EXISTENCE, UNIQUENESS AND PROPERTIES OF MARKOV PERFECT NASH EQUILIBRIUM IN A STOCHASTIC DIFFERENTIAL GAME OF A PRODUCTIVE ASSET

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Abstract. This paper analyzes a non-cooperative and symmetric dynamic game where players have free access to a productive asset whose evolution is a diffusion process with Brownian uncertainty. A Euler-Lagrange equation is found and used to provide necessary and sufficient conditions for the existence and uniqueness of a smooth Markov Perfect Nash Equilibrium. The Euler-Lagrange equation also provides a stochastic Keynes-Ramsey rule, which has the form of a forward-backward stochastic differential equation. It is used to study the properties of the equilibrium and to make some comparative statics exercises.

Keywords: Stochastic productive asset, Markov Perfect Nash Equilibrium, Euler-Lagrange equation

Journal of Economic Literature Classification Numbers: C61, G11, G23.

1. Introduction

The exploitation of resources under noncooperative management have received much attention in the literature. See the recent survey of Van Long (2011) for a modern and exhaustive account. The seminal paper of Levhari and Mirman (1980) studies a dynamic game in discrete time, finding an explicit solution of the equilibrium when the players have logarithm utilities. The equilibrium leads to an inefficient allocation of resources since, as a consequence of the noncooperative character of the game, the resource is overexploited. Further studies in the field are Clemhout and Wan (1985), Sundaram (1989), Benhabib and Radner (1992), Dutta and Sundaram (1993a,b), Dockner and Sorger (1996) and Sorger (1998). Of special relevance to us are the two latter papers. Both show that with an infinite horizon, infinitely many subgame perfect equilibria of the symmetric game exist. All of them are discontinuous with respect to the asset stock level. The reason is the assumption taken with respect to the elasticity of the marginal utility, which says that it is greater –Sorger (1998)– or equal –Dockner and Sorger (1996)– than the ratio $N/(N-1)$, where N is the number of players[1]. The willingness of the players to exchange consumption between time periods strengthens the fight for the resource and, as we prove, it makes the consumption rate jump to infinite, no matter the resource suffers stochastic fluctuations. To avoid this problem, the aforementioned papers impose an exogenous

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[1]Note that in particular the one-player game never fulfills these assumption.

upper bound on the consumption rate. We consider here just the opposite assumption, that is, that the willingness to exchange consumption across time is strictly less than $N/(N-1)$. We obtain in this way the existence of a unique smooth equilibrium.

As in most of the previous literature, we focus on the symmetric game and the symmetric Nash equilibrium. Another feature of our study is that we analyze mainly games with a finite horizon, allowing for rather general bequest functions at the final time. Then we show how the finite horizon equilibrium approaches the infinite horizon one as the horizon tends to infinite for an ample class of bequest functions. Finite horizon games serve to model problems where access to the fishing pool is valid only for a fixed period of time; after the time expires, the bequest function of each player could represent some tax that the owner of the resource imposes on the players for the use of the resource over the agreed period of time.

Our approach is based on the Euler-Lagrange equation (EL henceforth) associated with the dynamic game. EL equations are one of the more useful tools to study dynamic optimization problems. Introduced in the Calculus of Variations for the first time, EL equations have become a cornerstone in dynamic economic analysis. They provide a first order optimality condition for interior solutions of dynamic problems, avoiding the use of the value function. The value function is characterized by the Hamilton-Jacobi Bellman equation (HJB henceforth). HJB equations provide a general characterization of optimality, in the sense that they do not need interiority of the equilibrium to hold[2]. Once the value functions are known, the Markov Perfect Nash Equilibrium (MPNE henceforth) is recovered from the HJB equation as the fixed point of the best response mappings in the Hamiltonian game[3]. It is worth noting that the resolution of the EL equations provides the solution directly, with no need to compute the value function. In fact, the value function can be found once the EL equations are solved, as we will show below.

What we propose in this paper is a general method for obtaining EL equations in a class of stochastic differential games, in which the MPNE is interior, and the uncertainty, which is modeled as a standard Brownian motion, is independent of the strategies of the players. This is the most serious limitation of the method developed in the paper. Nevertheless, many interesting economic models other than those from finance, present this feature. In particular, the noncooperative game of exploitation of a productive asset that we study here belongs to this class. The idea of obtaining the EL equations in a differential game to determine the MPNE can be traced back to Case (1979). Tsutsui and Mino (1990) also use the EL equation to find infinitely many discontinuous MPNE in an oligopolistic differential game. Further concrete applications of the use of the EL equations system in differential game theory are Dockner and Sorger (1996) and Sorger (1998). In Rincón-Zapatero *et al.* (1998) and Rincón-Zapatero (2004), this approach has been made systematic. These two papers also provide sufficient conditions of optimality which are independent of the value function. An application to a general model of exploitation of a non renewable resource in a finite horizon is given. In Martin-Herrán and Rincón-Zapatero (2005), the EL equations are used to identify games of fishery where the MPNE is Pareto optimal.

[2]EL equations can also be formulated in the general case by introducing multipliers and inequalities instead of equalities, although obviously in this case they lose much of their direct applicability.

[3]Of course, this program succeeds in continuous time under some technical conditions guaranteeing the existence of solutions of the corresponding evolution equations.

The subsequent paper Josa-Fombellida and Rincón-Zapatero (2007) focuses on a stochastic control problem with Brownian uncertainty, where the players' decisions cannot affect the size of the uncertainty[4]. The present paper extends the above methodology to the game framework. Our main achievement is to prove the existence and uniqueness of smooth MPNE for the finite horizon game, under quite general hypotheses, allowing for the unboundedness of the functions intervening in the definition of the problem, as well as non Lipschitzianity of the utility function. This is necessary to cover the more popular cases, including those that allow an analytical resolution. However, no specific functional form is postulated in our results. To prove our main theorem we proceed in two steps. First, we show the existence and the uniqueness of solutions to the EL equation, which is a partial differential equation (PDE henceforth) in the finite horizon case. Classical results of PDEs are not directly applicable to the EL equation, due to the lack of good properties of the model (unboundedness and non Lipschitzianity), so we have to build a specific proof from known theorems. Our method of proof also provides useful bounds, above and below, for the solution, which are used later to prove that the solution is in fact a MPNE of the game, as well as to obtain the turnpike convergence result. Second, we impose further conditions such that the solution of the PDE is an MPNE. To show this, we use a verification theorem and, as a byproduct, we find an expression for the value function and show that it is a solution of the HJB equation[5].

A consequence of the EL equation is a Keynes-Ramsey (KR henceforth) rule that governs the equilibrium. Due to the stochastic nature of the problem, the rule consists of a pair of forward-backward stochastic differential equations. The representation allows us to measure the effect of uncertainty and to show that, under the standard hypothesis of concavity on the bequest and the recruitment functions, the larger the uncertainty, the larger the consumption rate of the players. Thus, uncertainty sharpens competition among players[6]. We also do some exercises on comparative statics, studying the effect on the equilibrium of a variation in the number of players and in the time preference rate.

Another important issue is extinction, that is, to analyze whether the competition could lead to an overexplotiation of the resource and eventually drive the resource stock to zero. It is also worth comparing the stochastic case we deal with here and the deterministic case, analyzed in Clemhout and Wan (1985). We show that uncertainty raises the possibility of extinction.

Finally, we also study the curvature of the equilibrium, finding conditions such that the consumption equilibrium is concave with respect to the state variable. The curvature of the

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[4]Here the approach was based on the stochastic maximum principle, which is a more general way than that used here, based on the value function; the second approach has the advantage of simplicity. In Josa-Fombellida and Rincón-Zapatero (2010) an EL equation of a Mayer problem, where the diffusion coefficient depends on the control, has been obtained and analyzed. It turns out that the EL equation in this case is much more complex.

[5]A direct approach, not based on verification theorems that use the value function is shown in Josa-Fombellida and Rincón-Zapatero (2007) for the single player case.

[6]Note carefully that this claim has nothing to do with precautionary savings, as the context is different. Precautionary savings means that the agent saves more today when there is uncertainty in his/her tomorrow's income than when the uncertainty is eliminated by adding the expected income to the wealth process. We do not carry out this exercise here, but compare the Markov equilibrium strategies of two games with different diffusion coefficients, without modifying the income process (in this case, the productive asset process). Usually, precautionary savings appear when the marginal utility is convex, which can be easily proved in two period, discrete time models by means of Jensen's inequality, see Leland (1968).

consumption rule gives us information concerning players' propensity to consume. A concave consumption rule implies a higher propensity to consume for poor people than for rich people. Carroll and Kimball (1996) proved concavity of the consumption function in a one player game of finite horizon and discrete time, where uncertainty comes from three sources: labor income, gross interest rate and discount factor. The family of utility functions considered by those authors were of the constant relative risk aversion class (CRRA henceforth), strictly increasing, concave, and with convex marginal utility. We consider only one source of uncertainty, but use continuous time and allow for a rather general bequest function, whereas Carroll and Kimball (1996) consider no bequest function at the end of the game. We obtain definite results only for the game we call the linear game, which will be defined below.

The paper has the following structure. It presents the game of exploitation of the stochastic productive asset in the main text, relegating the proofs of auxiliary results to Appendix C. In Appendix A is the definition of a general stochastic game where player's actions do no affect the size of uncertainty and the definition of admissible strategies and of the MPNE. In Appendix B is the deduction of the EL equation system from the HJB equation. Then this result is applied to the productive asset game and all further results are built upon the EL equation, or its consequence, the KR rule. The main text is organized as follows: After the Introduction, in Section 2 we define the game and some related concepts, find the EL equation of the game and establish precise assumptions. In Section 3 we prove the existence and uniqueness of MPNE in two steps, as explained above. Some insights into the infinite horizon game are also provided. Section 4 focuses on the KR rule. Section 5 is devoted to study some properties of the equilibrium. Besides the comparative statics of the equilibrium, we obtain a turnpike result and analyze the question of the extinction of the resource. Finally some conclusions are extracted from the paper in Section 6.

