaccour[5,6] constitute two recent updates of the literature on differential games that show the huge variety of models that can be analyzed with the help of the theory of differential games. The aim of this paper is to identify games for which the certainty equivalence principle holds. According to Theil,[7] certainty equivalence means that a decision agent who maximizes expected utility and takes actions based on the information available at the time of taking the decision, may neglect the disturbances and to suppose that the uncertain elements are settled at their mean values. This is an important property, as it means that the equilibrium of the deterministic game is robust, in the sense that it continues to be optimal, even if the system becomes exposed to zero mean random shocks in the state variable. We say that a Markov Perfect Nash Equilibrium (MPNE henceforth) is robust if it is also an MPNE of an associated stochastic differential game.

It is well known that linear quadratic stochastic dynamic games in which the random source is independent from both the state variable and the player's strategies, satisfy the certainty equivalence principle*. The question we address in this paper is whether the certainty equivalence principle holds in other classes of differential games, and how we can identify them. There are some results in this direction already. Games with linear value functions satisfy this principle. This is because the second-order derivatives of the value function are null, so the Hamilton-Jacobi-Bellman equation system (HJB system of equations henceforth) is the same for both the deterministic and the stochastic game. Relevant examples are Sorger,[9] where a non-linear marketing game of advertising is studied, and Yeung,[10] where a class of games with linear value functions is identified. In Kuwana,[11] it is shown that logarithmic utility is the only utility specification that satisfies this property in Merton's model with partially observable drift†. Our investigations show that, in a variety of games, beyond those with logarithmic, quadratic or linear value functions, the certainty equivalence principle holds. Our starting point, Theorem 2.1 establishes a necessary condition. It points out that, for an MPNE to be robust, it must be

*This is true only for the equilibrium based on linear strategies. Tsutsui and Mino[8] is one of the first papers dealing with nonlinear equilibria in linear quadratic deterministic differential games.

†In the class of games we analyze here, all variables are observable for every player.

the case that changes in shadow prices have constant variance. Theorem 2.1 is complemented with Theorem 3.1, which solves an inverse problem to determine utility functions that make the certainty equivalence principle hold.

The study of inverse optimum problems in economics has a long history, starting with Hahn[12] and Kurz;[13] Chang[14] extended the approach to the stochastic optimal growth model, and He and Huang[15] discussed a quite general inverse Merton's model. When a policy function can be rationalized by a well behaved utility function (i.e., strictly concave), it means that the prescribed behavior is consistent with an optimizing behavior. Inverse problems are easily handled with the Euler-Lagrange system of equations that directly characterizes the MPNE. Working with these equations in stochastic differential games constitutes a novel approach introduced in Josa-Fombellida and Rincón-Zapatero.[16] The Euler-Lagrange system is obtained from the HJB system upon differentiating the equations that constitute the HJB system with respect to the state variables[‡]. Working with the Euler-Lagrange system of equations is advantageous for studying the question we address here, since the equations are independent of the players' value functions, depending solely on their strategies (and the rest of the elements of the game). In Josa-Fombellida and Rincón-Zapatero,[16] the question of identifying games where the certainty equivalence principle holds was not addressed. Hence, although the current paper is based on the Euler-Lagrange system approach, it presents new results and techniques of independent interest.

Theorem 3.1 below applies to a class of scalar differential games with linear dynamics in the strategies of the players and separable payoffs. This class encompasses many interesting differential games. For instance, our method allows us to extend the aforementioned dynamic advertising model studied in Sorger[9§] and Prasad and Sethi.[20] In addition, we study the non-cooperative management of a stochastic productive asset. This enables resource games with a linear recruitment function and constant elasticity of variance dynamics (that is, the dynamics is driven by a CEV stochastic process)to satisfy the certainty equivalence principle for constant relative risk aversion (CRRA) utility functions with a suitable coefficient of risk aversion. Our approach allows us to discover new solutions in closed-form that, to our knowledge, are not available in the literature[¶].

The paper is organized as follows. Section 2 is devoted to the definition of the game and to presenting some general results, including the Euler-Lagrange equations and the necessary con-

[‡]Rincón-Zapatero[17] develops this methodology in deterministic games, allowing for non-smooth MPNE. Josa-Fombellida and Rincón-Zapatero[18] derive the Euler-Lagrange equations from the Maximum Principle in stochastic optimal control problems instead of using the HJB system.

[§]The model is the duopoly extension of a model first proposed in Sethi[19] in a single-player framework.

[¶]We have limited ourselves to the cases where we are able to find closed-form solutions, but Theorem 3.1 is a general result.

dition established in Theorem 2.1. Section 3 studies a class of scalar games with linear dynamics in the players' strategies and separable payoffs that satisfies the certainty equivalence principle. Theorem 3.1 gives sufficient conditions for the existence of strictly concave utility functions for which the MPNE is robust. An explicit expression for the player's value function is also provided. Section 4 studies two applications: a general advertising game and a productive asset game. Section 5, which establishes the conclusions of the paper, includes possible extensions and lines for further research.

2 Description of the game and general results

In this section, we describe the stochastic differential game model as well as the method employed to analyze whether the game enjoys the certainty equivalence principle. As stated in the Introduction, we use the Euler-Lagrange system of partial differential equations that directly characterizes the MPNE instead of the classical HJB system.

We consider an N–person differential game over a bounded or unbounded time interval. In the former case, T denotes the final date. The state process X, where $X(s) \in \mathcal{X} \subseteq \mathbb{R}^n$, $\forall s \in [0,T]$, satisfies the system of stochastic differential equations (SDEs henceforth)

$$dX(s) = f(s, X(s), u(s))\, ds + \sigma(s, X(s))\, dw(s), \quad t \leq s \leq T, \tag{1}$$

where f and σ are both assumed to be of class C^1 with respect to t, f is of class C^2 with respect to (x,u), and σ is of class C^2 with respect to x. Players' strategies are denoted by u^i, where $u^i(s) \in U^i \subseteq \mathbb{R}^n$, $\forall s \in [0,T]$, $i = 1,\ldots,N$, and $u = (u^1,\ldots,u^N)$ is a profile of strategies. As it is common in many games, we will assume that the equilibrium strategies are interior to U^i for each $i = 1,\ldots,N$. The random source is given by a d–dimensional Brownian motion $w(s)$ defined on a complete probabilistic space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by the Brownian motion w. The instantaneous utility function of player i is L^i and the bequest function, S^i. Given initial conditions $(t,x) \in [0,T] \times \mathcal{X}$ and an admissible profile u, the payoff function of each player to be maximized is

$$J^i(t,x;u) = \mathrm{E}_{tx} \left\{ \int_t^T e^{-\rho^i(s-t)}\, L^i(s, X(s), u(s))\, ds + e^{-\rho^i(T-t)} S^i(T, X(T)) \right\}, \tag{2}$$

where E_{tx} denotes conditional expectation, under the probability measure $\mathbb{P}$, given the initial condition $X(t) = x$. The functions L^i and S^i are both of class C^1 with respect to t, L^i is of class C^2 with respect to (x,u), and S^i is of class C^2 with respect to x. The constant $\rho^i \geq 0$ is the rate of discount. In the infinite horizon case, the bequest functions S^i are null, and ρ^i is supposed to be strictly positive for all i. In this case, if the problem is autonomous and the strategies are

Markov stationary, the value function is independent of time and the initial condition is simply x.

