Stochastic pension funding when the benefit and the risky asset follow jump diffusion processes

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Abstract

We study the asset allocation of defined benefit pension plans of the type designed and sponsored by firms with the aim of providing a lifetime pension to the employees at the age of retirement. Benefits are stochastic, combining Poisson jumps with Brownian uncertainty. The sponsor dynamically forms portfolios where the risky asset is also subjected to Poisson jumps and Brownian uncertainty, possibly correlated with the evolution of benefits. The objective is to assure future benefits, while controlling the contribution made to the fund reserves. The problem is solved analytically using dynamic programming techniques.

Keywords: Optimization in Financial Mathematics; Pension funding; Stochastic control; Poisson process

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1 Introduction

The objective of this paper is to study the asset allocation of defined benefit pension plans of the type designed and sponsored by firms with the aim of providing a lifetime pension to the employees at the age of retirement. A pension plan may serve as an instrument to real-locate individuals' wealth from their working life to retirement, a period where typically there might be no source of wealth to sustain consumption other than a low support from the state. Moreover, pension plans have become influential institutions in the financial markets for their high capitalization. However, in most developed countries the pension system is been subject to controversy and concern because the age pyramid is changing due to the reduction of the birth rate and a longer life expectancy, which, in our context, means that a smaller workforce should fund a large number of retirees. This demographic change has caused a steady increase in contributions, resulting in excessive costs to the pension plan sponsor. However, depending on the performance of the real economy, even the most aggressive contribution scheme cannot guarantee a real level of future benefits. To mitigate this problem the sponsor may form a risky portfolio to earn higher returns, assuming then some risk.

Our model is purely actuarial, leaving aside the evolution of the real economy. All information is supposed to be captured in the evolution of the financial market and in the pension benefits. Note also that we do not include any explanation of labor supply shortfalls caused by current demographic trends, and we take these as given. The sponsor does not care about the overall economic progress, nor does it have tools to influence it. The sponsor is only concerned with the evolution of the pension plan, and we cannot infer any consequence for the output of the economy. Hence we are examining here the "book balancing" problem, which is an interesting endeavor for actuaries.

There are two major types of pension plans: Defined Benefit (DB therefore) and Defined Contribution. In a DB plan, the benefits are fixed in advance and contributions are designed to maintain the fund in balance, that is, to fund employees' promised benefits. Usually, benefits are linked to salaries, and the contributions are shared by employer and employee. The sponsor bears the risk of funding the pension fund to assure future benefits, and the employee does not suffer possible investment losses. In contrast, in the latter scheme, the individual builds his/her own pension fund, selecting a fixed contribution rate and an investment strategy across assets, such as equities and bonds. Benefits are not fixed anymore, but the inherent risk is entirely borne by the individual. It is important to note that the decisions regarding contributions and investment depends, of course, on the member's preferences. In a DB plan, it is also necessary to set preferences that capture appropriate objectives of the sponsor. We choose, as is customary in the literature, quadratic preferences as in Haberman and Sung (1994) or Josa–Fombellida and Rincón–Zapatero (2001). Quadratic preferences are adequate to model the sponsor's concern towards the solvency risk and the amortization rate risk. These risks are defined as quadratic deviations of the fund level and the contribution rate from liabilities and normal cost, respectively. Whereas solvency risk is related with the security of the pension fund in attaining the promised liabilities, the contribution risk takes care of the stability of the pension fund scheme. Liabilities are simply the aggregation of all future benefits of retirees. Note that we avoid the consideration of different costs of being over/underfunded, although the asymmetry of costs might be more appropriate in some contexts. For example, firms with different tax shelters or different financial slack, as in Bodie *et al* (1987), could value differently the over/underfunded phases. From a technical point of view, asymmetry leads to piecewise quadratic preferences, henceforth to a non–smooth cost function. The problem becomes harder to solve, and no explicit solution seems to be available.

We are interested in aggregated pension plans of DB type, where benefits are stochastic and include both Brownian motions and Poisson jumps. The interest in considering jumps in the evolution of benefits is justified from several facts. First, since benefits are generally linked with salaries, sudden shocks in the latter give rise shocks in the former. The recent economic and financial crisis, which especially affects the peripheral members of the Euro zone, has motivated a significative reduction in salaries of civil servants. Wage reduction could not have been planned in advance, thus it can be considered as a random event resulting in a jump down of benefits. This random event seems to have originated in the financial markets, thus it is necessary to model benefits as correlated with jumps affecting the risky assets. Second, changes in legislation may lead to changes in the valuation of benefits. An example is a recent UK government edict, that base future cost–of–living increases on the consumer price index rather than the generally higher retail price index; in our context, this implies a change in the technical rate of actualization to be defined below. Consequently, liabilities would fall suddenly.

This basic framework has already been explored by us with dynamic programming methods, Josa–Fombellida and Rincón–Zapatero (2001, 2004, 2008a, 2008b, 2010), and by many other authors, such as Battocchio *et al* (2007), Berkelaar and Kouwenberg (2003), Cairns (2000), Chang (1999), Chang *et al* (2003), Haberman and Sung (1994, 2005), Haberman *et al* (2000) or Taylor (2002). Motivated by the possibilities of jumps in the real evolution of pension plans, in this paper, we add jumps to the stochastic processes governing the evolution of benefits and risky assets, with the aim of obtaining a closed–form solution that allows us to isolate the effects of the jumps both in the optimal investment strategy and the optimal contribution rate.

It is likely that the first paper considering Poisson jumps in dynamic asset allocation in continuous time is Merton (1971). In the pension funding framework, Ngwira and Gerrard (2007) considers the optimal management of a defined benefit pension plan, built on the model of Josa–Fombellida and Rincón–Zapatero (2004), but in a finite horizon and where jumps only appear in the risky asset, as in Wu (2003), but without considering correlation with benefits.

The inclusion of Poisson uncertainty requires the use of a more general Hamilton–Jacobi–Bellman equation (HJB therefore) and verification theorems *á la* Fleming and Soner (2006). The monograph of Oksendal and Sulem (2005) provides a framework to analyze these problems with dynamic programming methods that cover general Lévy processes.

We find that Poisson jumps affect the optimal solutions and the optimal fund evolution in a simple way, and that their effects can be neatly isolated. The solution obtained in Josa–Fombellida and Rincón–Zapatero (2004) now contemplates an additional summand that includes various parameters of the Poisson jumps. Moreover, we show how it is possible to select the technical rate of actualization in order to get an optimal contribution proportional to the unfunded liabilities, even in this complicated framework. This rule of amortization is well known and has drawn attention among practitioners for its ease of implementation, and has been proposed in the literature as having good properties of stabilization of the fund. We found it there as a byproduct of the minimization of solvency risk and contribution rate risk.

The paper is organized as follows. Section 2 defines the elements of the pension scheme and describes the financial market where the fund operates. To simplify matters, we consider that the fund's wealth is invested in a portfolio with a single risky asset and a bond. Section 3 is devoted to accurately formulating the management of the DB plan as a stochastic optimal control problem with the objective of minimizing the solvency risk and the contribution rate risk over an infinite horizon. We suppose that the pension plan stands forever and that the pay–as–you–go scheme never breaks. The optimal solutions are obtained by solving the HJB equation, and the optimal contribution and optimal investment strategy are provided, together with explanations of their main properties, as well as of the optimal fund. Section 4 serves as a numerical illustration of previous results. Finally, Section 5 establishes some conclusions. All proofs are relegated to Appendix A.

2 The pension model

Consider a DB pension plan of aggregated type where, at every instant of time, active participants coexist with retired participants. We suppose that the benefits paid to the participants at the age of retirement are fixed in advance by the sponsoring plan, in the sense that they are indexed to e.g. salaries, so that the benefit promise is real rather than nominal. We provide a general framework where benefits are given by a stochastic process possibly correlated with the financial market.

It is supposed that every participant enters the plan at the same age a and retires at age d. But this takes place along time, thus we need to consider both the time elapsed since the plan started and the age of the participants. Figure 1 below illustrates this point. We suppose that both the plan's rate of entrance and rate of retirement are exogenous and constant, so that the population of the pension plan remains constant or increases/decreases at a constant rate. This makes the model tractable and is an acceptable assumption in the long run.