2. Competition for consumption of a stochastic productive asset: Game, assumptions and Euler-Lagrange Equations

2.1. **The game.** We consider a continuous time non-cooperative game where N agents consume a stochastic productive asset. Asset stock at time $t \geq 0$ is denoted by $X(t)$ and the consumption rate of player $i \in \{1, \dots, N\}$, denoted $c^i(t)$, is given through a Markov strategy$\phi^i : [0,T] \times [0,\infty) \longrightarrow [0,\infty)$, that is, $c^i(t) = \phi^i(t, X(t))$.[7] Given a consumption profile of Markov strategies of the other players, $\phi_{-i} = (\phi^1, \dots, \phi^{i-1}, \phi^{i+1}, \dots, \phi^N)$, player i chooses consumption $c^i \geq 0$ to maximize his/her payoff

$$J_i(t, x, \phi_{-i}; c^i) = \mathrm{E}_{tx} \int_t^T e^{-r_i(s-t)} L_i(c^i(s))\, ds + e^{-r_i(T-t)} \mathrm{E}_{tx} S_i(X(T)).$$

This is given by the expected total utility of consumption, $L_i(c^i)$, over a fixed time horizon $[0,T]$, plus the utility derived from the asset stock at the end of the period, S_i, both discounted at the rate $r^i > 0$. The utility L_i could be equal to sale revenue, $L_i(u^i) = u^i p_i(u^i)$, where p_i is an inverse demand function. The bequest function S_i reflects the fact that the asset has externality effects over the players. The asset stock must satisfy $X(s) \geq 0$ almost surely (a.s.)

[7]The game is a particular case of the general framework we develop in Appendix A. The class of admissible strategies is given in Definition A.1 of the Appendix.

for all s. Some of the assumptions listed below guarantee that both constraints $c^i(s) \geq 0$ and $X(s) \geq 0$ a.s. for all s are satisfied in equilibrium. The asset evolves according to the stochastic differential equation (SDE henceforth)

$$dX(s) = \left(F(X(s)) - c^i - \sum_{j \neq i} \phi^j(s, X(s)) \right) ds + \sigma(X(s)) dw(s), \tag{1}$$

where w is a standard Brownian motion defined on a complete probabilistic space $(\Omega, \mathcal{F}, \mathbf{P})$. We will denote by E_{tx} the conditional expectation with respect to the initial condition (t, x) under the probability measure $\mathbf{P}$, where $x = X(t) > 0$. The asset stock reproduces at the rate given by the production/recruitment function F, which may be a natural growth function describing the dynamics of a renewable resource, such as a fish population. In this case it is common to consider F with a maximum sustainable yield and with a maximum carrying capacity. We do not restrict ourselves to this case. The evolution of the asset is affected by stochastic fluctuations given by the diffusion term $\sigma(X(s))$. The uncertainty may come from inaccurate estimation of the resource reserves, which need to be continuously updated by the players.

In the following sections we will study the MPNE of this game. See Appendix A for the general definition of an MPNE.

2.2. **The Euler-Lagrange Equations.** The starting point of our approach to the game is the system of EL equations (26) given in Appendix B. We suppose that the functions involved are smooth enough for the computations needed to obtain the EL Equations. Precise assumptions will be established shortly after Lemma 2.1 below. They also ensure the existence of a well behaved solution to the equations when the game is symmetric. To obtain the EL equations, consider the current adjoint function and the current Hamiltonian of each player:

$$\text{Adjoint function: } \Lambda_i(x, (c^i | c_{-i})) = L_i'(c^i),$$

$$\text{Hamiltonian: } H^i(x, (c^i | c_{-i}), p^i) = L_i(c^i) + \left(F(x) - c^i - \sum_{j \neq i} c^j(s) \right) p_i.$$

Let $\varphi_i = (L_i')^{-1} \circ S_i'$, $i = 1, \ldots, N$, that defines the optimal strategy of player i at the end of the game. In the following lemma the symbols ∂_t, ∂_x, denote total derivative with respect to t, x; details are given in Appendix A.

Lemma 2.1. *A smooth MPNE given by Markov strategies $\phi = (\phi^1, \ldots, \phi^N)$ satisfies the following system of EL equations:*

$$r_i L_i'(\phi^i(t,x)) = \partial_t L_i'(\phi^i(t,x)) + \partial_x \left(H^i(x, \phi(t,x), L_i'(\phi^i(t,x))) + \frac{1}{2}\sigma^2(x) \partial_x L_i'(\phi^i(t,x)) \right) \tag{2}$$

for $i = 1, \ldots, N$, with final value

$$\phi^i(T, x) = \varphi_i(x), \qquad x > 0 \tag{3}$$

and boundary condition

$$\phi^i(t, 0) = 0, \quad \forall t \leq T. \tag{4}$$

Proof. Apply equation (26) in Appendix B to obtain the EL equation (2), and equation (27) to get the final condition (3). The boundary condition (4) is a requirement imposed by feasibility. □

In the infinite horizon case the EL equation system is still (2), but there is no terminal condition. For stationary Markov strategies the term $\partial_t L_i' = L_i'' \phi_t^i$ vanishes.

Let $R_i = -L_i'/L_i''$ be the *absolute risk tolerance* index (the inverse of the absolute risk aversion index of Arrow–Prat) and $P_i = -L_i'''/L_i''$ be the *absolute prudence* index of player i as defined in Kimball (1990). Taking total derivatives in (2) we get, for $i = 1, \dots, N$

(5)
$$\phi_t^i(t,x) + \Big(F(x) - \sum_{j=1}^N \phi^j(t,x) + \sigma'(x)\sigma(x)\Big)\phi_x^i(t,x) - \frac{1}{2}\sigma(x)^2 P_i(\phi^i(t,x))(\phi_x^i(t,x))^2 + \frac{1}{2}\sigma(x)^2\phi_{xx}^i(t,x)$$
$$+ R_i(\phi^i(t,x))\Big(r_i - F'(x) + \sum_{j\neq i}\phi_x^j(t,x)\Big) = 0.$$

In the rest of the paper we center *on the symmetric game and on the symmetric MPNE.* Thus, for all $i = 1, \dots, N$,
$$L_i = L, \quad S_i = S, \quad r_i = r$$
and hence we will drop the index in the functions defined above. The symmetry condition leads to the same risk tolerance and same prudence indexes for each player, $R_i = R$ and $P_i = P$ for any i, as well as the same terminal value at time T, $\varphi_i = \varphi$. Under this assumption, the symmetric MPNE leads, after rearrangement, to the single EL equation

(6)
$$\phi_t(t,x) + \Big(F(x) - N\phi(t,x) + (N-1)R(\phi(t,x)) + \sigma'(x)\sigma(x)\Big)\phi_x(t,x) - \frac{1}{2}\sigma(x)^2 P(\phi(t,x))\phi_x(t,x)^2$$
$$+ \frac{1}{2}\sigma(x)^2\phi_{xx}(t,x) + R(\phi)(r - F'(x)) = 0,$$

with final and boundary conditions

$$\phi(T,x) = \varphi(x) = (L')^{-1}(S'(x)), \quad x > 0,$$
$$\phi(t,0) = 0, \quad t < T, \tag{7}$$

respectively.

Let $\rho(c) = R(c)/c$ be the elasticity of intertemporal substitution for riskless consumption paths and $\pi(c) = P(c)c$ the relative prudence index.

2.3. **Assumptions.** To get our results of existence and uniqueness of a symmetric MPNE of the game, we need to impose several assumptions. They are justified after they are stated. The following table clarifies the generality with which we attain our results.

- (A1) Functions L, S, F and σ are continuous in $[0,\infty)$, with $L(0) = F(0) = \sigma(0) = 0$, $\sigma(x) > 0$ for $x > 0$. Function L is of class C^6 and S, F and σ are of class C^4 in $(0,\infty)$. Moreover, both F' and σ' are bounded in $(0,\infty)$.
- (A2) The instantaneous utility function L is strictly concave, with $L''' \geq 0$ and $R(0) = 0$.

Table 1. List of Results obtained and assumptions employed

Result	Functions L, S, F, σ, φ
Existence, uniqueness	General
Turnpike	L CRRA
Monotonicity	S and F concave
Concavity	Linear game, φ concave, non-decreasing
Dependence on uncertainty	S and F concave
Dependence on N and r	General
Extinction	F and σ linear

(A3) (a) There exist constants $0 \le a^- \le a^+ < \frac{N}{N-1}$ such that

$$a^- \le \rho(c) \le a^+ \quad \text{for all } c \ge 0.$$

(b) There exist constants $0 \le b^- \le b^+$ such that

$$b^- \le \pi(c) \le b^+ \quad \text{for all } c > 0.$$

(A4) Function φ satisfies $\varphi(0) = 0$ and $\varphi(x) > 0$ for $x > 0$.

(A5) There is a function f, continuous in $[0, \infty)$ and of class C^4 in $(0, \infty)$ with f' bounded, satisfying $f(x) > 0$ for $x > 0$ such that

(a) $f'(0^+) = \lim_{x \to 0^+} f(x)/x$ exists and is finite.