Definition 2.1 (Admissible strategies) *A profile u is admissible if $u^i(t) \in U^i$, all $t \in [0,T]$, for $i = 1, \ldots, N$ and*

(i) *for every $(t,x) \in [0,T] \times \mathcal{X}$, (1) admits a pathwise unique strong solution;*

(ii) *for each $i = 1, \ldots, N$, there exists a function ϕ^i with ϕ^i_t, ϕ^i_{tx} and ϕ^i_{xx} continuous, such that $u^i(s) = \phi^i(s, X(s))$ for every $s \in [0,T]$.*

Let $\mathcal{U}^i$ be the set of admissible strategies of player i and let $\mathcal{U} = \mathcal{U}^1 \times \cdots \times \mathcal{U}^N$.

Definition 2.2 (MPNE) *An N–tuple of strategies $\phi \in \mathcal{U}$ is called a Markov Perfect Nash Equilibrium if for every $(t,x) \in [0,T] \times \mathcal{X}$, for every $u^i \in \mathcal{U}^i$*

$$J^i(t,x;(u^i|\phi_{-i})) \leq J^i(t,x;\phi),$$

for all $i = 1, \ldots, N$.

In the above definition, $(u^i|\phi_{-i})$ denotes $(\phi^1, \ldots, \phi^{i-1}, u^i, \phi^{i-1}, \ldots, \phi^N)$.

Along the paper, we will use subscripts to denote partial derivatives, and primes to denote scalar derivatives. Also, ∂_z denotes total derivation with respect to the variable z; and for a matrix A, $\mathrm{tr}(A)$ is the trace of A, $A^\top$ denotes the transpose of A and $A^{-\top}$ denotes the transpose of the inverse of A, A^{-1}.

Given an MPNE ϕ, the value function of player i is $V^i(t,x) := J^i(t,x;\phi)$, the deterministic Hamiltonian is $H^i(s,x,u,p^i) := L^i(s,x,u) + (p^i)^\top f(s,x,u)$, and the costate function is

$$\Gamma^i(t,x,u) := -f_{u^i}^{-\top} L^i_{u^i}(t,x,u), \tag{3}$$

for all $i = 1, \ldots, N$.

As shown in Josa-Fombellida and Rincón-Zapatero,[16] an interior and smooth MPNE ϕ satisfies the Euler-Lagrange system of differential equations

$$\begin{aligned} -\rho^i \Gamma^i_j(t,x,\phi(t,x)) \quad &+ \quad \partial_t \Gamma^i_j(t,x,\phi(t,x)) + \partial_{x_j} H^i(t,x,\phi(t,x),\Gamma^i(t,x,\phi(t,x))) \\ &+ \quad \tfrac{1}{2}\partial_{x_j} \mathrm{tr}\left(\sigma(t,x)\sigma(t,x)^\top \partial_x \Gamma^i(t,x,\phi(t,x))\right) = 0, \end{aligned} \tag{4}$$

with final conditions $\phi^i(T,x) = \varphi^i(x)$, given implicitly by

$$\Gamma^i(T,x,\phi(T,x)) + S^i_x(T,x) = 0,$$

for $i = 1, \dots, N$ and $j = 1, \dots, n$. In the case where the game is autonomous, of infinite horizon, and the MPNE is stationary, the term $\partial_t \Gamma^i(x, \phi(x)) = 0$ vanishes.

Now, consider the associated deterministic game by taking $\sigma = 0$, so that the state equation becomes

$$dX(s) = f(s, X(s), u(s))ds, \quad s \in [t, T], \tag{5}$$

with initial condition $X(t) = x$. The objective of the ith player is to maximize in $u^i \in \mathcal{U}^i$

$$J^i(t, x; u^i | u_{-i}) = \int_t^T e^{-\rho^i(s-t)} L^i(s, X(s), u(s))ds + e^{-\rho^i(T-t)} S^i(T, X(T)), \tag{6}$$

$i = 1, \dots, N$, once the remaining players have fixed their strategies $u_{-i} \in \mathcal{U}_{-i}$.

Definition 2.3 (Robust MPNE) *We say that an MPNE of the associated deterministic game* (5), (6) *is robust if it is also an MPNE of the stochastic game* (1), (2).

Theorem 2.1 *Suppose that ϕ is a robust MPNE of the deterministic game. Then there exist functions A^i, such that for all $i = 1, \dots, N$, for all $x \in \mathcal{X}$, for all $t \in [0, T)$*

$$\operatorname{tr}\left((\sigma\sigma^\top)(t, x)\partial_x \Gamma^i(t, x, \phi(t, x))\right) = A^i(t). \tag{7}$$

Proof. System (4) also characterizes the deterministic game, which is obtained when σ is the null matrix. This is because the maximization condition of the Hamiltonian, for both the deterministic and the stochastic game, is the same, since σ is independent of the strategies of the players. This implies that $\Gamma^i(t, x, u)$ is also the costate variable of player i in the deterministic game. Hence, if ϕ is an MPNE of both the deterministic and the stochastic game, then the vector

$$\partial_x \operatorname{tr}\left((\sigma\sigma^\top)(t, x)\partial_x \Gamma^i(t, x, \phi(t, x))\right)$$

must be null for all $i = 1, \dots, N$; hence, $\operatorname{tr}\left((\sigma\sigma^\top)(t, x)\partial_x \Gamma^i(t, x, \phi(t, x))\right)$ depends only on t. □

3 A class of differential games satisfying the certainty equivalence principle

In the rest of the paper we focus on a particular family of stochastic differential games within the class described in Section 2. The games have the following features: the time horizon is unbounded, the dynamics is linear with respect to the players' strategies, the payoffs are separable, there is one state variable, and each player has only one strategy at his/her disposal. We will identify conditions on the elements that define the game, such that the certainty equivalence principle holds true. To carry out this identification, we solve an inverse problem, which consists of, given the rest of the elements that define the game, finding well behaved utility functions (*i.e.*, utility functions that are twice differentiable, as well as strictly concave in each player's strategies) for which the certainty equivalence principle is satisfied. To solve the inverse problem, we use the Euler-Lagrange equation (4), both for the deterministic and for the stochastic game, and the necessary condition established in Theorem 2.1.