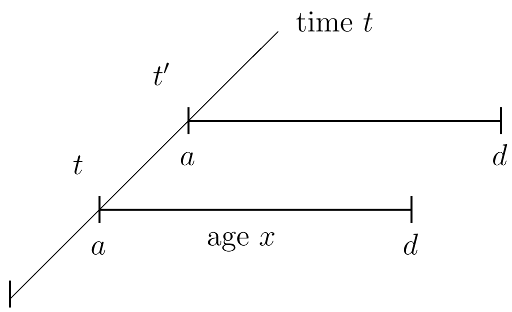


Figure 1: The time–age structure of the pension plan.

Participants accumulate benefits as they get older according to some pre–fixed distribution function $M(x)$ depending on age x. Roughly speaking, the main concern of the sponsoring plan is to set aside a reserve large enough to ensure that the benefit promise can be honored when the worker retires. To this end, actuaries must forecast future benefits and determine the contribution stream needed to match future benefits needs. The so prescribed yearly accruals take care of the pension benefit obligation of the employer and are called the plan's normal cost. However, the employer's contribution must also contemplate past unfunded pension liabilities as well as the expected rate of return on pension fund investment. Finally, stochastic shocks can affect the plan. Typically, if the current reserves of the pension fund do not suffice to cover projected liabilities (stock underfunding), then actuarial practice prescribes employers to reduce this gap over time by making contributions in excess of normal cost.

The main elements intervening in the DB plan are the following.

$F(t)$: value of fund assets at time t;

$P(t)$: benefits promised to the participants at time t; they are related with the salary at the moment of retirement;

$C(t)$: contribution rate made by the sponsor at time t to the funding process;

$AL(t)$: actuarial liability at time t, that is, total liabilities of the sponsor;

$NC(t)$: normal cost at time t; if the fund assets match the actuarial liability, and if there are no uncertain elements in the plan, the normal cost is the value of the contributions allowing equality between asset funds and liabilities;

$UAL(t)$: unfunded actuarial liability at time t, equal to $AL(t) - F(t)$;

$SC(t)$: supplementary cost at time t, equal to $C(t) - NC(t)$;

$M(x)$: proportional value of the future benefits accumulated until age $x \in [a, d]$, where a is the common age of entrance in the fund and d is the common age of retirement; M is a probability distribution function on $[a, d]$;

δ : constant rate of valuation of the liabilities, which can be specified by the regulatory authorities;

r : constant risk–free market interest rate.

2.1 The actuarial functions

Following Josa–Fombellida and Rincón–Zapatero (2004), we suppose that random disturbances affect the evolution of benefits and hence the evolution of the normal cost and the actuarial liability. The novelty here is that benefits and the risky asset (we consider a single risky asset to gain clarity in the exposition) are jump diffusion processes where the uncertainty is given by a Brownian motion and a Poisson process. To model the randomness of the pension plan and the financial market, we consider a probability space $(\Omega^w, \mathscr{F}^w, \mathbb{P}^w)$, where $\mathbb{P}^w$ is a probability measure on Ω^w and $\mathscr{F}^w = \{\mathscr{F}_t^w\}_{t\geq 0}$ is a complete and right continuous filtration generated by the two–dimensional standard Brownian motion (w_0, w_1), that is to say, $\mathscr{F}_t^w = \sigma\{(w_0(s), w_1(s)); 0 \leq s \leq t\}$, $t \geq 0$. We also consider a two–dimensional Poisson process (N_1, N_2) with constant intensity (λ_1, λ_2), $\lambda_1, \lambda_2 \in \mathbb{R}_+$, defined on a complete probability space $(\Omega^N, \mathscr{F}^N, \mathbb{P}^N)$, where $\mathscr{F}_t^N = \sigma\{(N_1(s), N_2(s));\, 0 \leq s \leq t\}$, $t \geq 0$. It is known that the process $H_i(t) = N_i(t) - \lambda_i t$, $i = 1, 2$, is a $\mathscr{F}^N$–martingale, which is called the compensated Poisson process; see Jeanblanc–Picqué and Pontier (1990) and García and Griego (1994). Let $(\Omega, \mathscr{F}, \mathbb{P}) = (\Omega^w \times \Omega^N, \mathscr{F}^w \otimes \mathscr{F}^N, \mathbb{P}^w \otimes \mathbb{P}^N)$ denotes the product probabilistic space. We suppose (N_1, N_2) and (w_0, w_1) are independent processes on this space.

The stochastic actuarial liability and the stochastic normal cost are defined as in Josa–Fombellida and Rincón–Zapatero (2004). It is simply the conditional expected value with respect to the objective probability measure of the accumulated projected benefits, discounted at the technical rate of actualization. Recall that we are adopting an actuarial viewpoint to define liabilities and that the financial market is incomplete since benefits are not tradeable, and thus cannot be used as an instrument to hedge the risk. Benefits cannot be replicated due to the existence of Poisson jumps and because the Brownian motions in the asset returns and benefits are correlated. See Section 2.2 below. Thus, the actuarial liability and the normal cost are

$$
\begin{aligned}
AL\,(t) &= \mathbb{E}\left(\int_a^d e^{-\delta(d-x)}\, M(x)P(t+d-x)\,dx \,|\, \mathscr{F}_t\right),\\
NC\,(t) &= \mathbb{E}\left(\int_a^d e^{-\delta(d-x)}\, M'(x)P(t+d-x)\,dx \,|\, \mathscr{F}_t\right),
\end{aligned}
$$

for every $t \geq 0$, where $\mathbb{E}(\cdot|\mathscr{F}_t)$ denotes conditional expectation with respect to the filtration $\mathscr{F}_t$ and where M' denotes the derivative of M.

Using basic properties of conditional expectation, the previous definitions can be expressed as

$$
\begin{aligned}
AL\,(t) &= \int_a^d e^{-\delta(d-x)}\, M(x)\mathbb{E}\left(P(t+d-x) \,|\, \mathscr{F}_t\right)\,dx,\\
NC\,(t) &= \int_a^d e^{-\delta(d-x)}\, M'(x)\mathbb{E}\left(P(t+d-x) \,|\, \mathscr{F}_t\right)\,dx.
\end{aligned}
$$

For analytical tractability, we suppose that benefits are given by a jump–diffusion process, where the diffusion part increases on average at an exponential rate, extending in this way the results obtained previously in Josa–Fombellida and Rincón–Zapatero (2004), where P is supposed to be a geometric Brownian motion, and in Bowers *et al* (1986), where P is an exponential deterministic function. This assumption is natural since, in general, benefits depend on salary, which on average show exponential growth subject to random disturbances that may be supposed proportional to the variables' size. This is the content of the following hypothesis.

Assumption 1 *The benefit P satisfies the stochastic differential equation (SDE therefore)*

$$dP(t) = \mu P(t)\,dt + \beta P(t)\,dB(t) + \eta_1 P(t-)\,dN_1(t) + \eta_2 P(t-)\,dN_2(t), \quad t \geq 0,$$

where B is a standard Brownian motion on $(\Omega^w, \mathscr{F}^w, \mathbb{P}^w)$, and where $\mu \in \mathbb{R}$, $\beta \neq 0$ and $\eta_i > -1$, $i = 1, 2$. The initial condition $P(0) = P_0$ is a random variable that represents the initial liabilities.

Note that B and (N_1, N_2) are independent stochastic processes.