(b) The function $\varphi_0 = \varphi/f$ satisfies

$$M \equiv \sup_{x \in [0,\infty)} \varphi_0(x) < \infty, \qquad m \equiv \inf_{x \in [0,\infty)} \varphi_0(x) > 0.$$

(c) For $x > 0$, let

$$\gamma^+(x) = \max\{a^-(r - F'(x)), a^+(r - F'(x))\},$$
$$\gamma^-(x) = \min\{a^-(r - F'(x)), a^+(r - F'(x))\}$$

and let

$$\beta^+(x) = (F(x) + \sigma'(x)\sigma(x)) \left(\frac{f'(x)}{f(x)}\right) + \frac{1}{2}\sigma^2(x) \left(\frac{f''(x)}{f(x)}\right) - \frac{b^-}{2}\sigma^2(x) \left(\frac{f'(x)}{f(x)}\right)^2 + \gamma^+(x),$$
$$\beta^-(x) = (F(x) + \sigma'(x)\sigma(x)) \left(\frac{f'(x)}{f(x)}\right) + \frac{1}{2}\sigma^2(x) \left(\frac{f''(x)}{f(x)}\right) - \frac{b^+}{2}\sigma^2(x) \left(\frac{f'(x)}{f(x)}\right)^2 + \gamma^-(x).$$

We assume that

$$0 < \beta^- \equiv \inf_{x \in [0,\infty)} \beta^-(x), \quad \beta^+ \equiv \sup_{x \in [0,\infty)} \beta^+(x) < \infty.$$

We will say that a function f satisfying this assumption is a *limiting function* for the equilibrium.

Let us explain the assumptions.

- Assumption (A1) establishes the required smoothness of the data to apply our results and imposes finite marginal productivity at 0 and at $+\infty$. Concavity of F is not needed. Note that (A1) implies the existence of constants A and σ such that $F(x) \leq Ax$ and $\sigma(x) \leq \sigma x$, for all $x \geq 0$.
- Assumption (A2) imposes strict concavity of the instantaneous utility L, a standard property that seems to be unavoidable for the existence of an interior and smooth equilibrium. The assumption $L''' \geq 0$ is typical in consumer theory and, for a concave utility L, it implies a positive prudence index. Consumers with a positive prudence index tend to make extra savings in the present date due to future income being random, a behavior known as precautionary savings. The condition $R(0) = 0$ guarantees that the solution of the EL equation is non–negative.
- Assumption (A3) is key for proving existence of a smooth solution of the EL equation (6) and boundary conditions (7), and allows us to get convergence of the finite horizon approximations to the solution of the infinite horizon case with constant relative risk aversion instantaneous utility (CRRA henceforth). It assumes that both the marginal elasticity of substitution and the relative prudence index are bounded. However, we allow for unbounded utility functions L. In particular, the CRRA utility case is covered in our framework, as in the linear game defined below. Condition (a) means that the willingness of the players to substitute consumption across time is bounded by the ratio $N/(N-1)$, which depends on the number of players. As will be shown in Corollary 3.1 below, this condition cannot be relaxed if one seeks for interior and smooth equilibria, since it prevents the blow up of the solution of the EL equation (6) in finite time. The upper bound decreases as N increases. In the limit when $N \to \infty$, it becomes 1, but in the other extreme case with only one player, it places no constraint. Note that whereas we consider the stochastic game with $\rho(c) < N/(N-1)$, Dockner and Sorger (1996) studied the deterministic game with utility $L(c) = \sqrt{c}$ and two players, where $\rho(c) = N/(N-1) = 2$ for all c, leading to a continuum of symmetric discontinuous MPNE in the game with infinite horizon. Sorger (1998) attains the same result for more general utility functions satisfying $\rho(c) \geq N/(N-1)$. In both cases the equilibrium exists due to the imposition of an upper bound in the consumption rate. The MPNE prescribes consumption at the maximal allowed rate for high enough values of the resource stock, so that the equilibrium is no longer interior. We study here the case left aside in the aforementioned references, in a stochastic environment.
- Assumption (A4) takes care of feasibility: it establishes that, at the final time T, consumption is positive as soon as there is something to consume, $\phi(T, x) > 0$ if $x > 0$, and that it is feasible, $\phi(T, 0) = 0$.
- Assumption (A5) is a way to generalize our results including in our analysis games with the unbounded bequest function S, the production function F and the diffusion coefficient σ. The selection of a given f will depend on the form of these functions. Moreover, it provides lower and upper estimates for the consumption rule. The role of the constants β^+ and β^- will be made clear in the proof of the theorem of the existence of smooth solutions of the EL equation, given in Appendix C.

We now introduce new notation that will be used throughout the paper

$$\alpha^+ = a^+(N-1) - N, \quad \alpha^- = a^-(N-1) - N. \tag{8}$$

Assumption (A3) (a) implies that $\alpha^+ < 0$.

We illustrate the above assumptions in a particular game, that we will call the *linear game with CRRA instantaneous utility*, or simply *the linear game*, which is defined now for further reference.

Definition 2.1. The linear game *corresponds to a CRRA instantaneous utility*

$$L(c) = \frac{c^{1-\delta}}{1-\delta}, \quad \textit{if } 0 < \delta < 1,$$

where both F and σ are linear, $F(x) = Ax$, $\sigma(x) = \sigma x$, for some constants $A \geq 0$, $\sigma > 0$.

Note that *no specific functional form is imposed on the bequest function S.* We will use the linear game as a touchstone to illustrate the results obtained along the paper.

For the linear game, $f(x) = x$ is an adequate selection as a limiting function whenever the bequest value S satisfies, in accordance with assumption (A4) $S'(0^+) = \infty$ and $S'(x) > 0$ for $x > 0$, and $\varphi(x) = S'(x)^{-1/\delta}$ satisfies assumption (A5), that is

$$\inf_{x>0} \frac{S'(x)^{-1/\delta}}{x} > 0, \qquad \sup_{x>0} \frac{S'(x)^{-1/\delta}}{x} < \infty.$$

The remaining elements are $R(c) = \frac{c}{\delta}$, $P(c) = \frac{1+\delta}{c}$, $\rho(c) = \frac{1}{\delta}$ and $\pi(c) = 1+\delta$, and the several constants defined above are

$$\begin{aligned} a &\equiv a^- = a^+ = \frac{1}{\delta}, \\ b &\equiv b^- = b^+ = 1+\delta, \\ \alpha &\equiv \alpha^+ = \alpha^- = (N-1)/\delta - N < 0 \quad \text{iff} \quad \delta > 1 - 1/N, \\ \beta &\equiv \beta^- = \beta^+ = A + \frac{\sigma^2}{2}(1-\delta) + \frac{r-A}{\delta}. \end{aligned}$$

3. Existence of a symmetric MPNE

In the following results we will suppose that the assumptions imposed in the above section hold. To show existence of an MPNE, we proceed in two steps. First we establish a result about the existence of a solution to the EL equation, providing at the same time upper and lower bounds for the solution. In a second step, we will prove that under suitable additional conditions the solution qualifies as an MPNE of the symmetric game. The consideration of a limiting function f in the estimates found for the solution of the EL equation in the next result allows us to drop the boundedness hypothesis that is commonly assumed in the PDE literature. Boundedness is a severe limiting assumption in economic models, since it eliminates CRRA utility functions and linear production functions from the analysis, even though they are by far the more popular and widely used in applications. Our approach will be useful in proving that solutions of the EL equation are in fact an MPNE of the game for a wide range of utility and production functions, as well as in analyzing the turnpike properties of the finite–horizon MPNE.

3.1. **Existence of smooth solutions to the Euler-Lagrange equation.** The proof of the theorem below uses classical results of existence of solution to PDEs, adapted to our framework with unbounded functions and unbounded state space. For this reason (unboundedness), those results cannot be applied directly, so we combine them with an approach that uses the maximum and minimum values of solutions existing in finite time and bounded intervals of the state space. These solutions provide upper and lower estimates for the solution that prevent the blow up, or that the solution could become zero or negative, in terms of the limiting function f defined in (A5) above. Thus, besides proving existence of solution, we find a “window” moving with t where the solution is confined for any x. As we will see, in the CRRA case, the window stretches as the final date T tends to infinity, so that the finite horizon solution converges to a solution of the infinite horizon EL equation. To find the upper and lower bounds, we use a maximum principle for nonlinear parabolic PDEs together with the well known envelope theorem due to Danskin (1966).

Recall the definitions made in the previous section of the constants α^+, α^- in (8), as well as of $M = \sup_{x\in(0,\infty)} \frac{\varphi(x)}{f(x)}$, $m = \inf_{x\in(0,\infty)} \frac{\varphi(x)}{f(x)}$ and β^+, β^- in (A5). We will also use the following functions of t

$$
\begin{aligned}
k_-(t) &= \frac{m\beta^- e^{\beta^-(T-t)}}{\alpha^- m(\sup_{(0,\infty)} f')(1-e^{\beta^-(T-t)}) + \beta^-},\\
k_+(t) &= \frac{M\beta^+ e^{\beta^+(T-t)}}{\alpha^+ M(\inf_{(0,\infty)} f')(1-e^{\beta^+(T-t)}) + \beta^+}.
\end{aligned} \tag{9}
$$

Note that thanks to our assumptions, these functions are well defined and are obviously smooth, with $0 < k_-(t) < k_+(t) < \infty$ for all $t \le T$.

Theorem 3.1. *Let assumptions* (A1)–(A5) *hold, with limiting function f. Then there is a unique non–negative solution ϕ of the Cauchy problem* (6) *of class $C^{2,4}$ that satisfies $\phi(t,0)=0$ for all $t\in[0,T]$. Moreover, ϕ satisfies for all $x>0$ and $t\in[0,T]$ the estimates*

$$0 < k_-(t)f(x) \le \phi(t,x) \le k_+(t)f(x).$$

Proof. See Appendix C. □

For further reference, note that because $\alpha^+ < 0$, it holds that

$$0 < k_- \equiv \min\left\{m, -\frac{\beta^-}{\alpha^- \sup_{(0,\infty)} f'}\right\} \le k_-(t) \le k_+(t) \le \max\left\{M, -\frac{\beta^+}{\alpha^+ \inf_{(0,\infty)} f'}\right\} \equiv k_+,$$

for any $0 \le t \le T$.