We now describe the inverse problem. We suppose that, when not explicitly stated, the functions that appear below satisfy the differentiability conditions required in the previous section. The evolution of the state variable is given by the scalar stochastic process

$$dX(t) = \left(-\sum_{i=1}^{N} a_i(X(t))u^i(t) + b(X(t)) \right) dt + \sigma(X(t))dw(t), \tag{8}$$

where the functions a_i and b are given, and have a continuous derivative. Moreover, we assume that the functions a_i are not null and that they are monotone (not necessarily strictly), for all $i = 1, \ldots, N$. Our aim is to find a strictly concave utility function $\ell_i(u^i)$ such that, given the functions $h_i(x)$, the game with payoffs

$$\mathrm{E}_x \int_0^\infty e^{-\rho^i t} \Big(\ell_i(u^i(t)) + h_i(X(t)) \Big) dt, \tag{9}$$

for $i = 1, \ldots, N$, admits a robust MPNE. Hence, the utility function of the ith player is $L^i(x, u^i) = \ell_i(u^i) + h_i(x)$. Since that the game is autonomous, we have eliminated the time dependence in the interval of integration as well as in the expectation operator. Summing up, we set and study the following inverse problem:

Determine strictly concave and continuously differentiable utility functions, $\ell_1, \ldots, \ell_N$*, such that the stochastic differential game*

$$\mathrm{SDG} := \left((\ell_i)_{i=1}^N, (h_i)_{i=1}^N, (a_i)_{i=1}^N, b, \sigma \right),$$

and its associated deterministic differential game

$$\mathrm{DDG} := \left((\ell_i)_{i=1}^N, (h_i)_{i=1}^N, (a_i)_{i=1}^N, b, 0 \right),$$

have the same MPNE.

As defined in Section 2, an MPNE that solves both SDG and DDG, is a robust MPNE. In what follows we will denote

$$\Theta(x) = \int^x \frac{1}{\sigma^2(v)} dv, \tag{10}$$

the antiderivative of $1/\sigma^2(x)$, with null constant. The next proposition establishes that a robust and smooth MPNE must satisfy a system of linear differential equations where the expression of the unknowns functions $\ell_1, \ldots, \ell_N$ do not appear explicitly.

Proposition 3.1 *If ϕ is a robust MPNE, then there exist constants A^i, B^i such that ϕ satisfies the system of linear linear differential equations*

$$\begin{aligned}\Big(A^i\Theta(x)+B^i\Big)\sum_{j\neq i}^{N}(\phi^j)'(x)a_j(x) &= -\sum_{j=1}^{N}\Big(\tfrac{A^i}{\sigma^2(x)}a_j(x)+(A^i\Theta(x)+B^i)a_j'(x)\Big)\phi^j(x) \\ &+ (b'(x)-\rho^i)(A^i\Theta(x)+B^i)+h_i'(x)+\tfrac{A^i}{\sigma^2(x)}b(x),\end{aligned} \tag{11}$$

for all $x \in \mathcal{X}$, for all $i = 1, \ldots, N$.

Proof. If ϕ is a robust MPNE, then it satisfies the Euler-Lagrange equation (4) with $\sigma = 0$, which for this game becomes (we occasionally omit the dependence of ϕ^i on x in what follows to simplify the notation)

$$0 = -\rho^i\Gamma^i(x,\phi^i) + \partial_x\left(\ell_i(\phi^i) + h_i(x) + \Gamma^i(x,\phi^i)\left(-\sum_{i=1}^{N}a_i(x)\phi^i + b(x)\right)\right), \tag{12}$$

equation that, after expanding the x-derivative, becomes

$$\begin{aligned}0 = -\rho^i\Gamma^i + \ell_i'(\phi^i)(\phi^i)' + h_i'(x) + \partial_x\Gamma^i(x,\phi^i)\left(-\sum_{j=1}^{N}a_j(x)\phi^j + b(x)\right) \\ + \Gamma^i(x,\phi^i)\left(-\sum_{j=1}^{N}a_j'(x)\phi^j - \sum_{j=1}^{N}a_j(x)(\phi^j)' + b'(x)\right),\end{aligned} \tag{13}$$

where

$$\Gamma^i(x,u^i) = \frac{\ell_i'(u^i)}{a_i(x)}. \tag{14}$$

On the other hand, by Theorem 2.1, ϕ satisfies

$$\partial_x\Gamma^i(x,\phi^i) = \frac{A^i}{\sigma^2(x)} \tag{15}$$

for some constant A^i, and for all $i = 1, \ldots, N$. Integrating expression (15), we obtain

$$\Gamma^i(x,\phi^i(x)) = A^i\Theta(x) + B^i, \tag{16}$$

for another arbitrary constant B^i, for all $i = 1, \ldots, N$. Equating (14) and (16) with $u^i = \phi^i$, we find the following expression for the derivative of the utility function evaluated at the equilibrium strategy of the ith player

$$\ell_i'(\phi^i(x)) = a_i(x)(A^i\Theta(x) + B^i). \tag{17}$$

Substituting Γ^i, $\partial_x \Gamma^i$ and $\ell_i'(\phi^i)$ given in (16), (15) and (17), respectively, into (13), for $i = 1, \ldots, N,$ we obtain that ϕ satisfies the linear system of differential equations

$$\begin{aligned} 0 &= -\rho^i(A^i\Theta(x) + B^i) + (A^i\Theta(x) + B^i)a_i(x)(\phi^i)' + h_i'(x) + \frac{A^i}{\sigma^2}\left(-\sum_{j=1}^N a_j(x)\phi^j + b(x)\right) \\ &\quad + (A^i\Theta(x) + B^i)\left(-\sum_{j=1}^N a_j'(x)\phi^j - \sum_{j=1}^N a_j(x)(\phi^j)' + b'(x)\right). \end{aligned}$$

Rearranging terms, we obtain (11). □

Note that (11) is linear due to the linearity of the dynamics (8) with respect to the strategies of the players and to the separability of the players' payoffs. We could have set a more general framework, but at the cost of obtaining a nonlinear system of differential equations for the robust MPNE. As we wish, not only to give a theoretical result, but to find explicitly utilities and equilibria, we make the simplifying assumption (8). Another advantage of our game problem specification is that, as the system is linear, suitable assumptions on the coefficients would guarantee the existence and uniqueness of a global solution. We postulate the existence of a unique solution to (11) in the next theorem, which establishes sufficient conditions for the solvability of the inverse problem.