The actuarial functions AL and NC are related in a simple way, shown in the next proposition, which will be used in what follows. We introduce new notation:

$$\psi_{AL} = \int_a^d e^{(\mu-\delta+\lambda_1\eta_1+\lambda_2\eta_2)(d-x)} M(x)\,dx,$$
$$\psi_{NC} = \int_a^d e^{(\mu-\delta+\lambda_1\eta_1+\lambda_2\eta_2)(d-x)} M'(x)\,dx.$$

Proposition 2.1 *Under* Assumption 1*, the actuarial functions satisfy $AL(t) = \psi_{AL} P(t)$ and $NC(t) = \psi_{NC} P(t)$, and they are linked by the identity*

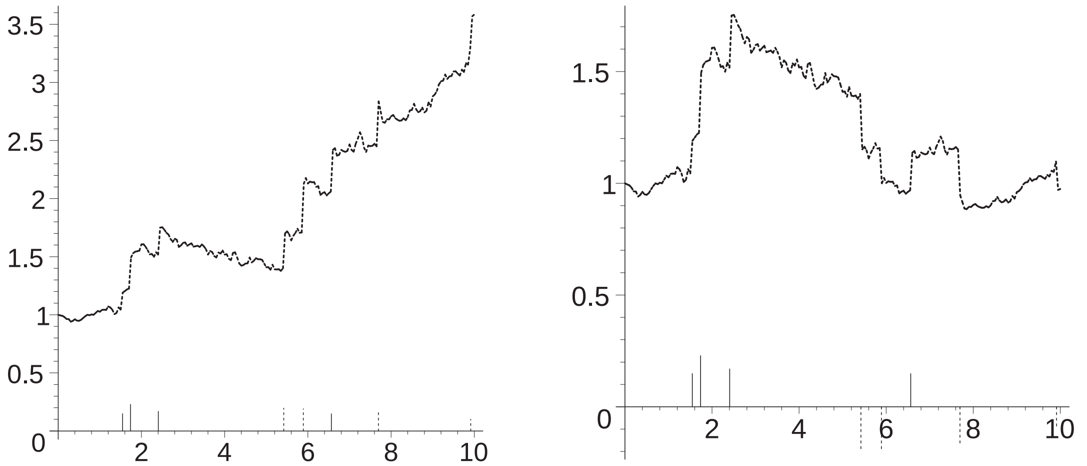
$$(\delta - \mu - \lambda_1\eta_1 - \lambda_2\eta_2) AL(t) + NC(t) - P(t) = 0, \tag{1}$$

for every $t \geq 0$. Moreover, the actuarial liability satisfies the SDE

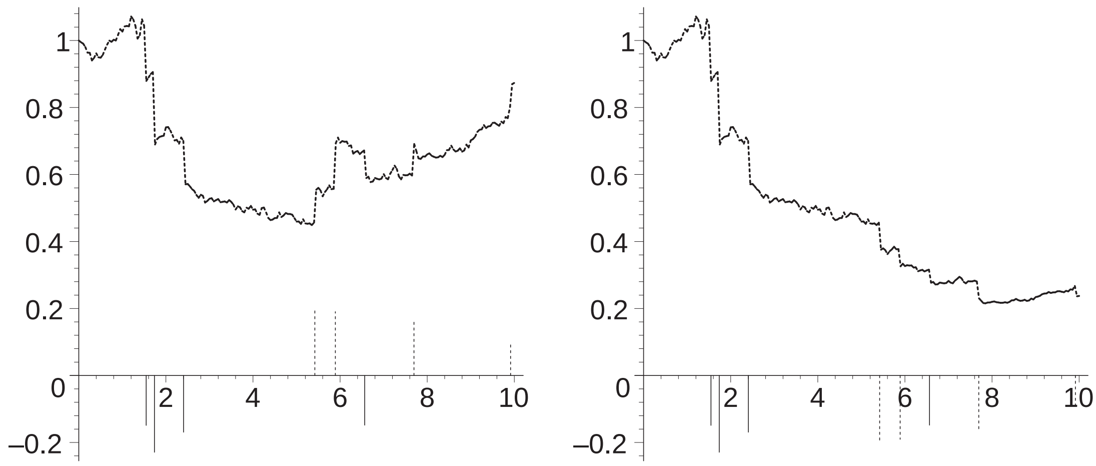
$$dAL(t) = \mu AL(t)\,dt + \beta AL(t)\,dB(t) + \eta_1 AL(t)\,dN_1(t) + \eta_2 AL(t) dN_2(t), \tag{2}$$

with initial condition $AL(0) = AL_0 = \psi_{AL} P_0$.

Figure 2 shows a solution of (2) with $AL_0 = 1$ and with N_1, N_2 as specified in Section 4 below. The vertical segments with basis in the horizontal axis point out the time and magnitude of the jumps due to N_1 (dotted) and N_2 (solid).



(a) $\eta_1 = \eta_2 = 0.1$ (left) and $\eta_1 = -0.1, \eta_2 = 0.1$ (right).



(b) $\eta_1 = 0.1, \eta_2 = -0.1$ (left) and $\eta_1 = \eta_2 = -0.1$ (right).

Figure 2: Paths of AL.

2.2 The financial market

In this section, we describe the financial market where the fund operates. The plan sponsor manages the fund in the planning interval $[0, \infty)$ by means of a portfolio formed by a risky asset S, which is a jump–diffusion process correlated with the benefit process and is generated by (w_1, N_2) and a riskless asset S^0

$$dS^0(t) = rS^0(t)dt, \quad S^0(0) = 1, \tag{3}$$

$$dS(t) = bS(t)dt + \sigma S(t)dw_1(t) + \varphi S(t-)dN_2(t), \quad S(0) = s > 0. \tag{4}$$

Recall that $r > 0$ denotes the short risk–free rate of interest. The mean rate of return of the risky asset is $b > 0$, and $\sigma > 0$ and $\varphi > -1$ are constant parameters. Note that we assume that the volatility is constant and we do not consider the more general case of a stochastic volatility; this would require the introduction of an additional state variable, breaking down the linear–quadratic structure of the model. We denote by θ the Sharpe ratio or market price of risk for this portfolio, that is

$$\theta = \frac{b - r + \lambda_2 \varphi}{\sqrt{\sigma^2 + \lambda_2 \varphi^2}}, \tag{5}$$

see for instance Björk and Slinko (2006), Appendix A. It is assumed that $b + \lambda_2 \varphi > r$, so the sponsor has incentives to invest in the risky asset. We suppose that there exists a correlation $q \in [-1, 1]$ between B and w_1. As a consequence, B is expressed in terms of (w_0, w_1) as $B(t) = \sqrt{1 - q^2}\, w_0(t) + q w_1(t)$, where $q^2 \leq 1$. In this way, the influence of salary and inflation on the evolution of liabilities P is taken into account, as well as the effect of inflation on the prices of the assets. It is worth noting that with this formulation the benefit process P depends on the financial market.

2.3 The fund wealth

For properly funding the liabilities promised, the sponsoring plan adopts an amortization scheme and proceeds actively in the financial market to form suitable portfolios. The share of portfolio invested in the risky stock S at time t is denoted by $\pi(t)$. The remainder, $F(t) - \pi(t)$, is invested in the bond. Borrowing and shortselling are allowed. A negative value of π means that the sponsor sells a part of his risky asset S short while, if π is larger than F, then the sponsor gets into debt to purchase the stocks, borrowing at the riskless interest rate r. As is common in the literature, we restrict strategies to fulfill some technical conditions: the investment strategy $\{\pi(t) : t \geq 0\}$ is a control process adapted to the filtration $\{\mathscr{F}_t\}_{t \geq 0}$, $\mathscr{F}_t$–measurable, Markovian and stationary, satisfying

$$\mathbb{E} \int_0^T \pi^2(t) dt < \infty, \tag{6}$$

and the contribution rate process $C(t)$ is also an adapted process with respect to $\{\mathscr{F}_t\}_{t\geq 0}$ verifying

$$\mathbb{E}\int_0^T SC^2(t)dt < \infty. \tag{7}$$

Under the investment/contribution policy chosen, the dynamic fund evolution is given by

$$dF(t) = \pi(t)\frac{dS(t)}{S(t)} + (F(t) - \pi(t))\,\frac{dS^0(t)}{S^0(t)} + (C(t) - P(t))\,dt. \tag{8}$$

By substituting (3) and (4) in (8), we obtain that

$$dF(t) = \Big(rF(t) + (b-r)\pi(t) + C(t) - P(t)\Big)dt + \sigma\pi(t)\,dw_1(t) + \varphi\pi(t-)dN_2(t), \tag{9}$$

with initial condition $F(0) = F_0 > 0$, determines the fund evolution.

3 The optimal strategies

In this section, we analyze how the sponsor selects the optimal contribution rate and investment strategy. As explained in the Introduction, we model the sponsor's preferences as quadratic, penalizing deviations from prescribed targets, identified with the normal cost and the actuarial liability. These quadratic deviations are clearly related with the solvency and stability objectives of the funding process. In the optimization process, the sponsor faces two elements of randomness: one due to the benefits, which is inherent to the pension plan; the other being financial market variables, specifically the risky asset.