3.2. **A non-existence result.** The lower estimate provided in Theorem 3.1 is useful to give a negative criterion for the existence of an interior MPNE. To fix ideas, we consider a CRRA utility $L(c) = c^{1-\delta}/(1-\delta)$ with $\rho(c) > N/(N-1)$ (equivalently, $\alpha = (N-1)/\delta - N > 0$). The same analysis can be done for a general utility function, using the lower bound α^- defined in the previous section. As the following result shows, the (local) smooth solution of the EL equation explodes in finite time, so no global smooth solution exists. Since the EL equation is necessary for the optimality of a smooth MPNE, we can conclude that no equilibrium exists in this case. The intuition for this behavior is as follows: the willingness of the players to substitute consumption across time is too high and this motivates a strong competition to obtain the resource. Eventually, the consumption rate blows up in finite time. As in the discrete-time model game studied in Dutta and Sundaram (1993b), the Markovian first-best solution ($N = 1$) always exist but in the competition version, the existence is not guaranteed, unless an upper bound is imposed in the intertemporal rate of substitution of the players, bound that is related with the number of players.

Corollary 3.1. *Suppose that in the conditions of Theorem 3.1, $\alpha > 0$. Then, for T large enough, there is no smooth solution of the EL equation.*

Proof. We reason by contradiction, assuming that a smooth solution exists. The bounds we have found in the proof of Theorem 3.1 above are still valid. In particular, the lower bound with $\alpha > 0$ implies that the denominator of $k_-(t)$ in (9) vanishes at time

$$\widehat{t} = T - \frac{1}{\beta} \ln \left(1 + \frac{\beta}{\alpha m \sup_{x \in (0,\infty)} f'(x)} \right),$$

with $0 < \widehat{t} < T$, hence if a solution exists, it would satisfy

$$\phi(t,x) \geq k_-(t) f(x) \to \infty \text{ as } t \to \widehat{t}^{+},$$

thus, the solution becomes infinite at a finite instant of time, reaching a contradiction. □

The corollary shows an extreme sensitivity of the MPNE with respect to variations in the number of players that have free access to the asset, as explained in the corollary just above. In fact, it implies the non existence of an interior MPNE, and thus the imposition of an upper bound in consumption is needed for the equilibrium to exist. In the CRRA case, with δ denoting the elasticity of the marginal utility, there is a critical number, $\hat{N} =$ integer part of $(1-\delta)^{-1}$, such that if the number of players is $\hat{N}$ and a new player enters the game, then the game changes drastically, since the MPNE blows up in finite time. For instance, if $\delta < 1/2$, then $\hat{N} = 1$, so the solution in the single player case is interior (as is always the case for the one player model under our assumptions); but when a new identical player enters the game, so that the game becomes a duopoly, the competition is so intense that the players would like to consume all the resource instantaneously, a characteristic already revealed by Reinganum and Stokey (1985) in a related game. Dockner and Sorger (1996) and Sorger (1998) handle this case in the deterministic game by imposing an upper bound in the maximal consumption rate, proving the existence of a continuum of discontinuous MPNE.

3.3. **Markov Perfect Nash Equilibrium and value function.** We next show that the solution of the EL equation is indeed an MPNE of the game. Given $\phi(t,x)$ a solution of the EL equation (6) satisfying the boundary conditions (7), let us define for $x > 0$

$$\lambda(t,x) = \Lambda(x, \phi(t,x)) = L'(\phi(t,x))$$

and let $H(x,c,p) = L(c) + p(F(x) - Nc)$. We will prove that λ is the costate variable or asset shadow price, that is $V_x = \lambda$, where V denotes the value function of the symmetric game. We will also show that ϕ is an MPNE and we will provide the following expression for the value function in terms of $\phi(t,x)$: for $x>0$

$$\begin{aligned} V(t,x) &= \int_\ell^x \lambda(t,z)\,dz \\ &\quad + \int_t^T e^{-r(s-t)}\left(H(\ell,\phi(s,\ell),\lambda(s,\ell)) + \frac{1}{2}\sigma^2(\ell)\lambda_x(s,\ell)\right) ds + e^{-r(T-t)}S(\ell), \end{aligned} \tag{10}$$

where $0<\ell<x$ is an arbitrary constant, and $V(t,0)=0$ for any t. The corresponding expression for the value function in the infinite-horizon case is given below the theorem.

Theorem 3.2. *Assume that the conditions of Theorem 3.1 hold with a limiting function f for which there exist constants $D>0$ and $a>-1$ such that*

$$L'(k_- f(x)) \le D(1+k_- x^a), \tag{11}$$

$$\lim_{x\to 0^+} \sigma(x)^2 L''(k_\pm f(x)) = 0 \tag{12}$$

and

$$\lim_{x\to 0^+} x^a F(x) = 0. \tag{13}$$

Then, a non-negative solution ϕ of the Cauchy problem (6) *is an MPNE of the differential game. Moreover, the value function of the players, given by* (10)*, is of class $C^{2,5}$ in $(0,\infty)$, continuous in $[0,\infty)$ and $V(t,0)=0$ for all $t\ge 0$.*

Proof. We first claim that for any T there exists a unique strong solution of

$$dX(s) = \big(F(X(s)) - N\phi(s,X(s))\big)\,ds + \sigma(X(s))\,dw(s), \quad t<s\le T, \quad X(t)=x, \tag{14}$$

which is positive almost surely. To show this, observe that both F and σ are locally Lipschitz by assumption. On the other hand, the monotone condition

$$xF(x) - Nx\phi(t,x) + \frac{1}{2}\sigma^2(x) \le Kx^2$$

holds for some constant K, since $\phi(t,x)\ge 0$, $x\ge 0$ and because by (A.1) $F(x)\le Ax$ and $\sigma(x)\le\sigma x$ for some constants A and σ. Thus, according to Theorem 3.6 in Mao (1997), there exists a unique strong solution of (14). Moreover, since $F(0)=\sigma(0)=\phi(t,0)=0$, Lemma 3.2 in Mao (1997) implies $X\ge 0$ a.s.

To continue with the proof, let $W(t,x)$ be the right hand side of (10). From the regularity of ϕ, W is of class $C^{2,5}$ in $[0,T)\times(0,\infty)$. Let us show that $\lim_{x\to 0^+} W(t,x)=0$, so that $W(t,0)$ can be defined as $0=V(t,0)$. Note that by (11) and since L' is decreasing and $\phi(t,x)\ge k_-(t)f(x)\ge k_- f(x)$ for all $t\ge 0$ and for all $x>0$

$$\lambda(t,x) = L'(\phi(t,x)) \le D(1+k_- x^a). \tag{15}$$

Hence

$$\int_0^x \lambda(t,z)\,dz \le D\left(x + k_-\frac{x^{a+1}}{a+1}\right) \to 0, \quad \text{as } x\to 0^+.$$

Since the integral $\int_0^x z^a dz$ is convergent, then $\int_\ell^x \lambda(t,z)\,dz$ tends to zero as $x\to 0^+$ (and thus $\ell\to 0^+$), which is the first integral in the definition of W. The second integral defining W is with respect to time. Regarding the Hamiltonian H, note that $L(0) = 0$, so by concavity of L, $L'(c)c \le L(c)$. Hence

$$\begin{aligned} L'(\phi(t,\ell))|F(\ell) - N\phi(t,\ell)| &\le L'(\phi(t,\ell))F(\ell) + NL'(\phi(t,\ell))\phi(t,\ell) \\ &\le D(1 + k_-\ell^a)F(\ell) + NL(\phi(t,\ell)) \end{aligned}$$

tends to 0 uniformly since $\ell \to 0^+$ by (13) and $\phi(t,\ell) \to 0$. The second summand in the integral is $\frac{1}{2}\sigma^2(\ell)\lambda_x(s,\ell)$. Let us show that it tends to 0 as $\ell \to 0$. Note that $\phi(t,0) = 0$ and

$$k_- \frac{f(x)}{x} \le \frac{\phi(t,x)}{x} \le k_+ \frac{f(x)}{x}$$

imply $k_- f'(0^+) \le \phi_x(t,0^+) \le k_+ f'(0^+)$, thus $\phi_x(t,\cdot)$ is bounded around 0 and in consequence

$$\sigma^2(\ell)L''(k_+f(\ell))\phi_x(s,\ell) \ge \sigma^2(\ell)\lambda_x(s,\ell) \ge \sigma^2(\ell)L''(k_-f(\ell))\phi_x(s,\ell)$$

where we have used that $L''' \ge 0$. Now, by (12), both sides tend to 0 as $\ell \to 0$.

Now let us show that W satisfies the Hamilton-Jacobi-Bellman equation for any $x > 0$ and $t < T$. Observe that, for $x > 0$, $W_x = \lambda$ and $W_{xx} = \lambda_x$ by the definition of W. Let $g(t)$ be the function of t given in the second summand of (10). The derivative is

$$g'(t) = rg(t) + H(\ell, \phi(t,\ell), \lambda(t,\ell)) + \frac{1}{2}\sigma^2(\ell)\lambda_x(t,\ell).$$

Also notice that

$$W_t(t,x) = \int_\ell^x \lambda_t(t,z)\,dz + g'(t).$$

Now, integrating with respect to x in (2), recalling the definition of W and rewriting in terms of W_x, W_{xx} we have

$$\begin{aligned} 0 &= -rW(t,x) + rg(t) + W_t(t,x) - g'(t) + H(x,\phi(t,x),\lambda(t,x)) - H(l,\phi(t,l),\lambda(t,l)) \\ &\quad + \frac{1}{2}\sigma^2(x)\lambda_x(t,x) - \frac{1}{2}\sigma^2(l)\lambda_x(t,l) \\ &= -rW(t,x) + W_t(t,x) + H(x,\phi(t,x),\lambda(t,x)) + \frac{1}{2}\sigma^2(x)\lambda_x(t,x) \\ &= -rW(t,x) + W_t(t,x) + H(x,\phi(t,x),W_x(t,x)) + \frac{1}{2}\sigma^2(x)W_{xx}(t,x). \end{aligned}$$