In what follows we use the notation ζ^i for the inverse of ϕ^i, that is, $x = \zeta^i(\phi^i(x))$, for all $x \in \mathcal{X}$, for all $i = 1, \ldots, N$. The inverse of ϕ^i exists under the assumptions of the theorem below.

Theorem 3.1 *Let ϕ be a solution of the system* (11) *for which each ϕ^i is twice continuously differentiable and strictly monotone in $\mathcal{X}$, and such that* (8)*, with initial condition $X(t) = x$, admits a unique strong solution X^ϕ for each $(t, x) \in [0, \infty) \times \mathbb{R}$. Let*

$$\ell_i(u^i) = \int^{u^i} \Big(A^i\Theta(\zeta^i(v)) + B^i\Big)a_i(\zeta^i(v))dv, \tag{18}$$

where A^i and B^i are constants, and let V^i be defined by

$$\begin{aligned} \rho^i V^i(x) &= \ell_i(\phi^i(x)) + h_i(x) - \Big(A^i\Theta(x) + B^i\Big)\sum_{j=1}^N a_j(x)\phi^j(x) \\ &\quad + (A^i\Theta(x) + B^i)b(x) + \frac{A^i}{2} \end{aligned} \tag{19}$$

for all $i = 1, \ldots, N$. Suppose that $a_i\Theta$ has a continuous derivative for all $i = 1, \ldots, N$, where Θ is defined in (10) *and that the functions a_i and $a_i\Theta$ are monotone, with at least one of them strictly monotone, for all $i = 1, \ldots, N$. Suppose further that the following transversality condition holds: for all $u^i \in \mathcal{U}^i$,*

$$\liminf_{T\to\infty} e^{-\rho^i T}\mathrm{E}_{tx}V^i(X^{u^i|\phi_{-i}}(T)) \geq 0, \tag{20}$$

and

$$\limsup_{T\to\infty} e^{-\rho^i T}\mathrm{E}_{tx}V^i(X^{\phi}(T)) \le 0, \tag{21}$$

for all $i = 1,\ldots,N$. Then

(a) The function ℓ_i is twice continuously differentiable and strictly concave for suitable constants A^i, B^i, for all $i = 1,\ldots,N$.

(b) The strategy profile ϕ is a robust MPNE.

(c) The function V^i is the (strictly concave) value function of player i, for all $i = 1,\ldots,N$.

Proof. It is clear that ℓ_i, as defined in (18), is twice continuously differentiable, for $i = 1,\ldots,N$. Taking the derivative in (18) with respect to u^i, we have

$$\ell_i'(u^i) = a_i(\zeta^i(u^i))(A^i\Theta(\zeta^i((u^i)) + B^i),$$

and differentiating again this expression and collecting terms, we obtain

$$\ell_i''(u^i) = (\zeta^i)'(u^i)\Big(A^i(a_i\Theta)'(\zeta^i((u^i)) + B^i a_i'(\zeta^i(u^i))\Big). \tag{22}$$

We want to show that it is possible to choose suitable constants A^i and B^i such that $\ell_i''(u^i) < 0$, for $i = 1,\ldots,N$. This will prove (a). Note that $(\zeta^i)'$ has the same sign of $(\phi^i)'$. By assumption, this is positive or negative for all $x \in \mathcal{X}$, since ϕ^i is strictly monotone. Assume, without loss of generality, that $(\zeta^i)' > 0$ for some player $i \in \{1,\ldots,N\}$. Since that both $a_i\Theta$ and a_i are monotone, and that at least one of these two functions is strictly monotone, it is possible to select constants $A^i \neq 0$ and $B^i \neq 0$ such that $A^i(a_i\Theta)' \le 0$ and $B^i a_i' \le 0$ for all $x \in \mathcal{X}$, and at least one of these two expressions is negative for all $x \in \mathcal{X}$. Thus, $\ell_i''(u^i) < 0$ as claimed. The case $(\zeta^i)' < 0$ is handled in the same way. This shows (a). We will prove (b) and (c) at once. We start by showing that (19) defines a solution of the HJB system of the stochastic differential game. To this end, we compute the first and second order derivatives of V. Plugging $\Gamma^i = A^i\Theta + B^i$ into (19), we have (again we occasionally omit the dependence of ϕ^i on x in what follows to simplify the notation)

$$\rho^i V^i(x) = \ell_i(\phi^i) + h_i(x) + \Gamma^i(x,\phi^i)\left(-\sum_{j=1}^{N} a_j(x)\phi^j + b(x)\right) + \frac{A^i}{2}. \tag{23}$$

Since ϕ satisfies (12), $-\rho^i\Gamma^i(x,\phi^i) + \partial_x\rho^i V^i(x) = 0$ holds true for all $x \in \mathcal{X}$; thus, $(V^i)'(x) =$

$\Gamma^i(x,\phi(x)) = A^i\Theta(x) + B^i$, and differentiating again, $(V^i)''(x) = \frac{A^i}{\sigma^2(x)}$. Then

$$\begin{aligned}
&-\rho V^i(x) + H^i(x,\phi(x),(V^i)'(x)) + \frac{1}{2}\sigma^2(x)(V^i)''(x)\\
&= -\rho^i V^i(x) + \ell_i(\phi^i) + h_i(x) + \left(-\sum_{i=1}^N a_i(x)\phi^i + b(x)\right)(V^i)'(x) + \frac{1}{2}\sigma^2(x)(V^i)''(x)\\
&= -\rho^i V^i(x) + \ell_i(\phi^i) + h_i(x) + \left(-\sum_{i=1}^N a_i(x)\phi^i + b(x)\right)\Gamma^i(x,\phi^i) + \frac{A^i}{2},
\end{aligned}$$

which is equal to zero by (23). Hence, we have proved that V^i satisfies the second order differential equation

$$-\rho^i V^i(x) + H^i(x,\phi(x),(V^i)'(x)) + \frac{1}{2}\sigma^2(x)(V^i)''(x) = 0 \tag{24}$$

for all $x \in \mathcal{X}$, for all $i = 1,\ldots,N$. Consider the Hamiltonian

$$H^i(x,(u^i|\phi_{-i}(x)),(V^i)'(x)) = \ell_i(u^i) + h_i(x) + \left(-u^i a_i(x) - \sum_{j\neq i}^N a_j(x)\phi^j(x) + b(x)\right)(V^i(x))'.$$