The objective functional to be minimized over the class of admissible controls $\mathcal{A}_{F_0,AL_0}$, is given by

$$J((F_0, AL_0); (SC, \pi)) = \mathbb{E}_{F_0,AL_0}\int_0^\infty e^{-\rho t}\left(\kappa SC^2(t) + (1-\kappa)(AL(t) - F(t))^2\right)dt. \tag{10}$$

Note that we choose $SC = C - NC$ as the control variable instead of C, leading to an equivalent control problem. Here, $\mathcal{A}_{F_0,AL_0}$ denotes the set of Markovian processes (SC, π), adapted to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ where C satisfies (7), π satisfies (6), and where F and AL satisfy (9) and (2), respectively. In the above, $\mathbb{E}_{F_0,AL_0}$ denotes conditional expectation with respect to the initial conditions (F_0, AL_0).

Notice that the objective functional is a convex combination of parameter κ, $0 < \kappa \leq 1$, which reflects the relative importance of solvency against stability for the sponsor. With this formulation, the optimal solution is efficient, in the sense that it is not possible to improve one of the objectives without worsening the other, that is, the solution is in the Pareto frontier of the attainable payoffs in the solvency risk–contribution rate risk plane. The time preference of the sponsor is given by $\rho > 0$.

The value function is defined as

$$\widehat{V}(F, AL) = \min_{(SC,\pi)\in\mathcal{A}_{F,AL}} J((F, AL); (SC, \pi)).$$

Since the problem is autonomous and the horizon unbounded, we may suppose that $\widehat{V}$ is time independent. It is clear that the value function so defined is non–negative and strictly convex. The connection between value functions and optimal feedback controls in stochastic control theory under Poisson–diffusion setting is accomplished by the HJB; see Sennewald (2007).

The following result characterizes the solution under the assumption of a time preference rate satisfying a lower bound. We shall use the Sharpe ratio given in (5).

Theorem 3.1 *Suppose that Assumption 1 holds. If the inequality*

$$2\mu + \beta^2 + 2(\lambda_1\eta_1 + \lambda_2\eta_2) + \lambda_1\eta_1^2 + \lambda_2\eta_2^2 < \rho \tag{11}$$

is satisfied, then the optimal contribution rate and the optimal investment in the risky assets are given by

$$C^* = NC - \frac{\alpha_{FF}}{\kappa}F - \frac{\alpha_{F,AL}}{2\kappa}AL\,, \tag{12}$$

$$\pi^* = -\left(\frac{\theta}{\sqrt{\sigma^2+\lambda_2\varphi^2}}\right)F - \frac{\alpha_{F,AL}}{2\alpha_{FF}}\left(\frac{\theta}{\sqrt{\sigma^2+\lambda_2\varphi^2}} + \frac{\beta\sigma q + \lambda_2\eta_2\varphi}{\sigma^2+\lambda_2\varphi^2}\right)AL\,, \tag{13}$$

respectively, where α_{FF} is the unique positive solution to the equation

$$\alpha_{FF}^2 + \kappa\left(\rho - 2r + \theta^2\right)\alpha_{FF} - \kappa(1-\kappa) = 0, \tag{14}$$

and $\alpha_{F,AL}$ is the unique solution to the equation

$$\kappa\left(-\rho+r+\mu+\lambda_1\eta_1+\lambda_2\eta_2-\theta^2-\frac{\theta(\beta\sigma q+\lambda_2\eta_2\varphi)}{\sqrt{\sigma^2+\lambda_2\varphi^2}}\right)\alpha_{F,AL}$$

$$-\alpha_{FF}\alpha_{F,AL}+2\kappa(\mu-\delta+\lambda_1\eta_1+\lambda_2\eta_2)\alpha_{FF}-2\kappa(1-\kappa)=0. \tag{15}$$

Remark 3.1 The optimal strategies SC^* and π^* are linear functions of the fund assets F and the actuarial liability AL , and depend on the parameters of the financial market and the benefit process, and also, through $\alpha_{F,AL}$, depend on the technical rate of interest δ, and on μ, η_1 and λ_1.

We can distinguish two terms in the optimal investment decisions (13). The first summand is proportional to F, with a coefficient proportional to the opposite of θ, while the second one is proportional to AL, with a coefficient that also depends on the randomness parameters and the correlation between benefit and risky asset. One of the summands of this coefficient, $\theta/\sqrt{\sigma^2+\lambda_2\varphi^2}=(b-r+\lambda_2\varphi)/(\sigma^2+\lambda_2\varphi^2)$, is that corresponding to the so called optimal–growth portfolio strategy, $(b-r)/\sigma^2$, of the model without jumps. An interesting consequence is that there exists a linear relationship between the optimal supplementary cost and the optimal investment strategy,

$$\pi^*=\left(\frac{\theta}{\sqrt{\sigma^2+\lambda_2\varphi^2}}\right)\left(\frac{\kappa}{\alpha_{FF}}\right)SC^*-\left(\frac{\beta\sigma q+\lambda_2\varphi\eta_2}{\sigma^2+\lambda_2\varphi^2}\right)\left(\frac{\alpha_{F,AL}}{2\alpha_{FF}}\right)AL\,.$$

Thus, for each unit of additional amortization with respect to the normal cost, the manager must invest $\frac{\theta}{\sqrt{\sigma^2+\lambda_2\varphi^2}}\frac{\kappa}{\alpha_{FF}}$ units in the risky assets, plus an additional amount (positive or negative) of

$$-\left(\frac{\beta\sigma q+\lambda_2\varphi\eta_2}{\sigma^2+\lambda_2\varphi^2}\right)\left(\frac{\alpha_{F,AL}}{2\alpha_{FF}}\right)AL\,,$$

which depends directly on liabilities.

Remark 3.2 The manager must borrow at rate r to invest in the risky asset, that is to say $\pi^*>F^*$, when the level of the fund is below $k_0 AL$, where the constant k_0 is defined as

$$k_0=-\left(\frac{b-r+\lambda_2\varphi+\beta\sigma q+\lambda_2\varphi\eta_2}{b-r+\lambda_2\varphi+\sigma^2+\lambda_2\varphi^2}\right)\left(\frac{\alpha_{F,AL}}{2\alpha_{FF}}\right),$$

and he/she needs to short sell asset, that is to say $\pi^* < 0$, when the fund is above the value $k_1 AL$, where

$$k_1 = -\left(\frac{b - r + \lambda_2\varphi + \beta\sigma q + \lambda_2\varphi\eta_2}{b - r + \lambda_2\varphi}\right)\left(\frac{\alpha_{F,AL}}{2\alpha_{FF}}\right).$$

Thus, the manager does not need either to short–sell or to borrow, $0 \leq \pi^* \leq F^*$, when the fund reserves F^* are between $k_0 AL$ and $k_1 AL$. In other way, when the liquidity of the fund, $L = F/AL$, is below k_0, the optimal strategy demands to borrow to invest in the risky asset and to contribute in excess of the normal cost, whereas if L is above k_1, it is optimal selling short the risky asset and to contribute well below the normal cost.

Now we explore how the optimization process helps to lessen the solvency risk and provides stability to the funding process. This will be done in terms of expected values. To this end, let us rewrite the SDE (2) for AL as

$$\begin{aligned} dAL(t) = (\mu + \lambda_1\eta_1 + \lambda_2\eta_2)AL(t)dt + \beta\sqrt{1-q^2}AL(t)dw_0(t) + \beta q AL(t)dw_1(t) \\ + \eta_1 AL(t)dH_0(t) + \eta_2 AL(t)dH_1(t), \end{aligned} \tag{16}$$

where H_0, H_1 are the compensated Poisson processes defined in Section 2.1. Consider the optimal fund time path F^* given in (24) in the Appendix and take conditional expectations both in (24) and in (16) to obtain

$$\begin{aligned} \mathbb{E}_{F_0,AL_0}F^*(t) - \mathbb{E}_{F_0,AL_0}AL(t) = (F_0 - aAL_0)e^{\left(r-\theta^2-\frac{\alpha_{FF}}{\kappa}\right)t} \\ + (a-1)AL_0 e^{\left(2\mu+\beta^2+2(\lambda_1\eta_1+\lambda_2\eta_2)+\lambda_1\eta_1^2+\lambda_2\eta_2^2\right)t}, \end{aligned} \tag{17}$$

with

$$a = \frac{\mu - \delta + \lambda_1\eta_1 + \lambda_2\eta_2 - \frac{\alpha_{F,AL}}{2\kappa} - \left(\theta^2 + \frac{\theta(\beta\sigma q + \lambda_2\eta_2\varphi)}{\sqrt{\sigma^2+\lambda_2\varphi^2}}\right)\frac{\alpha_{F,AL}}{2\alpha_{FF}}}{\mu + \lambda_1\eta_1 + \lambda_2\eta_2 - r + \theta^2 + \frac{\alpha_{FF}}{\kappa}}.$$

Given the plethora of parameters, convergence of the difference of the expected values to zero cannot be guaranteed without imposing additional conditions. This convergence is of course a desirable property of the managing process. For instance, if $\mu \geq 0$, $\eta_1 \geq 0$, $\eta_2 \geq 0$ and $a \neq 1$, then the second term of (17) is not bounded.