Given Hypothesis (A2), the function $c^i \longmapsto H(x,(c^i|c_{-i}),\lambda)$ is concave and, by the definition of λ,

$$H_{c^i}(t,x,(c^i|c_{-i}),\lambda) = H_{c^i}(t,x,(c^i|c_{-i}),L'(c^i)) = L'(c^i) - L'(c^i) = 0.$$

Since critical points of concave functions are global maximum, we get that for arbitrary admissible consumption strategies c^i and for $i = 1,\dots,N$

$$\begin{aligned} 0 &= -rW(t,x) + W_t(t,x) + h(x,\phi,W_x(t,x)) + \frac{1}{2}\sigma^2(x)W_{xx}(t,x) \\ &\ge -rW(t,x) + W_t(t,x) + H(t,x,(c^i|\phi_{-i}),W_x(t,x)) + \frac{1}{2}\sigma^2(x)W_{xx}(t,x). \end{aligned}$$

Hence W is a solution of the HJB equation (25). Moreover, $W(T,x)=S(x)$. This is easily seen as follows

$$W(T,x)=\int_\ell^x \lambda(T,z)\,dz+S(\ell)=\int_\ell^x L'(\phi(T,z))\,dz+S(\ell)=\int_\ell^x L'(\varphi(z))\,dz+S(\ell)=S(x),$$

since $\varphi=(L')^{-1}\circ S'$. Now, condition (15), identity $V_x=\lambda$, together with the continuity of $W(t,x)$ at $x=0$ proved above, imply that W is indeed the value function of the problem and ϕ^i is the MPNE of player i, $i=1,\dots,N$. This is because V is polynomially bounded in x, so we can apply the verification theorems of Dockner *et al* (2000) or Fleming and Soner (2006). □

The result is extended to the infinite horizon game with the help of the usual transversality condition. We analyze only the stationary case. From (6), the stationary MPNE ϕ is characterized by the EL equation

(16)

$$\big(F(x)-N\phi(x)+(N-1)R(\phi(x))+\sigma'(x)\sigma(x)\big)\phi'(x)-\frac{1}{2}\sigma(x)^2P(\phi(x))\phi'(x)^2+\frac{1}{2}\sigma(x)^2\phi''(x)$$
$$+R(\phi(x))(r-F'(x))=0.$$

Now, no terminal condition is imposed, but we still have to consider the boundary condition $\phi(0)=0$. The value function can be expressed in terms of the MPNE as

(17)

$$V(x)=\int_\ell^x L'(\phi(z))\,dz+\frac{1}{r}\Big(L(\phi(\ell))+L'(\phi(\ell))(F(\ell)-N\phi(\ell))+\frac{1}{2}\sigma^2(\ell)L''(\phi(\ell))\phi'(\ell)\Big),$$

where $0<\ell<x$ is an arbitrary constant. In the following result, X^c denotes the solution of (1) when the admissible profile of strategies $(c,\dots,c)$ is played.

Theorem 3.3. *Assume that the conditions of Theorem 3.2 hold and that for all $c\in\mathcal{U}$, the transversality condition*

$$\limsup_{T\to\infty} \mathrm{E}_{tx}\left\{e^{-r(T-t)}V(X^c(T))\right\}=0$$

holds. Then, a stationary and smooth solution ϕ of the Cauchy problem (16) *is an MPNE of the differential game. Moreover, the value function of the players is given by* (17).

Proof. We can follow the same arguments as for the finite horizon case shown in Theorem 3.2 above, by selecting a bounded function S fulfilling the hypotheses required by that theorem, so that $\lim_{T\to\infty}e^{-rT}S(x)=0$. For a Markov stationary strategy $\phi(x)$, the costate variable λ is also independent of time. Taking the limit as $T\to\infty$ in the function g obtained from (10) in the proof of Theorem 3.2, we have

$$\begin{aligned}
g(t,\ell)&=\lim_{T\to\infty}\int_t^T e^{-r(s-t)}\left(H(\ell,\phi(s,\ell),\lambda(s,\ell))+\frac{1}{2}\sigma(\ell)^2\lambda_x(s,\ell)\right)ds\\
&=\left(H(\ell,\phi(\ell),L'(\phi(\ell)))+\frac{1}{2}\sigma(\ell)^2L''(\phi(\ell))\phi'(\ell)\right)\lim_{T\to\infty}\int_t^T e^{-r(s-t)}\,ds\\
&=\frac{1}{r}\left(H(\ell,\phi(\ell),L'(\phi(\ell)))+\frac{1}{2}\sigma(\ell)^2L''(\phi(\ell))\phi'(\ell)\right).
\end{aligned}$$

We need the transversality condition of the theorem to apply a verification theorem of Dockner *et al* (2000) or Fleming and Soner (2006) for the infinite horizon case. □

4. The stochastic Keynes-Ramsey Rule

In this section we describe how the EL equation (6) defines a stochastic KR rule for the MPNE. This has the form of a forward–backward stochastic differential equation (FBSDE)[8]. Note that the representation is by no means unique and that the process Y below can be chosen in multiple ways.

Given a solution $\phi(t,x)$ of the PDE (6) satisfying $\phi(T,x)=\varphi(x)$, let $C(s)=\phi(s,Y(s))$ for $s\geq t$, where the stochastic process Y satisfies the SDE

$$dY(s)=\big(F(Y(s))+g_N(C(s))+\sigma'(Y(s))\sigma(Y(s))\big)\,ds+\sigma(Y(s))\,dw(s),\qquad Y(t)=x, \tag{18}$$

where $g_N(c)=-Nc+(N-1)R(c)$. Since we are supposing in (A3) (a) (see also (8)) that $\alpha^+=-N+(N-1)a^+\leq 0$ and $R(c)\leq a^+c$, then $g_N(c)\leq 0$ for all $N\geq 1$ and all $c\geq 0$.

A solution of the stochastic KR rule we will show below, is a triplet (Y,C,Z) of $\{\mathcal{F}_s\}_{s\geq t}$–adapted processes.

Theorem 4.1. *Assume that the hypotheses of Theorem 3.2 hold and that σ'' is bounded. Then, the symmetric MPNE is described by the stochastic KR rule given by the forward SDE* (18) *and the backward SDE*

$$dC(s)=C(s)\left(\rho(C(s))(F'(Y(s))-r)+\frac{1}{2}\pi(C(s))Z^2(s)\big)\,ds+Z(s)\,dw(s)\right), \tag{19}$$

with final value

$$C(T)=\varphi(Y(T)),$$

where Z is a square–integrable $\{\mathcal{F}_s\}_{s\geq t}$–adapted process. Moreover, Z is non–negative a.s. in the case that both F and S are concave.

Proof. Under the assumptions and by Theorem 3.1, there is a unique non-negative classical solution $\phi(t,x)$ of equation (6) satisfying $\phi(T,x)=\varphi(x)$ and $\phi(s,0)=0$ for any $s\geq t$. Consider the forward SDE (18). Since ϕ, F, R, σ and σ' are uniformly Lipschitz, by Lemma 3.2 in Mao (1997), the SDE admits a unique strong solution $Y(s)$, $s\geq t$ for all $t\geq 0$, which satisfies $Y(s)\geq 0$.

By Itô's formula applied to $C(s)=\phi(s,Y(s))$ we get

$$\begin{aligned}dC(s)=\Big(&\phi_t(s,Y(s))+\big(F(Y(s))+g_N(C(s))+(\sigma\sigma')(Y(s))\big)\phi_x(s,Y(s))+\frac{1}{2}\sigma^2(Y(s))\phi_{xx}(s,Y(s))\Big)\,ds\\&+\phi_x(s,Y(s))\sigma(Y(s))\,d\omega(s).\end{aligned}$$

[8]FBSDEs were introduced by Bismut (1973) for stochastic control problems and studied with more generality in Pardoux and Peng (1990). FBSDEs play a central role in the statement of the stochastic maximum principle, see Peng (1990), Yong and Zhou (1999) or Ma and Yong (1999). FBSDEs in optimization models play the same role as the Hamiltonian system for the state-costate variable in deterministic dynamics, that is, they constitute a part of the necessary conditions for optimality. In our game, the forward part corresponds to the new state Y and the backward part to the optimal strategy C. The definition of a backward SDE is not straightforward, as the filtration of the Brownian motion is an increasing family of σ-algebras, but the stochastic process C evolves in the opposite direction to time. The measurability problems are overcome by introducing the process Z, which is an integral part of the definition of the solution.

Defining $Z(s) = \sigma(Y(s))\phi_x(s, Y(s))/\phi(s, Y(s))$ for $s \geq t \geq 0$, and using the EL equation (6), the equality reduces to (19). Note that Z is square–integrable and $\{\mathcal{F}_s\}_{s\geq t}$–adapted by the regularity of the functions involved. When $T < \infty$ the terminal condition for C at T comes from (7). In the case of an infinite horizon, we follow the definition given in Ma and Yong (1999), considering finite–horizon approximations where the terminal condition for C is given by selecting previously a suitable bequest function S, as explained in the sections above. Finally, since by Theorem 5.2 (below) $\phi_x \geq 0$, we have $Z(s) = \sigma(Y(s))\phi_x(s, Y(s))/\phi(s, Y(s)) \geq 0$ a.s. □

Abusing the notation, from (19) and the definitions of ρ and π, we have that for $s \geq t$ and any $t \geq 0$

$$E_s\left(\frac{dC}{C}\right) = \left(\rho(C)(F'(Y) - r) + \tfrac{1}{2}\pi(C)Z^2\right)ds. \tag{20}$$

Thus, the expected mean increase in future consumption with respect to current consumption is a correction of the Ramsey Keynes rule of the deterministic case[9] ($Z \equiv 0$), where the correction term, $\frac{1}{2}\pi Z^2$, collects the uncertainty effects. Notice that the larger the relative prudence index is, the larger the effect of uncertainty, and that the sign of π determines whether the stochastic KR rule implies mean increments in consumption above ($\pi > 0$) or below ($\pi < 0$) the deterministic rule, since $Z^2 > 0$. Since we are supposing that $\pi > 0$, then uncertainty speeds up mean consumption in comparison with the deterministic case.