The partial derivative with respect to u^i evaluated at $u^i = \phi^i(x)$ is

$$\frac{\partial H^i}{\partial u^i}(x,(\phi^i(x)|\phi_{-i}(x)),(V^i)'(x)) = \ell_i'(\phi^i(x)) - a_i(x)(V^i)'(x),$$

for all $x \in \mathcal{X}$, for all $i = 1,\ldots,N$. Taking into account (14) and the identity $(V^i)'(x) = \Gamma^i(x,\phi(x))$, we have that $\ell_i'(\phi^i(x)) - a_i(x)(V^i)'(x) = 0$, for all $x \in \mathcal{X}$. Thus, $u^i = \phi^i(x)$ is a critical point of the function $u^i \longmapsto H^i(x,u^i|\phi_{-i}(x),(V^i)(x))$, for all $x \in \mathcal{X}$. Since

$$\frac{\partial^2 H^i}{\partial (u^i)^2}(x,(u^i|\phi_{-i}(x)),(V^i)'(x)) = \ell_i''(u^i) < 0,$$

for all u^i, $u^i = \phi^i(x)$ is the unique global maximum of $u^i \longmapsto H^i(x,u^i|\phi_{-i})(x),(V^i)(x))$. Hence

$$H^i(x,(\phi^i(x)|\phi_{-i}(x)),(V^i)'(x)) = \max_{u^i} H^i(x,(u^i|\phi_{-i}(x)),(V^i)'(x))$$

for all u^i, for all $x \in \mathcal{X}$ and for all $i = 1,\ldots,N$. Thereby, (24) is the HJB equation for an MPNE

$$0 = -\rho^i V^i(x) + \max_{u^i} H^i(x,(u^i,\phi_{-i}(x)),(V^i)'(x)) + \frac{1}{2}\sigma^2(x)(V^i)''(x).$$

Finally, the transversality conditions (20) and (21) allow us to apply a Verification Theorem, see Fleming and Soner[21] Ch. III Th. 9.1—turning minimizing to maximizing— or Dockner et al.[3] Th. 8.5. □

Regarding the assumptions imposed in Theorem 3.1, the monotonicity of the functions a_i and $a_i\Theta$, for all $i = 1,\ldots,N$, play an important role to show that the functions $\ell_1,\ldots,\ell_N$ that solve the inverse problem are strictly concave. The transversality conditions (20) and (21) are adapted from Fleming and Soner[21] Ch. III Th. 9.1—turning minimizing to maximizing—from an optimal control problem to a differential game problem. They are the counterpart of the boundary conditions satisfied by the costate variable and the optimal controls in finite horizon games in infinite horizon problems. Contrary to the finite horizon case, however, (20) and (21) are not necessary conditions of optimality for infinite horizon problems any more. They are only sufficient conditions. Dockner et al.[3] discuss the role of transversality conditions in infinite horizon deterministic and stochastic differential games, and establish several forms depending on the optimality criteria chosen. It is worth mentioning that condition (20) holds true if the value function V^i is bounded below. The two differential games models that we study in Section 4 satisfy this requirement, since the player's value functions are nonnegative. Regarding (21), it imposes that the growth rate of the expected optimal utility, that a player may obtain along the equilibrium in the long run, is smaller than the player's discount rate, $\rho^i > 0$. We check that this property holds true in the two differential games that we analyze in Section 4.

Remark 3.1 (Optimal Control Problem) *In control problems, where* $N = 1$, (11) *is an algebraic equation, not a differential one. Introducing the notation* $a_i = a$, $h_i = h$, $A^i = A$ *and* $B^i = B$ *for all* $i = 1, \dots, N$ *and solving, we have that*

$$\phi(x) = \frac{(b' - \rho)(A\Theta(x) + B) + h'(x) + \frac{A}{\sigma^2(x)} b(x)}{\frac{A}{\sigma^2(x)} a(x) + (A\Theta(x) + B)a'(x)},$$

is a candidate to be a robust control.

Remark 3.2 (Symmetric Game) *Consider a symmetric game with* $N > 1$ *players and let* ϕ *be a robust symmetric MPNE. As in the remark above, we denote all functions and constants defining the game without indexes, as well as the introduced constants* A *and* B*. Observe that, with the assumptions of Theorem 3.1, the sign of* $A\Theta(x) + B$ *is well defined, positive or negative for all* x*. The linear ODE* (11) *takes the form*

$$\phi'(x) = P(x)\phi(x) + Q(x), \tag{25}$$

where the coefficients are

$$\begin{aligned} P(x) &= \frac{N}{1-N}\left(\frac{A}{\sigma^2(x)(A\Theta(x)+B)} + \frac{a'(x)}{a(x)}\right); \\ Q(x) &= \frac{1}{(N-1)a(x)}\left(b'(x) - \rho + \frac{h'(x)}{A\Theta(x)+B} + \frac{Ab(x)}{\sigma^2(x)(A\Theta(x)+B)}\right). \end{aligned}$$

An integrating factor is $|A\Theta(x)+B|^{\frac{N}{1-N}}a(x)^{\frac{N}{1-N}}$. *In consequence, the general solution of* (25) *is*

$$\phi(x)=|A\Theta(x)+B|^{\frac{N}{N-1}}a(x)^{\frac{N}{N-1}}\left(\int^x Q(z)|A\Theta(z)+B|^{\frac{N}{1-N}}a(z)^{\frac{N}{1-N}}dz+C\right),$$

where C *is an arbitrary constant. This is a candidate for robust MPNE, for the game with the utility function* ℓ_i *as given in* (18). *We will use this formula in Section 4.2 below.*