A similar comment applies to the contribution rate and the normal cost

$$\mathbb{E}_{F_0,AL_0} C^*(t) - \mathbb{E}_{F_0,AL_0} NC(t) = -\frac{\alpha_{FF}}{\kappa}\mathbb{E}_{F_0,AL_0} F^*(t) - \frac{\alpha_{F,AL}}{2\kappa}\mathbb{E}_{F_0,AL_0} AL(t).$$

We will choose the technical rate of actualization δ to assure convergence. Moreover, this selection simplifies the amortization scheme to a popular and successful method, widely studied in the literature and used by practitioners, that consists in taking the supplementary cost *SC* proportional to the unfunded actuarial liability *UAL*, which is often quoted as a "spread method" of funding; see Owadally and Haberman (1999). From (12), to achieve this, the identity $\alpha_{F,AL} = -2\alpha_{FF}$ must be fulfilled. Substituting the identity into (15) and comparing with (14), we obtain an expression for the technical actualization rate, which must coincide with the rate of return of the bond modified to get rid of the sources of uncertainty. Specifically we assume:

Assumption 2 *The technical rate of actualization is*

$$\delta = r + \theta\left(\frac{\beta\sigma q + \lambda_2\eta_2\varphi}{\sqrt{\sigma^2 + \lambda_2\varphi^2}}\right)$$

According to this selection, the liabilities are valued with r plus a summand that involves the Sharpe ratio times a factor depending on the correlation parameters and the standard deviation of the portfolio. It is important that δ does not depend on parameters μ, λ_1 and η_1 associated to P, but only on the parameters defining the financial market and the correlation parameter between benefits and the risky asset. Notice that, if there is no correlation, $q = 0 = \varphi$, then δ is simply the risk–free rate of interest.

Corollary 3.1 *Suppose that Assumptions 1 and 2 hold. If the inequality* (11) *is satisfied, then the optimal contribution rate and the optimal investment strategy are given by*

$$\begin{aligned} C^* &= NC + \frac{\alpha_{FF}}{\kappa} UAL, \\ \pi^* &= \left(\frac{\theta}{\sqrt{\sigma^2 + \lambda_2\varphi^2}}\right) UAL + \left(\frac{\beta\sigma q + \lambda_2\eta_2\varphi}{\sigma^2 + \lambda_2\varphi^2}\right) AL\,, \end{aligned} \tag{18}$$

respectively, where α_{FF} is the unique positive solution to the equation (14).

Remark 3.3 By construction, the supplementary cost SC^* is proportional to the unfunded actuarial liability UAL, whereas for the optimal investment decisions π^*, (18), we can distinguish two terms. The first is again proportional to UAL, but the second is a correction term which depends on the risk parameters of the model and of AL. This second term is zero when there is no uncertainty in the benefits, as in Josa–Fombellida and Rincón–Zapatero (2001), or when there is no correlation between benefits and the risky asset. Another feature is that the optimal strategies do not depend on μ, η_1 and λ_1, which are the randomness coefficients of P. In fact, $\varphi = 0$ gives the same strategies found in Josa–Fombellida and Rincón–Zapatero (2004), where there were no Poisson jumps.

The following result establishes the stability and solvency of the pension plan in the long–run, in terms of the expected values.

Proposition 3.1 *Suppose that Assumptions 1, 2 and the inequality* (11) *are satisfied. If the inequality*

$$\alpha_{FF} > \kappa \left(r - \theta^2 \right), \tag{19}$$

holds, then the expected unfunded actuarial liability and the expected supplementary cost converge in the long term to zero, that is to say,

$$\lim_{t \to \infty} \mathbb{E}_{F_0, AL_0} UAL^*(t) = \lim_{t \to \infty} \mathbb{E}_{F_0, AL_0} SC^*(t) = 0.$$

Attending to the definition of α_{FF}, inequality (19) is automatically fulfilled when $\rho \leq \theta^2$. In the case $\rho > \theta^2$, it reduces to

$$\kappa < \frac{1}{1 + (\rho - r)\left(r - \theta^2\right)}.$$

4 A numerical illustration

This offers some numerical explorations to illustrate the dynamic evolution of the optimal fund, the optimal contribution rate and the optimal portfolio strategy. The simulation is built with a numerical algorithm for systems of jump–diffusion SDEs borrowed from Cyganowski *et al*

(2002), adapted to our specific setting, that contemplates several Poisson jumps and a correlation between Brownian motions. The system is formed by the linear SDEs given in (24) in the Appendix for F^* and in (2) for AL. The method used for simulation is a generalized Euler scheme proposed by Maghsoodi (1996). We refer the interested reader to the cited reference Cyganowski *et al* (2002) for further details on the method and for an implementation of the algorithm[1] in MAPLE. The continuous–time stochastic processes N_1 and N_2 carry information on the number of jumps until time t, on the distribution of the jump times, and on the distribution of the jump magnitudes. It is assumed that the jump times are independent and identically distributed, and the same property is true for the jump magnitudes. Moreover, we assume that the former distribution is lognormal with mean μ_i and variance σ_i, $i = 1, 2$, that we select as $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 0.1$ to perform the experiments.

We consider the following values for the parameters:

- Time interval: $T = 10$;
- Benefits: $\mu = 0.1$, $\beta = 0.08$, $\eta_1 = \pm 0.1$, $\lambda_1 = 0.25$, $\eta_2 = \pm 0.1$, and $\lambda_2 = 0.3$;
- Technical rate of actualization: Given by assumption 2, $\delta = 0.03423$; we could have considered the general case, not imposing Assumption 2 over the technical rate of actualization;
- Financial market: $r = 0.03$, $b = 0.1$, $\sigma = 0.2$, $\varphi = 0.06$; this implies a Sharpe ratio $\theta = 0.3352$;
- Correlation between the risky asset and benefits is $q = 0.5$;
- Time preference: $\rho = 0.9$;
- Relative weight of risks: $\kappa = 0.5$;
- Initial values: $AL_0 = 1$, $F_0 = 0.5$; thus, we consider that the plan is 50% underfunded at the initial date.

[1] We can provide our own algorithm to the interested reader, upon request.

All cases satisfy the conditions imposed in Theorem 3.1 and Proposition 3.1.

Figures 3 and 4 below show the evolution of the ratio of fund reserves to the actuarial liability, the relative investment made in the risky stock and the evolution of supplementary cost. The jump times and jump magnitudes are also drawn in the graphs as vertical segments with their basis in the horizontal axis. In all cases the optimal fund approaches the actuarial liability, and especially at the beginning, the gap between F and AL is rapidly diminishing with an aggressive investment strategy, as can be seen in the middle graph. In Figure 3 (a) the parameters are $\eta_1 = \eta_2 = 0.1$ (upward jumps both in benefits and in risky assets). As predicted by the theory, see Remark 3.2, the fund borrows to invest in the risky asset when the optimal fund ratio $L = F/AL$ is below $k_0 = 0.54719$. However, with the selected parameters it is not optimal selling short the risky asset, since the liquidity is never over $k_1 = 1.35$. The supplementary cost shows that the fund contributes in excess to the normal cost when it is underfunded. Panel (b) shows a similar picture, but now with $\eta_1 = -0.1$ and $\eta_2 = 0.1$ (downward jumps in benefits and upward jumps in benefits). Notice that k_0 and k_1 are the same as before. Now, it is optimal selling short the risky asset in some periods. This property becomes more pronounced in Figures 4 (a) and (b), where the parameters are $\eta_1 = 0.1$, $\eta_2 = -0.1$ and $\eta_1 = -0.1$, $\eta_2 = -0.1$, respectively. In both cases, $k_0 = 0.49508$ and $k_1 = 1.22143$.