To better understand the stochastic KR , let us rewrite the couple of FBSDE in integral form. For $s \geq t$,

$$Y(s) = x + \int_t^s \left(F(Y(z)) + g_N(C(z)) + (\sigma\sigma')(Y(z))\right)\,dz + \int_t^s \sigma(Y(z))\,d\omega(z),$$

$$C(s) = \varphi(Y(T)) - \int_s^T C(z)\left(\rho(C(z))(F'(Y(z)) - r) + \frac{1}{2}\pi(C(z))Z^2(z)\right)\,dz - \int_s^T C(z)Z(z)\,d\omega(z).$$

In particular, for $s = t$ we have that $C(t) = \phi(t, x)$ is deterministic. Taking conditional expectations and due to the adaptiveness of the solution one gets the following non–linear Feynman–Kac representation for the MPNE

$$\phi(t, x) = E_{t,x}\left\{\varphi(Y(T)) - \int_t^T C(z)\left(\rho(C(z))(F'(Y(z)) - r) + \frac{1}{2}\pi(C(z))Z^2(z)\right)\,dz - \int_t^T C(z)Z(z)\,d\omega(z)\right\}$$

We can also interpret the above in the infinite horizon case, by selecting a suitable function φ, and considering that the pair of FBSDEs above hold for any T.

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[9]The deterministic KR rule is obtained setting $\sigma \equiv 0$ in (18) and (19), that implies $Z \equiv 0$, to get

$$\frac{\dot{C}}{C} = \rho(C)(F'(Y) - r),$$

where the (deterministic) process Y satisfies

$$\dot{Y} = F(Y) + g_N(C).$$

Of course, in the one–player game where $N = 1$ and $g_1(c) = -c$, this is the classical Ramsey rule established by Ramsey (1928) in the classical growth model.

For illustration purposes only, let us show the KR rule for the linear game. Assuming an infinite horizon, the smooth symmetric MPNE is given by the linear consumption function, $c(x) = \mu x$, where $\mu > 0$. Then one has $Z(s) = \mu\sigma Y(s)$ and (19) becomes

$$dC(s) = \left(\frac{A-r}{\delta} - \frac{1}{2}\sigma^2(1-\delta)\right) C(s)\, ds + \sigma C(s)\, dw(s).$$

and (20) is

$$E_s\left(\frac{dC}{C}\right) = \frac{A-r}{\delta} - \frac{\sigma^2}{2}(1-\delta).$$

The process Y complementing the representation of the consumption process satisfies

$$d(\mu Y)(s) = (A + \mu\alpha)\mu Y(s)\, ds + \mu\sigma Y(s)\, dw(s)$$

by (18). Note that by (4)

$$d(\mu Y)(s) = dC(s) = \mu\left(\frac{A-r}{\delta} - \frac{1}{2}\sigma^2(1-\delta)\right) Y(s)\, ds + \mu\sigma Y(s)\, dw(s).$$

Equating both expressions we get

$$A + \alpha\mu = \frac{A-r}{\delta} - \frac{1}{2}\sigma^2(1-\delta),$$

which solving for μ, of course coincides with Corollary 5.1.

5. Comparative statics and further properties of the MPNE

The EL Equation is useful to reveal important properties of the symmetric MPNE. In this section, we study: i) the turnpike property of the MPNE; ii) the monotonicity of the equilibrium consumption strategy with respect to the resource stock; iii) the curvature of the equilibrium consumption strategy; iv) the dependence with respect to variations in the number of players and with respect to the preference rate; v) the dependence with respect to the size of the uncertainty; and finally, vi) the issue of extinction.

5.1. Finite horizon approximations of the stationary MPNE.

Now we investigate the turnpike properties of the game in the case of CRRA preferences (but not necessarily the linear game, that is, F and σ do not need to be linear). We prove next that for any smooth solution $f(x)$ of the stationary EL equation, one can find bequest functions S such that the associated finite horizon MPNE converges to f. This is useful in computing approximated solutions of the stationary EL equation based on well known methods for the Cauchy problem. In what follows, we study the limit as $T \to \infty$ of the solution of the EL equation for a finite horizon T, although we do not make the dependence of $\phi(t, x)$ on T explicit so as to shorten notation.

Theorem 5.1. *Consider the game with CRRA instantaneous utility function L and $\alpha < 0$. Suppose that f is a solution of the stationary EL equation* (16)*. Let S be any bequest function of the finite horizon game such that f serves as a limiting function. Then, the solution of the finite horizon EL equation, $\phi(t, x)$, converges to $f(x)$ as $T \to \infty$.*

Proof. See Appendix C. □

The reverse question, that is, whether the finite horizon MPNE $c(t,x)$ converges to a solution of the stationary EL equation is more difficult. A result in this direction is obtained below for the linear game.

Corollary 5.1. *If c is an MPNE for the linear game then*

$$\lim_{T\to\infty}\phi(t,x) = -\frac{\beta+\gamma}{\alpha}x,$$

where $\beta = A+\sigma^2(1-\delta)/2$, $\gamma = (r-A)/\delta$, $\alpha = -N+(N-1)/\delta$, that is, it converges to the stationary MPNE of the autonomous game.

Proof. Notice that $f(x) = -\dfrac{\beta+\gamma}{\alpha}x$ is a solution of the stationary EL equation (16), thus the result is a consequence of Theorem 5.1, simply by taking this limiting function f. □

5.2. **Monotonicity.** Under mild assumptions, the MPNE is monotonous non-decreasing in the asset stock: the higher the stock of the stochastic productive asset, the higher the consumption is in equilibrium. Once this is shown, we prove that the value function of the players is concave. A direct proof of this fact in a differential game framework (even in the symmetric case) does not seem to follow easily from standard arguments.

Theorem 5.2. *Assume that the assumptions of Theorem 3.1 hold and that both F and S are concave. Then the MPNE is non–decreasing in x.*

Proof. See Appendix C. □

Corollary 5.2. *Assume that the assumptions of Theorem 5.2 hold. Then the value function is concave in x.*

Proof. This follows from the shadow price characterization given in the proof of Theorem 3.2 and from Theorem 5.2 on the monotonicity of the optimal consumption program, since

$$V_{xx}(t,x) = \frac{\partial}{\partial x}\lambda(t,x) = \frac{\partial}{\partial x}L'(\phi(t,x)) = L''(\phi(t,x))\,\phi_x(t,x) \le 0.$$

□

5.3. **Concavity of the MPNE.** In this section we study whether the MPNE is concave with respect to the asset level. Carroll and Kimball (1996) proved concavity of the consumption function in a one-player game of finite horizon and discrete time, where uncertainty comes from three sources: labor income, gross interest rate and discount factor. The family of utility functions considered by those authors were the CRRA class, strictly increasing, concave, and satisfying $L''' \ge 0$. We limit ourselves to the linear game, but allowing for general functions S. Note that Carroll and Kimball (1996) choose $S = 0$. Another difference, of course, is that we work in the continuous-time case, with Brownian uncertainty. To prove concavity of the equilibrium consumption function we will impose that φ is concave. This implies a condition both for the instantaneous utility function and the bequest function that is provided in Lemma C.1.

Theorem 5.3. *Consider the linear game and suppose also that φ is concave. Then the MPNE is concave with respect to x.*

Proof. See Appendix C. □

5.4. **Dependence of the MPNE on the uncertainty.** In this section we study the way the symmetric MPNE depends on the size of uncertainty. We will show that the MPNE is monotonous increasing in σ, more precisely, we prove that under our assumptions plus a technical hypothesis to allow a change of measure, $\sigma_1 \leq \sigma_2$ implies $\phi^{\sigma_1} \leq \phi^{\sigma_2}$, where we denote ϕ^{σ_i} the MPNE strategy when the diffusion coefficient $\sigma \in \Sigma$, with Σ defined below. To obtain this result is not straightforward, as the equilibrium is driven by a backward SDE, which is coupled with a forward SDE. In our analysis we will use a variation of the KR rule found in Section 4 that represents the equilibrium. Let $\mu = \frac{F}{\sigma} + \frac{\sigma'}{2}$ and let $\Sigma = \{\sigma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \; : \; \mu \text{ bounded}\}$. For $\sigma \in \Sigma$, let Y^σ be the process that satisfies (we omit the time argument to simplify notation)

$$dY = (F(Y) + (\sigma\sigma')(Y))\, ds + \sigma(Y)\, dw(s), \qquad Y(t) = x.$$

Along Y^σ, the equilibrium satisfies the BSDE

$$\begin{aligned} dC &= \left(R(C)(F'(Y) - r) + \frac{1}{2} P(C) C^2 Z^2 - \frac{g_N(C) Z}{\sigma(Y)} \right) ds + Z dw(s), \\ C(T) &= \varphi(Y(T)), \end{aligned} \tag{21}$$

where Z is given by $Z = \sigma(Y^\sigma(s))\phi_x(s, Y^\sigma(s))$ and ϕ is the solution of the EL equation. The proof follows the same lines as the proof of the KR rule in Theorem 4.1.

Theorem 5.4. *If both F and S are concave, and φ is monotonous non-decreasing, then the MPNE is monotonous non decreasing with respect to $\sigma \in \Sigma$.*

Proof. See Appendix C. □

In the limit, as the diffusion coefficient tends to zero, and assuming that an MPNE exists of the deterministic game and that convergence holds, we get that the players are more conservative in the deterministic case than when some noise is present. This conclusion holds at every pair of date and stock of the resource, and not only in terms of expected mean growth. Thus, in this model, independently of the sign of the relative prudence index, the players consume at a higher rate as the uncertainty is larger (as measured by the function σ). This result also holds for the one-player case.