Remark 3.3 (Linear Value Functions) *It has been proven in Theorem 3.1 that* $(V^i)'(x)=\Gamma^i(x)=A^i\Theta(x)+B^i$. *Hence, the value function is linear in* x *for player* i *if it is possible to choose* $A^i=0$. *Note that it is not possible to take* $\Theta(x)$, *defined in* (10), *constant because* $1/\sigma^2(x)\neq 0$. *It is important to note that, from* (22) *in the proof of the theorem, the selection* $A^i=0$ *is possible only if* a_i' *does not vanish; otherwise, the function* ℓ_i *constructed in the theorem is not strictly concave, and there is no guarantee that the solution of the system* (11), *be a Nash equilibrium of the Hamiltonians of the players. See Section 4.1 below for a game with linear value function.*

4 Examples

In this section, we illustrate the theory developed in the previous sections by means of two stochastic differential games models. They are not academic models, but reflect important economic behavior of the agents that interact in a noncooperative environment under uncertainty. The first game deals with competition between firms to capture costumers through advertising. The second game models competition between economic agents to exploit a renewable resource or productive asset. Both games present a rather different structure. While the value function of the first game is linear, which easily explains why the MPNE is robust, the second game's value function is of the CRRA family. In this case, it is not evident whether it satisfies the certainty equivalence principle. Hence, this model constitutes a positive test of the usefulness of our approach.

4.1 A dynamic advertising game

Consider the stochastic differential game of competitive dynamic advertising of two firms studied in Sorger[9] in its infinite horizon formulation. Two firms compete for market shares through advertising effort. We denote the market share of firm 1 at time t by $X(t)$ and assume that the size of the total market is constant over time. Normalizing the total market to 1, we obtain that $1-X(t)$ is the market share of firm 2 at time t. Let us denote then $X_1(t)=X(t)$ and

$X_2(t) = 1 - X(t)$, so $X_i(t)$ represents the market shares of firm i at time t. The objective functional to be maximized for firm i is

$$J^i(t,x;u^1,u^2) = \mathrm{E}_{tx}\int_0^\infty e^{-\rho^i t}\left(q_i X_i(t) - C_i(u^i(t))\right)dt, \tag{26}$$

for $i = 1,2$, and dynamics

$$dX(t)\Big(\delta_1(1-X(t))^{\frac{1}{1+m_1}}u^1(t) - \delta_2 X(t)^{\frac{1}{1+m_2}}u^2(t) - \delta(2X(t)-1)\Big)dt + \sigma(X(t))dw(t), \tag{27}$$

with $X(0) = x \in \mathcal{X} = [0,1]$, and where $m_i > 0$, for $i = 1,2$. The cost function C_i has to be determined so that the MPNE of the game is robust. The model specification in Sorger[9] is obtained with $C_i(u^i) = c_i\frac{(u^i)^2}{2}$ and $m_1 = m_2 = 1$, $\delta_1 = \delta_2 = 1$, $\delta = 0$. Prasad and Sethi[20] allows for $\delta_1, \delta_2, \delta > 0$. In the above, $u^i(t)$ is the advertising rate at time t, $\rho^i > 0$ is the constant discount rate and $q_i > 0$ is the constant revenue per unit of market share of firm i, for $i = 1,2$. The diffusion parameter $\sigma(x) \geq 0$ satisfies $\sigma(0) = \sigma(1) = 0$. The state dynamics reflects two facts, already present in the classical Vidale-Wolfe advertising model, Vidale and Wolfe:[22] (i) a concave saturation effect in the capture of new costumers, and (ii) a positive (*resp.* negative) effect of own (*resp.* competitor) advertising spending. In comparison with the original game, we allow here for asymmetric market responses, even if the advertising effectiveness parameters δ_1 and δ_2 are equal. The churn parameter $\delta > 0$, accounts for declining effects in market shares due to other causes than advertising from the competitor firm, such as product obsolescence or lack of product differentiation.

In the notation of Section 3, $a_1(x) = -\delta_1(1-x)^{\frac{1}{1+m_1}}$, $a_2(x) = \delta_2 x^{\frac{1}{1+m_2}}$, $b(x) = -\delta(2x-1)$ and $h_1(x) = q_1 x$, $h_2(x) = q_2(1-x)$. By (3)

$$\Gamma^1(x,u^1) = C_1'(u^1)\delta_1(1-x)^{-\frac{1}{1+m_1}}, \quad \Gamma^2(x,u^2) = -C_2'(u^2)\delta_2 x^{-\frac{1}{1+m_2}}.$$

Linear value functions require (see Remark 3.3)

$$C_1'(u^1(x)) = B^1(1-x)^{\frac{1}{1+m_1}}, \quad C_2'(u^2(x)) = -B^2 x^{\frac{1}{1+m_2}}$$

for suitable constants $B^1 > 0$, $B^2 < 0$, so (7) is satisfied independently of $\sigma(x)$. We still have to check that the these two strategies solve the system (11), which in this particular game become (with $A^i = 0$)

$$\begin{aligned}
B^1\delta_2 x^{\frac{1}{1+m_2}}(u^2)' &= -B^1\frac{\delta_1}{1+m_1}(1-x)^{\frac{-m_1}{1+m_1}}u^1 - B^1\frac{\delta_2}{1+m_2}x^{\frac{-m_2}{1+m_2}}u^2 \\
&\quad + B^1(-2\delta - \rho^1) + q_1, \\
-B^2\delta_1(1-x)^{\frac{1}{1+m_1}}(u^1)' &= -B^2\frac{\delta_1}{1+m_1}(1-x)^{\frac{-m_1}{1+m_1}}u^1 - B^2\frac{\delta_2}{1+m_2}x^{\frac{-m_2}{1+m_2}}u^2 \\
&\quad + B^2(-2\delta - \rho^2) - q_2.
\end{aligned}$$

The structure of these equations suggests a solution (u^1, u^2) of the form $u^1(x) = \eta_1(1-x)^{\frac{m_1}{1+m_1}}$, $u^2(x) = \eta_2 x^{\frac{m_2}{1+m_2}}$, with $\eta_i > 0$, $i = 1, 2$. After substitution and collection of terms, the above differential system reduces to the following pair of algebraic relations

$$\begin{aligned} B^1\left(\eta_2\delta_2 + \eta_1\tfrac{\delta_1}{1+m_1} + 2\delta + \rho^1\right) - q_1 &= 0, \\ B^2\left(\eta_1\delta_1 + \eta_2\tfrac{\delta_2}{1+m_2} + 2\delta + \rho^2\right) + q_2 &= 0. \end{aligned} \tag{28}$$