FIGURES 3 AND 4 AROUND HERE

5 Conclusions

We have analyzed, by means of dynamic programming techniques, the management of an aggregated defined benefit pension plan, where the benefit and the risky asset are jump diffusion processes. The objective is to determine the contribution rate and the investment strategy, minimizing both the contribution and the solvency risk. We have found that there is a linear relationship between the optimal supplementary cost and the optimal investment strategy, and between this strategy and the optimal fund, with correction terms due to the random behavior of benefits.

The parameters associated to the Poisson processes intervene in the optimal strategies and in the optimal fund evolution. However, it is possible to select the technical rate of interest such that the optimal contribution does not depend on the parameters of the benefit process, getting a spread amortization and the stability and security of the plan in the long term. Moreover, this selection makes the optimal investment policy depends on benefits only through the correlation with the financial market.

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A Appendix

Proof of Proposition 2.1. We consider the SDE for P in Assumption 1. The processes $H_i(t) = N_i(t) - \lambda_i t$ are $\mathscr{F}_N$–martingales, $i = 1, 2$; see e.g. Jeanblanc–Picqué and Pontier (1990) and García and Griego (1994). Thus we take the conditional expectation in

$$P(s) = P(t) + \int_t^s \mu P(u)du + \int_t^s \beta P(u)dB(u) + \int_t^s \eta_1 P(u)dH_0(u) + \int_t^s \eta_1 P(u)\lambda_1 du + \int_t^s \eta_2 P(u)dH_1(u) + \int_t^s \eta_2 P(u)\lambda_2 du$$

and we obtain

$$m(s) = P(t) + \int_t^s \mu m(u)du + \int_t^s \eta_1\lambda_1 m(u)du + \int_t^s \eta_2\lambda_2 m(u)du,$$

with $m(s) = E\left(P(s)\,|\,\mathscr{F}_t\right)$, $s \in [t, T]$, that is to say

$$m'(s) = (\mu + \lambda_1\eta_1 + \lambda_2\eta_2)\, m(s), \qquad m(t) = P(t);$$

see García and Griego (1994). Then $m(s) = m(t)e^{(\mu+\lambda_1\eta_1+\lambda_2\eta_2)(s-t)}$, and for $s = t + d - x$, the conditional expectation is

$$\mathbb{E}\left(P(t + d - x)\,|\,\mathscr{F}_t\right) = P(t)e^{(\mu+\lambda_1\eta_1+\lambda_2\eta_2)(d-x)},$$

thus, recalling the definition of AL and ψ_{AL}, we get:

$$\begin{aligned} AL\,(t) &= \int_a^d e^{-\delta(d-x)}\, M(x)\, \mathbb{E}\left(P(t+d-x)\,|\,\mathscr{F}_t\right) dx \\ &= P(t) \int_a^d e^{(\mu+\lambda_1\eta_1+\lambda_2\eta_2-\delta)(d-x)}\, M(x)\, dx = P(t)\psi_{AL}. \end{aligned}$$

Analogously, $NC\,(t) = \psi_{NC} P(t)$.

Now, by means of an integration by parts, we have

$$\begin{aligned} \psi_{NC} &= \int_a^d e^{(\mu+\lambda_1\eta_1+\lambda_2\eta_2-\delta)(d-x)} dM(x) \\ &= e^{(\mu+\lambda_1\eta_1+\lambda_2\eta_2-\delta)(d-x)} M(x)\Big|_{x=a}^{x=d} \\ &\quad + \int_a^d e^{(\mu+(p_0\lambda_1+p_1\lambda_2)\eta-\delta)(d-x)} (\mu+\lambda_1\eta_1+\lambda_2\eta_2-\delta) M(x)\, dx \\ &= 1 + (\mu+\lambda_1\eta_1+\lambda_2\eta_2-\delta)\psi_{AL}. \end{aligned}$$

In consequence

$$\begin{aligned} NC\,(t) &= \psi_{NC} P(t) \\ &= P(t) + (\mu+\lambda_1\eta_1+\lambda_2\eta_2-\delta)\psi_{AL} P(t) \\ &= P(t) + (\mu+\lambda_1\eta_1+\lambda_2\eta_2-\delta)\; AL\,(t), \end{aligned}$$

which is (1). In order to deduce the stochastic differential equation that the actuarial liability satisfies, we use Assumption 1:

$$\begin{aligned} dAL\,(t) &= d(\psi_{AL} P)(t) \\ &= \psi_{AL} dP(t) \\ &= \psi_{AL}(\mu P(t)dt + \beta P(t) dB(t) + \eta_1 P(t-) dN_1(t) + \eta_2 P(t-) dN_2(t)) \\ &= \mu AL\,(t)\, dt + \beta AL\,(t)\, dB(t) + \eta_1 AL\,(t-) dN_1(t) + \eta_2 AL\,(t-) dN_2(t), \end{aligned}$$

with the initial condition $AL\,(0) = AL_0 = \psi_{AL} P_0$. $\square$

Proof of Theorem 3.1. For the problem of Section 3, the HJB equation is

$$
\begin{aligned}
-\rho V + \min_{SC,\pi} \big\{ \kappa SC^2 + (1-\kappa)(F - AL)^2 + \big(rF + (b-r)\pi + SC + NC - P\big) V_F \\
+\mu AL\, V_{AL} + \frac{1}{2}\sigma^2\pi^2 V_{FF} + \frac{1}{2}\beta^2 AL^2 V_{AL,AL} + \beta\sigma q\pi V_{F,AL} \\
+\lambda_1 \left(V(F, (1+\eta_1)AL) - V(F, AL) \right) \\
+\lambda_2 \left(V(F + \varphi\pi, (1+\eta_2)AL) - V(F, AL) \right) \big\} = 0.
\end{aligned} \tag{20}
$$

If there is a smooth solution V of the equation (20), strictly convex, then the maximizers values of the contribution rate and the investment rates are given by

$$
\widehat{SC} = -\frac{V_F}{2\kappa}, \tag{21}
$$

$$
\begin{aligned}
(b-r)V_F(F, AL) + \sigma^2\widehat{\pi} V_{FF}(F, AL) + \beta\sigma q AL\, V_{F,AL}(F, AL) \\
+ \lambda_2 \varphi V_F(F + \varphi\widehat{\pi}, (1-\eta_2)AL) = 0,
\end{aligned} \tag{22}
$$

respectively. The structure of the HJB equation obtained, once we have substituted these values for SC and π in (20), suggests a quadratic homogeneous solution

$$
V(F, AL) = \alpha_{FF}F^2 + \alpha_{F,AL}F\,AL + \alpha_{AL,AL}AL^2.
$$

Imposing this solution in (21) and (22), we obtain

$$
\begin{aligned}
\widehat{SC} &= -\frac{\alpha_{FF}}{\kappa}F - \frac{\alpha_{F,AL}}{2\kappa}AL, \\
\widehat{\pi} &= -\frac{b - r + \lambda_2\varphi}{\sigma^2 + \lambda_2\varphi^2}F - \frac{b - r + \beta\sigma q + \lambda_2\varphi(1+\eta_2)}{\sigma^2 + \lambda_2\varphi^2}\frac{\alpha_{F,AL}}{2\alpha_{FF}}AL,
\end{aligned}
$$

and, substituting in (20) and using (1), the following set of three equations for the coefficients is obtained: (14), (15) and

$$
\begin{aligned}
4\kappa(\rho - 2\mu - \beta^2 - \lambda_1(2+\eta_1)\eta_1 - \lambda_2(2+\eta_2)\eta_2)\alpha_{FF}\alpha_{AL,AL} \\
+\alpha_{FF}\alpha_{F,AL}^2 - 4\kappa(\mu - \delta + \lambda_1\eta_1 + \lambda_2\eta_2)\alpha_{F,AL}\alpha_{FF} \\
+\kappa\frac{(b - r + \beta\sigma q + \lambda_2\varphi(1+\eta_2))^2}{\sigma^2 + \lambda_2\varphi^2}\alpha_{F,AL}^2 - 4\kappa(1-\kappa)\alpha_{FF} = 0.
\end{aligned} \tag{23}
$$