5.5. **Variation in the number of players and on the time preference.** Now we will study the dependence of the MPNE with respect to variations in the number of players. The effect of the number of players on the consumption effort depends on the marginal substitution rate $\rho(c)$. When $\rho(c) > 1$, consumption increases with N, when $\rho(c) < 1$ it decreases, and when $\rho(c) = 1$ (the logarithm case), it is independent of the number of players. We will use the KR rule to show these facts.

Theorem 5.5. *Suppose that the hypotheses of Theorem 4.1 hold and let $N_1 \leq N_2$. Then $\phi^{N_1} \leq \phi^{N_2}$ if $\rho(c) \geq 1$ for any $c \geq 0$, and $\phi^{N_1} \geq \phi^{N_2}$ if $\rho(c) \leq 1$ for any $c \geq 0$.*

Proof. See Appendix C. □

In the linear game above the MPNE is non–decreasing with the number of agents if and only if $\delta \leq 1$.

Theorem 5.6. *Suppose that the hypotheses of Theorem 4.1 hold. If $r_1 \le r_2$, then $\phi^{r_1} \le \phi^{r_2}$.*

Proof. See Appendix C. □

5.6. **Extinction.** We explore here the question of extinction of the asset in the long run. This has already been investigated in Clemhout and Wan (1985) in the model without uncertainty. These authors proved that with the assumptions $L(0) = 0$, $L'(0) \in (0, \infty)$, $L'' < 0$, $Nr < F'(0) < \infty$, there exists an $\tilde{x} > 0$ such that in equilibrium, any initial stock level $x \in (0, \tilde{)}$ declines to extinction in finite time. This is not true under cartelization. Our main finding is that uncertainty strengthens the competition of the players compared to the deterministic case, and the resource declines to extinction with probability 1 *from any initial level of the resource* as the time horizon tends to infinity if the number of players is large enough.

For our purposes, we consider linear $F(x) = Ax$ and $\sigma(x) = \sigma x$, but an arbitrary utility function L satisfying our standing assumptions. For a solution $\phi(t, x)$ of the EL equation (6) we know from Theorem 3.1 that

$$k_- f(x) \le \phi(t, x) \le k_+ f(x),$$

whenever f is a suitable limiting function. Recall the definition of the constants k_- and k_+ just after Theorem 3.1.

$$k_- = \min\left\{m, -\frac{\beta^-}{\alpha^- \sup_{(0,\infty)} f'}\right\}, \qquad k_+ = \max\left\{M, -\frac{\beta^+}{\alpha^+ \inf_{(0,\infty)} f'}\right\},$$

where α^-, α^+ are defined in (8) and β^- and β^+ in Assumption (A5) (c). Notice that, given that both F and σ are linear, we can also take $f(x) = x$. Hence the MPNE satisfies

$$k_- x \le \phi(t, x) \le k_+ x \tag{22}$$

for any $t \le T$, for any T and for any $x \ge 0$. We are free to select a bequest function S such that the associated φ satisfies the standing assumption, since we are only interested in the limiting behavior of the solution. The asset follows the SDE

$$dX(s) = (AX(s) - N\phi(s, X(s)))\, ds + \sigma X(s)\, dw(s), \quad X(t) = x.$$

Let us denote by $X^\phi(s; t, x)$ the unique strong solution. By (22), the drift is bounded by

$$AX - Nc \le (A - Nk_-)X.$$

Consider the SDE

$$d\overline{X}(s) = (A - Nk_-)\overline{X}\, ds + \sigma \overline{X}\, dw(s).$$

By the classical Comparison Theorem for SDEs, Ikeda and Watanabe (1977), we have $0 \le X^\phi \le \overline{X}$ a.s. Hence, if the SDE for $\overline{X}$ is asymptotically stable, the same happens for the SDE for X.

Proposition 5.1. *Suppose that*

$$A - Nk_- < \frac{\sigma^2}{2}. \tag{23}$$

Then the equilibrium $X \equiv 0$ is asymptotically stable in the large, i.e. it is stochastically stable

$$\lim_{x \to 0} \mathbf{P}\{\sup_{s \ge t} |X(s; t, x) \ge 0|\} = 0$$

and for any initial level of stock $x > 0$

$$\mathbf{P}\{\lim_{T\to\infty} X(T;t,x) = 0\} = 1.$$

Proof. The proof is a direct application of Example 2.7 in Mao (1997). For $X = 0$ to be asymptotically stable, it suffices that the drift term should be smaller than $\sigma^2/2$. □

Inequality (23) depends on several parameters of the game and on the number of players. Since $-\beta^-/\alpha^-$ tends to 0 as $N \to \infty$ because $\alpha^- \to 0$, we have $k_- = -\beta^-/\alpha^-$ for a big enough N. Then, for a large number of players N

$$A - Nk_- = A + \frac{\beta^- N}{(a^- - 1)N - a^-}.$$

This expression is decreasing with N and in the limit as $N \to \infty$ inequality (23) is

$$A + \frac{\beta^-}{a^- - 1} < \frac{\sigma^2}{2}.$$

6. Conclusions

The purpose of this paper has been three-fold. First, to carry out a rigorous study of a symmetric stochastic dynamic game in continuous time where players consume from a productive asset in a noncooperative way. Second, to provide an easy method to obtain the EL equations of a stochastic differential game where the uncertainty is modeled as a diffusion process and players' decisions cannot influence the size of the uncertainty. Third and last, to show the usefulness of the EL equations in proving the existence and uniqueness of the MPNE in the game mentioned above, under quite general assumptions. Moreover, it has also been shown how the EL equations are specially suitable to make comparative statics exercises of the equilibrium and to answer important questions of the nature of the dependence of the equilibrium on uncertainty, its curvature and the issue of extinction. As we show, EL equations are equivalent to a stochastic KR rule that is a natural extension of the deterministic one and that shows in a neat way how the uncertainty changes the consumption-saving decisions of the players with respect to the deterministic case. To our knowledge, most of these questions are completely answered for the first time in this paper, thanks to the approach based on the EL equations. Our methods of proof combine methods of partial differential equations to show properties that concern first or second order derivatives of the consumption function, as well as the convergence of finite horizon approximations, together with comparison theorems of FBSDEs for making comparative statics.

With the exception of Sorger (1998), which proves the existence of MPNE in the deterministic infinite horizon game (but, as explained in the paper, with just the opposite hypothesis that we impose about the elasticity of the marginal utility, and assuming a specific recruitment function), other investigations focus on a particular form of the utility function, the recruitment function or the diffusion coefficient.

It is by no means trivial to prove the existence of the Nash equilibrium with the property of sub game perfection, even for what we can consider by now is a classical game that has received a lot of attention over the last few decades. As shown in the paper, in our non-existence result of the MPNE, one cannot freely work with any number of players and any strictly concave and smooth utility function, as it could be that no MPNE exists unless one sets an upper bound in the maximal consumption rate of the players. To address these questions seriously, avoiding heuristic claims that could not be supported on the existence of the object that is analyzed, it seems unavoidable to impose the correct amount of smoothness in the functions defining the game.

It is our hope to have had success conveying other researches about the usefulness of the EL equations approach to analyze models from economics, and that they include it as an additional tool for economic analysis. Our aim for the future is to study other relevant models with this technique, as well as to extend the method to problems where the players influence the size of the uncertainty through their decisions .

Appendix A. Description of the game

In this appendix we formulate a general stochastic differential game to which the model studied in the paper is a particular case. Standard references for differential games are Melhmann (1998), Başar and Olsder (1999) or Dockner *et al* (2000).

We shall use the following notation. The partial derivatives are indicated by subscripts and ∂_x stands for *total derivation*; the partial derivative of a scalar function with respect to a vector is a column vector; given a real vector function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a vector $z \in \mathbb{R}^n$, g_z is defined as the matrix $(\partial g^i/\partial z^j)_{i,j}$; for a matrix A, $A^{(i)}$ denotes the ith column and A^{ij} denotes the (i,j) element; vectors $v \in \mathbb{R}^n$ are column vectors and v^i is the ith component; $^\top$ denotes transposition.

We consider an N–person differential game over a fixed and bounded time interval $[0,T]$ with $0 < T \leq \infty$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. Assume that on this space a d–dimensional Brownian motion $\{w(t), \mathcal{F}_t\}_{t\in[0,T]}$ is defined with $\{\mathcal{F}_t\}_{t\in[0,T]}$ being the Brownian filtration. Let E denote expectation under the probability measure $\mathbf{P}$. We also consider the function space $L^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$ of all processes $X(\cdot)$ with values in $\mathbb{R}^n$ adapted to filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathrm{E}\int_0^T \|X(t)\|^2\,dt < \infty$.

The state space is a subset $\mathcal{X} \subseteq \mathbb{R}^n$ and the set of admissible profiles of the players is some subset $U = U^1 \times U^2 \times \cdots \times U^N$, where $U^i \subseteq \mathbb{R}^{m_i}$, with[10] $m_i = n$, for all $i = 1, \ldots, N$. *A U–valued process of strategic profiles* $\{(u(s);\mathcal{F}_s) = ((u^1(s), u^2(s), \ldots, u^N(s));\mathcal{F}_s)\}$ defined on $[t,T]\times\Omega$ is an $\mathcal{F}_s$–progressively measurable map $(r,\omega) \to u(r,\omega)$ from $[t,s]\times\Omega$ into U, that is, $u(t,\omega)$ is $\mathcal{B}_s \times \mathcal{F}_s$–measurable for each $s \in [t,T]$, where $\mathcal{B}_s$ denotes the Borel σ–field in $[t,s]$. For simplicity, we will denote $u(t,\omega)$ by $u(t)$.