Let $\zeta^1(v) = 1 - (\frac{v}{\eta_1})^{\frac{1+m_1}{m_1}}$ and $\zeta^2(v) = (\frac{v}{\eta_2})^{\frac{1+m_2}{m_2}}$, the inverse functions of u^1 and u^2, respectively. From (18), we have the cost functions (remember that $A^i = 0$)

$$\begin{aligned} \ell_1(u^1) &= -C_1(u^1) = -B^1\int^{u^1} a_1(\zeta^1(v))dv = B^1\frac{\delta_1}{\eta_1^{\frac{1}{m_1}}}\frac{(u^1)^{1+\frac{1}{m_1}}}{1+\frac{1}{m_1}}; \\ \ell_2(u^2) &= -C_2(u^2) = -B^2\int^{u^2} a_2(\zeta^2(v))dv = -B^2\frac{\delta_2}{\eta_2^{\frac{1}{m_2}}}\frac{(u^2)^{1+\frac{1}{m_2}}}{1+\frac{1}{m_2}}. \end{aligned} \tag{29}$$

If we denote $c_1 = B^1\frac{\delta_1}{\eta_1^{\frac{1}{m_1}}}$ and $c_2 = -B^2\frac{\delta_2}{\eta_2^{\frac{1}{m_2}}}$, then both $c_1, c_2 > 0$. Solving for B^1 and B^2 and plugging these values into the system (28), we obtain

$$\begin{aligned} c_1\eta_1^{\frac{1}{m_1}}\left(\eta_2\delta_2 + \eta_1\frac{\delta_1}{1+m_1} + 2\delta + \rho^1\right) - \delta_1 q_1 = &\ 0, \\ c_2\eta_2^{\frac{1}{m_2}}\left(\eta_1\delta_1 + \eta_2\frac{\delta_2}{1+m_2} + 2\delta + \rho^2\right) - \delta_2 q_2 = &\ 0. \end{aligned} \tag{30}$$

Given c_i, m_i, ρ^i, q_i, δ_i, $i = 1, 2$ and δ, the existence of positive solutions η_1, η_2 of this algebraic system guarantees the existence of a robust MPNE $\phi(x) = (\eta_1(1-x)^{\frac{m_1}{1+m_1}}, \eta_2 x^{\frac{m_2}{1+m_2}})$. This is because the SDE for the optimal path X^ϕ is linear, hence a unique strong solution exists. It is straightforward to check the rest of the conditions of Theorem 3.1. For instance, in order to check the transversality conditions we note that the value functions are linear, as can be easily realized from (19). Then, substituting the robust MPNE ϕ in (27), the transversality conditions are a consequence of

$$\begin{aligned} &\lim_{T\to\infty} e^{-\rho^i T}\mathrm{E}_x X^\phi(T) \\ &\quad = \lim_{T\to\infty}\left(\frac{\delta_1\eta_1+\delta}{\delta_1\eta_1+\delta_2\eta_2+2\delta}e^{-\rho^i T} + \left(x - \frac{\delta_1\eta_1+\delta}{\delta_1\eta_1+\delta_2\eta_2+2\delta}\right)e^{-\rho^i T}e^{-(\delta_1\eta_1+\delta_2\eta_2+2\delta)T}\right) = 0, \end{aligned}$$

for $i = 1, 2$, because $\delta_i, \delta, \eta_i, \rho^i > 0$, for $i = 1, 2$. We collect our findings in the following proposition.

Proposition 4.1 *Let the advertising game given by* (26) *and* (27)*, with cost functions* C_1, C_2 *given in* (29)*, be such that* (30) *admits a positive solution* (η_1, η_2)*. Then the certainty equivalence principle holds for the game and the strategy profile* $\phi(x) = (\eta_1(1-x)^{\frac{m_1}{1+m_1}}, \eta_2 x^{\frac{m_2}{1+m_2}})$ *is a robust MPNE. Moreover, the players' value functions are linear in* x.

4.2 A stochastic productive asset game

Let an N player symmetric noncooperative differential game where each player i consumes at rate $c^i \geq 0$ from a stochastic productive asset X. The payoff functional is

$$J^i(t,x;(c^1,\ldots,c^N)) = E_{tx}\left\{\int_t^\infty e^{-\rho^i(s-t)}\ell(c^i(s))ds\right\}, \tag{31}$$

subject to

$$dX(s) = \left(F(X(s)) - \sum_{i=1}^N c^i(s)\right)ds + \sigma(X(s))dw(s), \qquad X(t) = x > 0. \tag{32}$$

The function F is the recruitment/production function. The class of admissible strategies $\mathcal{U}^i$ for each player is as in Definition 2.1, with the additional condition that $X \geq 0$ almost sure is required, that is to say $\mathcal{X} = [0,\infty)$. Further conditions on $\mathcal{U}^i$ will be given in each of the specific cases we consider below. This game has been analyzed in detail in Josa-Fombellida and Rincón-Zapatero,[16] where we provide necessary and sufficient conditions for existence of a unique and smooth MPNE of the finite horizon game. Here we complete the study by analyzing whether the game satisfies the certainty equivalence principle.

According to Remark 3.2, and taking $B = 0$, a robust equilibrium must satisfy (25) with coefficients

$$P(x) = \frac{\frac{N}{1-N}}{\sigma^2(x)\Theta(x)}, \quad Q(x) = \frac{1}{1-N}\left(\rho - F'(x) - \frac{F(x)}{\sigma^2(x)\Theta(x)}\right). \tag{33}$$

Recall that Θ is the primitive of $1/\sigma^2(x)$ with null constant. We consider a linear production function[‖]

$$F(x) = \mu x,\ \ \mu \geq 0 \tag{34}$$

and assume a CEV model, that is

$$\sigma(x) = \sigma x^a,\ \text{with } 1 - \tfrac{1}{2N} < a < 1 \text{ and } \rho > 2\mu(1-a). \tag{35}$$

In this case, we have

$$\Theta(x) = \frac{1}{\sigma^2(1-2a)}x^{1-2a}, \quad P(x) = \frac{N}{1-N}\frac{1-2a}{x}, \quad Q(x) = \frac{\rho + 2\mu(a-1)}{1-N}.$$

Note that $\Theta(x) < 0$ for all $x > 0$, since $a > \frac{1}{2}$, hence we take $A < 0$ and $B = 0$. By Remark 3.2, $\phi(x) = \beta x + \eta x^{(2a-1)N/(N-1)}$ is a solution of (33), where $\beta = \frac{\rho - 2\mu(1-a)}{1-2N(1-a)}$ is positive given our