It is clear that (14) admits a positive solution, thus V is strictly convex. In order to prove that the solution of (20) is the value function and that C^* and π^*, given by (12) and (13) respectively, are the optimal strategies of the stochastic control problem, it is sufficient to check that the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0,AL_0} V(F^*(t), AL\,(t)) = 0$$

holds, where AL satisfies (2) and F^* is the optimal fund

$$\begin{aligned} dF^*(t) = {} & \Big(\Big(r-(b-r)\frac{b-r+\lambda_2\varphi}{\sigma^2+\lambda_2\varphi^2} - \frac{\alpha_{FF}}{\kappa}\Big)F^*(t) \\ & + \Big(-(b-r)\frac{b-r+\beta\sigma q+\lambda_2\varphi(1+\eta_2)}{\sigma^2+\lambda_2\varphi^2}\frac{\alpha_{F,AL}}{2\alpha_{FF}} - \frac{\alpha_{F,AL}}{2\kappa} + \mu-\delta+\lambda_1\eta_1+\lambda_2\eta_2\Big)AL\,(t)\Big)\,dt \\ & -\sigma\Big(\frac{b-r+\lambda_2\varphi}{\sigma^2+\lambda_2\varphi^2}F^*(t) + \frac{b-r+\beta\sigma q+\lambda_2\varphi(1+\eta_2)}{\sigma^2+\lambda_2\varphi^2}\frac{\alpha_{F,AL}}{2\alpha_{FF}}AL\,(t)\Big)\,dw_1(t) \\ & -\varphi\Big(\frac{b-r+\lambda_2\varphi}{\sigma^2+\lambda_2\varphi^2}F^*(t-) + \frac{b-r+\beta\sigma q+\lambda_2\varphi(1+\eta_2)}{\sigma^2+\lambda_2\varphi^2}\frac{\alpha_{F,AL}}{2\alpha_{FF}}AL\,(t-)\Big)\,dN_2(t), \end{aligned} \tag{24}$$

obtained after substitution in (9) of the expressions for C^* and π^*.

Following Jeanblanc–Picqué and Pontier (1990), we can apply Itô's formula with a Poisson jump to the processes $(F^*)^2, F^*AL$ and AL^2. Taking expected values, the functions defined by $\phi(t) = \mathbb{E}_{F_0,AL_0}(F^*)^2(t)$, $\psi(t) = \mathbb{E}_{F_0,AL_0}(F^*AL\,)(t)$ and $\xi(t) = \mathbb{E}_{F_0,AL_0}AL^2(t)$ satisfy the linear differential equations

$$\begin{aligned} \phi'(t) = {} & \Big(2r - 2\frac{\alpha_{FF}}{\kappa} - \theta^2\Big)\phi(t) + 2\Big(\frac{-\alpha_{F,AL}}{2\kappa} + \mu-\delta+\lambda_1\eta_1+\lambda_2\eta_2\Big)\psi(t) \\ & + \frac{(b-r+\beta\sigma q+\lambda_2\varphi(1+\eta_2))^2}{\sigma^2+\lambda_2\varphi^2}\frac{\alpha_{F,AL}^2}{4\alpha_{FF}}\xi(t), \\ \psi'(t) = {} & \Big(r - \frac{\alpha_{FF}}{\kappa} + \mu + \lambda_1\eta_1+\lambda_2\eta_2 - \theta^2 - \frac{(\beta\sigma q+\lambda_2\varphi\eta_2)(b-r+\lambda_2\varphi)}{\sigma^2+\lambda_2\varphi^2}\Big)\psi(t) \\ & + \Big(-\frac{(b-r+\beta\sigma q+\lambda_2\varphi(1+\eta p_1))^2}{\sigma^2+\lambda_2\varphi^2}\frac{\alpha_{F,AL}}{2\alpha_{FF}} - \frac{\alpha_{F,AL}}{2\kappa} + \mu-\delta+\lambda_1\eta_1+\lambda_2\eta_2\Big)\xi(t), \\ \xi'(t) = {} & \Big(2\mu+\beta^2+2(\lambda_1\eta_1+\lambda_2\eta_2)+\lambda_1\eta_1^2+\lambda_2\eta_2^2\Big)\xi(t), \end{aligned}$$

with initial conditions $\phi(0) = F_0^2$, $\psi(0) = F_0 AL_0$ and $\xi(0) = AL_0^2$, respectively. Therefore

$$\xi(t) = AL_0^2 e^{(2\mu+\beta^2+2(\lambda_1\eta_1+\lambda_2\eta_2)+\lambda_1\eta_1^2+\lambda_2\eta_2^2)t},$$

hence $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0,AL_0} AL^2(t) = 0$ if and only if (11) holds. On the other hand,

$$\psi(t) = (F_0 - a_1 AL_0) AL_0 e^{\left(r - \frac{\alpha_{FF}}{\kappa} + \mu + \lambda_1\eta_1 + \lambda_2\eta_2 - \theta^2 - \frac{(\beta\sigma q + \lambda_2\varphi\eta_2)(b - r + \lambda_2\varphi)}{\sigma^2 + \lambda_2\varphi^2}\right)t} + a_1\xi(t),$$

where a_1 is a constant depending on the parameters of the model. Then $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0,AL_0}(F^* AL)(t) = 0$, if and only if, both (11) and the inequality

$$r - \frac{\alpha_{FF}}{\kappa} + \mu + \lambda_1\eta_1 + \lambda_2\eta_2 - \theta^2 - \frac{(\beta\sigma q + \lambda_2\varphi\eta_2)(b - r + \lambda_2\varphi)}{\sigma^2 + \lambda_2\varphi^2} < \rho \tag{25}$$

simultaneously hold. The latter condition (25) follows from (11) and (14). To check this, we first observe that, by the definition, α_{FF} is the positive solution of (14), then $\alpha_{FF} > -\frac{\kappa}{2}(\rho - 2r + \theta^2)$, i.e. $r - \frac{\alpha_{FF}}{\kappa} < \frac{\rho}{2} + \frac{\theta^2}{2}$. Secondly, notice that

$$-\frac{b - r + \lambda_2\varphi}{\sigma^2 + \lambda_2\varphi^2}(\beta\sigma q + \lambda_2\varphi\eta_2) < \frac{b - r + \lambda_2\varphi}{\sigma^2 + \lambda_2\varphi^2}(-\beta\sigma) < \frac{1}{2}\frac{(b - r + \lambda_2\varphi)^2}{\sigma^2 + \lambda_2\varphi^2} + \frac{1}{2}\frac{\beta^2\sigma^2}{\sigma^2 + \lambda_2\varphi^2} < \frac{\theta^2}{2} + \frac{\beta^2}{2},$$

because $b - r + \lambda_2\varphi > 0$ and $-1 \le q \le 1$. These inequalities and (11) imply (25).

On the other hand,

$$\phi(t) = \left(F_0^2 + a_2 AL_0^2 - a_3 F_0 AL_0\right) e^{\left(2r - 2\frac{\alpha_{FF}}{\kappa} - \theta^2\right)t} - a_2\xi(t) + a_3\psi(t),$$

where a_2 and a_3 are constants. Hence $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0,AL_0}(F^*)^2(t) = 0$ by (11) and by the definition of α_{FF}.

Since V is a homogeneous quadratic polynomial in F and AL, $e^{-\rho t}\mathbb{E}_{F_0,AL_0} V(F^*(t), AL(t))$ converges to 0 when t goes to ∞.

Finally, we check that there exists a unique solution. The constant

$$\alpha_{FF} = -\frac{\kappa}{2}(\rho - 2r + \theta^2) + \frac{1}{2}\sqrt{\kappa^2(\rho - 2r + \theta^2)^2 + 4\kappa(1 - \kappa)},$$

is the unique positive solution to equation (14), there exists a unique solution $\alpha_{F,AL}$ to (15) because the coefficient of $\alpha_{F,AL}$ in (15) is $\neq 0$, by (25), and there exists a unique solution $\alpha_{AL,AL}$ to (23), by (11). $\square$

Proof of Proposition 3.1. Using $\alpha_{F,AL} = -2\alpha_{FF}$, we obtain $a = 1$ in (17). Thus, from (17), we obtain that

$$\mathbb{E}_{F_0,AL_0} UAL^*(t) = \mathbb{E}_{F_0,AL_0} AL(t) - \mathbb{E}_{F_0,AL_0} F^*(t) = (AL_0 - F_0) e^{\left(r - \theta^2 - \frac{\alpha_{FF}}{\kappa}\right)t}$$

converges to zero when t goes to ∞, by (19). Analogously,

$$\mathbb{E}_{F_0,AL_0} SC^*(t) = \frac{\alpha_{FF}}{\kappa} \mathbb{E}_{F_0,AL_0} UAL^*(t)$$

converges to zero when t goes to ∞. $\square$

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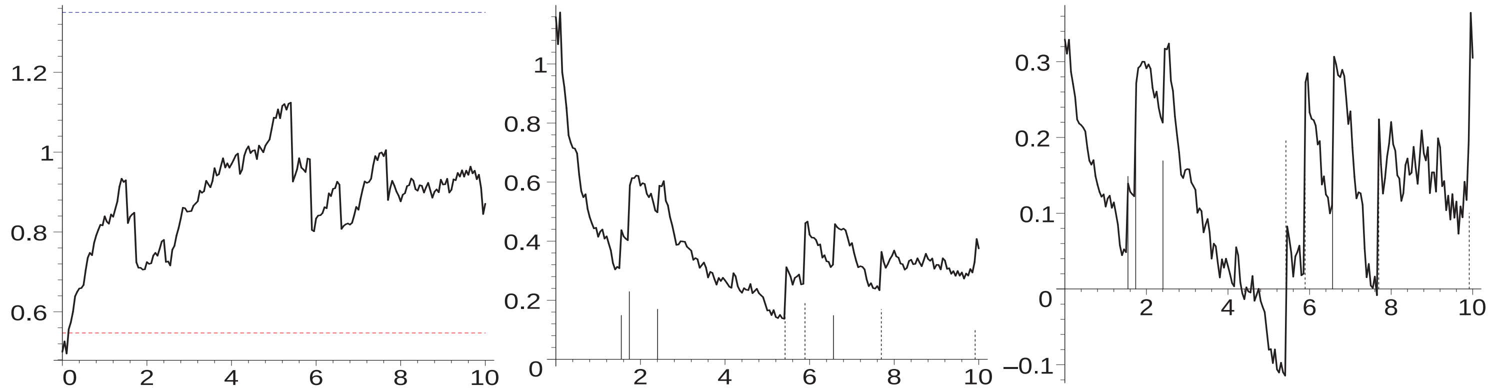
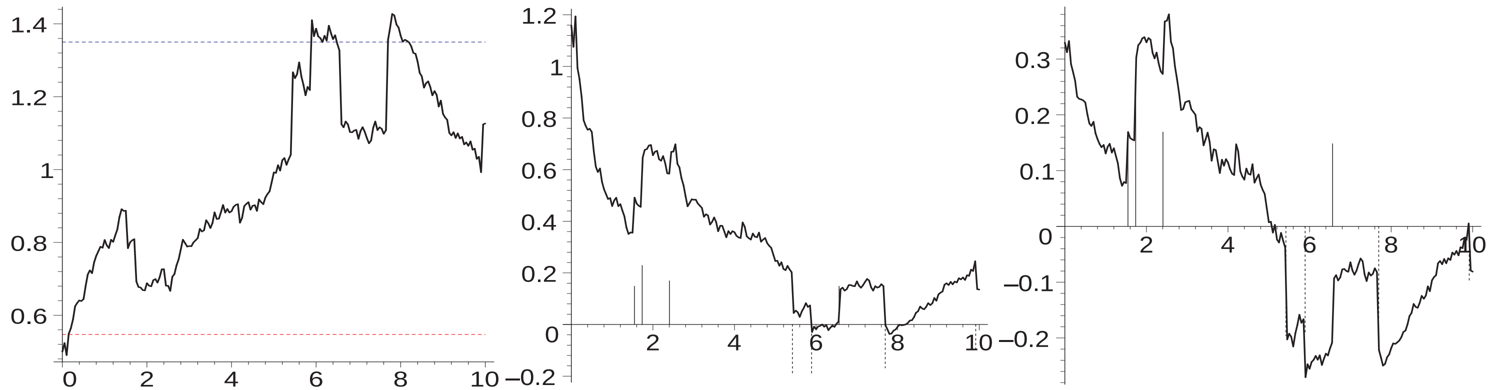
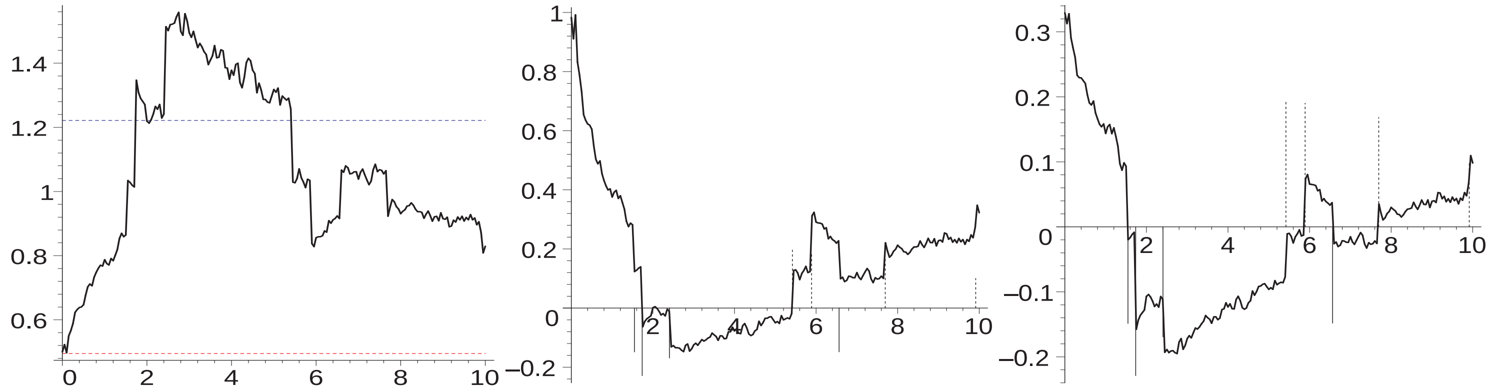


(a) Parameters $\eta_1 = \eta_2 = 0.1$.

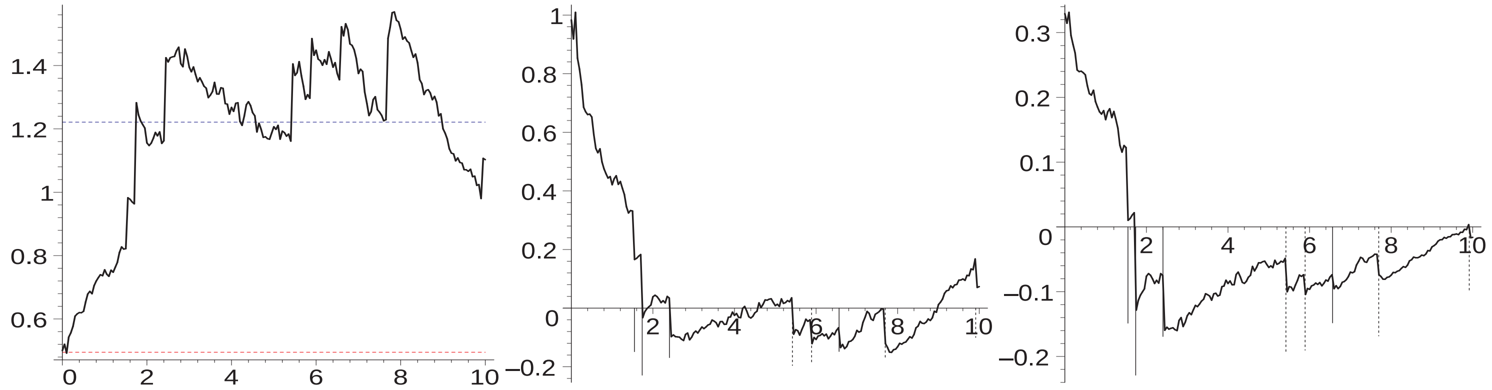


(b) Parameters $\eta_1 = -0.1$ and $\eta_2 = 0.1$.

Figure 3: Left: Liquidity F/AL with horizontal lines k_0 (bottom) and k_1 (top); middle: π/F; right: SC. Axis OX is time.



(a) Parameters $\eta_1 = 0.1$ and $\eta_2 = -0.1$.



(b) Parameters $\eta_1 = \eta_2 = -0.1$.

Figure 4: Left: Liquidity F/AL with horizontal lines k_0 (bottom) and k_1 (top); middle: π/F; right: SC. Axis OX is time.