‖ We have also solved the inverse problem for linear F and σ, as well as for a square root recruitment function, $F(x) = \mu\sqrt{x}$, and linear σ. Readers interested in the details will receive a copy of our computations and a proof of optimality, upon request.

assumptions, and $\eta \geq 0$. Note that ϕ is smooth, positive and increasing in $(0,\infty)$ and $\phi(0)=0$. We consider only the case with $\eta = 0$, so the inverse of ϕ is $\zeta(c) = c/\beta$. Substituting into (18) and taking $A = \sigma^2(1-2a)\beta^{1-2a}$, we find that the utility function is of the CRRA class, $\ell(c) = \frac{c^{2(1-a)}}{2(1-a)}$. Letting $\theta \equiv 2a-1$, this can be rewritten $\ell(c) = \frac{c^{1-\theta}}{1-\theta}$. The constraints on a imply $1-\frac{1}{N} < \theta < 1$. In terms of θ, the diffusion coefficient is then $\sigma(x) = \sigma x^{(1+\theta)/2}$. At equilibrium, the asset evolves according to the SDE

$$dX^{\phi}(s) \equiv dX(s) = (\mu - N\beta)X(s)ds + \sigma X(s)^{(1+\theta)/2}dw(s), \quad X(0) = x > 0. \tag{36}$$

Regarding the existence of solutions to (36), the functions $1/\sigma^2(x)$ and $x/\sigma^2(x)$ are locally integrable Borel functions in $(0,\infty)$. Indeed, both functions are continuous on any compact subset of $(0,\infty)$, and are thus integrable. Hence, the SDE (36) admits a unique-in-law weak solution, see Karatzas and Shreve[23] Ch. V, Th. 5.15. In fact, since the coefficients are locally Lipschitz, (36) admits a pathwise unique strong solution up to exit time of the interval $(0,\infty)$, see Karatzas and Shreve[23] Ch. IX, Ex. (2.10). To continue with the proof, we restrict the class of admissible strategies $\mathcal{U}^i$ to those elements c^i which satisfy $0 \leq c^i \leq kx$ for suitable k. For $c^i \in \mathcal{U}^i$, let the process X^{c^i} be given by

$$dX^{c^i}(s) = (\mu X^{c^i}(s) - c^i(s) - (N-1)\beta X^{c^i}(s))ds + \sigma(X^{c^i}(s))^{(1+\theta)/2}dw(s),$$

$X^{c^i}(0) = x >$. Let $\hat{X}$ with $\hat{X}(0) = x$ and

$$d\hat{X}(s) = (\mu - k - (N-1)\beta)\hat{X}(s)ds + \sigma\hat{X}(s)^{(1+\theta)/2}dw(s). \tag{37}$$

By a comparison theorem in Ikeda and Watanabe,[24] $X^{c^i} \geq \hat{X}$. Let $\mu_0 = \mu - k - (N-1)\beta$. By Example 3.2 in Mijatović and Urusov,[25] the exponential expression given by

$$M(t) = \exp\left(-\frac{\mu_0}{\sigma}\int_0^t \hat{X}(s)^{(1-\theta)/2}dw(s) - \frac{1}{2}\frac{\mu_0^2}{\sigma^2}\int_0^t \hat{X}(s)^{1-\theta}ds\right)$$

is a uniformly integrable martingale, since $\frac{1+\theta}{2} < 1$. This implies that $\hat{X} \in (0,\infty)$ with probability one, see Theorem 2.1 in Mijatović and Urusov.[25] In consequence, $V(X^{c^i}(T))$ is well defined and $V(X^{c^i}(T)) \geq V(\hat{X}(T)) \geq 0$ a.e., so (20) in Theorem 3.1 trivially holds. On the other hand, $X^{\phi} \leq \tilde{X}$, where $d\tilde{X}(s) = \mu\tilde{X}(s)ds + \sigma\tilde{X}(s)^{(1+\theta)/2}dw(s)$, and $\tilde{X}(0) = x$. Note that $e^{-\mu t}\tilde{X}(t) = x + \sigma\int_0^t e^{-\mu s}\tilde{X}^{(1+\theta)/2}(s)dw(s)$ is a nonnegative local martingale, thus a supermartingale, hence $E_x(e^{-\mu T}\tilde{X}(T)) \leq x$. This implies that X^{ϕ} does not exit at ∞. Moreover, since the value function is increasing and concave, then

$$E_x(V(X^{\phi}(T)) = \frac{A}{1-\theta}E_x(X^{\phi}(T))^{1-\theta} \leq \frac{A}{1-\theta}(E_x\tilde{X}(T))^{1-\theta} \leq \frac{A}{1-\theta}x^{1-\theta}e^{\mu(1-\theta)T},$$

by Jensen's inequality. Thus, condition (21) holds, because $\rho > \mu(1-\theta)$, by (35). We have proved the following proposition.

Proposition 4.2 *Let the symmetric productive asset game with N players be that given by* (31), (32), (34) *and* (35)*. Then the certainty equivalence principle holds for the game with the utility function $\ell(c) = \frac{c^{1-\theta}}{1-\theta}$, and the strategy profile $\phi(x) = (\beta x, \ldots, \beta x)$ is a robust MPNE, where*

$$\theta = 2a - 1, \quad \beta = \frac{\rho - 2\mu(1-a)}{1 - 2N(1-a)}.$$

5 Conclusions

This paper studies whether the certainty equivalence property holds in games beyond the well-known linear quadratic case and games with linear or logarithmic value functions. To approach the problem through the value function and the HJB equations is not easy, at least when the diffusion coefficient depends on the state variable, since then—with the exception of games where the value function is linear—the value function will be different in the stochastic and in the deterministic case. Hence, we have chosen to work with the Euler-Lagrange equations, which deal directly with the MPNE. As the MPNE is the same for both the deterministic and the stochastic differential games, the Euler-Lagrange equations constitute an overdetermined system. The existence of a common solution provides convenient information to solve an inverse problem that characterizes the utility functions of the players. We have shown how our approach can be used to find closed-form solutions to games for which a solution was not known. Further research can be conducted in three directions at least: (i) to work more examples than those analyzed here; (ii) to extend Theorem 3.1 to cover stochastic differential games where the players may affect the size of the uncertainty, and (iii) to consider stochastic processes where the uncertainty is more general than Brownian uncertainty.

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