The state process $X \in \mathbb{R}^n$ satisfies the system of controlled stochastic differential equations (SDEs)

$$dX(s) = f(s, X(s), u(s))\,ds + \sigma(s, X(s))\,dw(s), \quad t \leq s \leq T, \tag{24}$$

with initial condition $X(t) = x$, $t \in [0,T]$, $x \in \mathbb{R}^n$. Observe that the diffusion coefficient, σ, is independent of the control variable, u. The functions $f : [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$ are both assumed to be of class $\mathcal{C}^2$ with respect to (x,u) and of class $\mathcal{C}^1$

[10] The case $m_i > n$ could also be analyzed, by means of a reduction to the case $m_i = n$ as in Josa–Fombellida and Rincón–Zapatero (2007).

with respect to t. Since our aim is to work with the MPNE concept, we will consider the game for every initial condition (t,x).

Definition A.1 (Admissible strategies)**.** *A strategic profile* $\{(u(t);\mathcal{F}_t)\}_{t\in[0,T]} = \{((u^1(t),u^2(t),\dots,u^N(t));\mathcal{F}_t)\}_{t\in[0,T]}$ *is called admissible if*

(i) *for every (t,x) the system of SDEs* (24) *with initial condition $X(t)=x$ admits a pathwise unique strong solution;*
(ii) *for each $i=1,\dots,N$, there exists some function $\phi^i:[0,T]\times\mathbb{R}^n\longrightarrow U^i$ of class $\mathcal{C}^{1,2}$ with respect to (t,x) such that u^i is in relative feedback to ϕ^i, i.e. $u^i(s)=\phi^i(s,X(s))$ for every $s\in[0,T]$.*

Let $\mathcal{U}^i(t,x)$ denote the set of admissible strategies of player i and $\mathcal{U}=\mathcal{U}^1\times\cdots\times\mathcal{U}^N$ the set of admissible strategies profiles, corresponding to the initial condition $(t,x)\in[0,T]\times\mathbb{R}^n$.

Part (ii) in Definition A.1 means that players use Markov strategies. When ϕ^i is time independent, we will say that the strategy is a stationary Markovian strategy. In the case $T=\infty$, the feedback is simply a function of (t,x) and we will write in this case $\mathcal{U}^i(t,x)$ for the set of admissible strategies of player i.

The instantaneous utility function of player i is denoted by L^i and his or her bequest function by S^i. Given initial conditions $(t,x)\in[0,T]\times\mathbb{R}^n$ and an admissible strategic profile u, the *payoff function* of each player (to be maximized) is given by

$$J^i(t,x;u)=\mathrm{E}_{tx}\left\{\int_t^T e^{-r_i(s-t)}\,L^i(s,X(s),u(s))\,ds+e^{-r_i(T-t)}S^i(T,X(T))\right\},$$

where E_{tx} denotes conditional expectation with respect to the initial condition (t,x). In the following, the subscript will be eliminated if there is no confusion. The functions $L^i:[0,T]\times\mathbb{R}^n\times U\longrightarrow\mathbb{R}$ and $S^i:[0,T]\times\mathbb{R}^n\longrightarrow\mathbb{R}$, $i=1,\dots,N$, are both of class $\mathcal{C}^2$ with respect to (x,u) and of class $\mathcal{C}^1$ with respect to t. The constant $r_i\geq 0$ is the rate of discount. $J^i(t,x;u)$ denotes the utility obtained by player i when the game starts at (t,x) and the profile of strategies is u. Given that our aim is to solve the problem *for every* $(t,x)\in[0,T]\times\mathbb{R}^n$, $\mathcal{U}$ will often be written instead of $\mathcal{U}(t,x)$.

In the infinite horizon case, the bequest functions S^i are null. In this case, if the problem is autonomous and the strategies are Markov stationary, the value function is independent of time, and the initial condition is simply x.

In a non–cooperative setting, the aim of the players is to maximize their individual payoff J^i. Since this aspiration depends on the strategies selected by the other players also, it is generally impossible to attain[11]. An adequate concept of solution is Nash equilibrium, which prevents unilateral deviations of the players from its recommendation of play.

Definition A.2 (MPNE)**.** *An N–tuple of strategies $\widehat{\phi}\in\mathcal{U}$ is called a Markov perfect Nash equilibrium if for every $(t,x)\in[0,T]\times\mathbb{R}^n$, for every $\phi^i\in\mathcal{U}^i$*

$$J^i(t,x;(\phi^i|\widehat{\phi}_{-i}))\leq J^i(t,x;\widehat{\phi}),$$

for all $i=1,\dots,N$.

[11]But in some models the MPNE is also Pareto optimal; see Martín–Herrán and Rincón–Zapatero (2005).

In the above definition, $(\phi^i|\widehat{\phi}_{-i})$ denotes $(\widehat{\phi}^1,\dots,\widehat{\phi}^{i-1},\phi^i,\widehat{\phi}^{i-1},\dots,\widehat{\phi}^N)$. Note that with an MPNE no player has incentives to deviate unilaterally from the equilibrium, whatever the initial condition (t,x) is.

Definition A.3 (Value functions)**.** *Let $\widehat{\phi}$ be an MPNE of the game. The value function V^i of the ith player is*

$$V^i(t,x)=\sup_{\phi^i\in\mathcal{U}^i}\Big\{J^i(t,x;(\phi^i|\widehat{\phi}_{-i})):dX(s)=f(s,X(s),(\phi^i|\widehat{\phi}_{-i})(s,X(s)))\,ds+\sigma(s,X(s))\,dw(s),$$

$$X(t)=x,\quad \forall s\in[t,T]\Big\},$$

for all $(t,x)\in[0,T]\times\mathbb{R}^n$, for all $i=1,\dots,N$.

It is known that under suitable smoothness conditions, $V=(V^1,\dots,V^N)$ satisfies problem (25) and (**??**) below.

Appendix B. The system of EL equations

In what follows, the assumptions are that both V^i and ϕ^i are of class $C^{1,2}$, that $V^i_{xt}=V^i_{tx}$ and that ϕ^i is interior to the control region U^i. The smoothness of the value functions of the players guarantee that they satisfy the HJB equations with terminal conditions

$$r_iV^i(s,x)=V^i_t(s,x)+\max_{u^i\in U^i}H^i(s,x,(u^i|\phi_{-i}),V^i_x(s,x))+\frac{1}{2}\operatorname{Tr}(\sigma\sigma^\top V^i_{xx})(s,x), \tag{25}$$

$$V^i(T,x)=S^i(T,x),\quad t\le s\le T,$$

where H^i is the Hamiltonian of player i

$$H^i(s,x,u,p^i)=L^i(s,x,u)+(p^i)^\top f(s,x,u).$$

Since the MPNE is interior to $U=U^1\times\cdots\times U^N$, the maximization condition

$$\max_{u^i\in U^i}H^i(t,x,(u^i|\phi_{-i}),V^i_x),\qquad i=1,\dots,N$$

turns into

$$L^i_{u^i}(s,x,(\phi^i|\phi_{-i}))+f_{u^i}(s,x,(\phi^i|\phi_{-i}))^\top V^i_x=0,\qquad i=1,\dots,N$$

which is explicitly solvable for $V^i_x=\Lambda^i(s,x,\phi):=f_{u^i}^{-\top}L^i_{u^i}(t,x,\phi)$, supposing f^i_u to be invertible for all i.

On the other hand, by the Envelope Theorem, for each $j=1,\dots,n$ we get

$$r_iV^i_{x_j}(t,x)=V^i_{x_jt}(t,x)+\frac{\partial}{\partial x_j}H^i(t,x,\phi,V^i_x(t,x))+\frac{1}{2}\frac{\partial}{\partial x_j}\operatorname{Tr}(\sigma\sigma^\top V^i_{xx})(t,x)).$$

Substituting now $V^i_x=\Lambda^i(t,x,\phi)$ we get for the MPNE $\phi=(\phi^1,\dots,\phi^N)$ an EL system of equations (of differential type)

$$r_i\Lambda^i(t,x,\phi)=\frac{\partial}{\partial t}\Lambda^i(t,x,\phi)+\frac{\partial}{\partial x}H^i(t,x,\phi,\Lambda^i(t,x,\phi))+\frac{1}{2}\frac{\partial}{\partial x}\operatorname{Tr}\Big(\sigma\sigma^\top\frac{\partial}{\partial x}\Lambda^i(t,x,\phi)\Big). \tag{26}$$

Notice that $\partial/\partial x$ denotes total differential with respect to x. No explicit dependence of the value functions appears, as in the EL Equations in discrete dynamic programming. The final condition when T is finite is $\phi^i(T,x)=\varphi^i(x)$, given implicitly by

$$L^i_{u^i}(T,x,\varphi^1(x),\dots,\varphi^N(x))+S^i_x(T,x)f_{u^i}(T,x,\varphi^1(x),\dots,\varphi^N(x))=0. \tag{27}$$

Appendix C. Proofs

C.1. **Proof of Theorem 3.1.** Let $R_T=[0,T]\times(0,\infty)$. We first state an auxiliary result. Existence of solutions to quasilinear parabolic PDEs.

Theorem C.1. *There exists at least one bounded solution in R_T of the Cauchy problem*

$$u_\tau-\frac{\partial}{\partial x}a(\tau,x,u,u_x)=b(\tau,x,u,u_x),$$

with initial condition

$$u(0,x)=u_0(x),\quad x>0.$$

if all of the following conditions are satisfied.

C1.- *u_0 is of class C^4 and bounded.*
C2.- *Functions a and b are of class C^3 and C^2, respectively.*
C3.- *There are nonnegative constants b_1 and b_2 such that for all x and u*

$$\left(b(\tau,x,u,0)+\frac{\partial}{\partial x}a(\tau,x,u,0)\right)u\le b_1u^2+b_2.$$

C4.- *For all $M>0$, there are constants $\mu_2(M)\ge\mu_1(M)>0$ such that, if τ, x and u are bounded in modulus by M, then for arbitrary p*

$$\mu_1(M)\le\frac{\partial}{\partial p}a(\tau,x,u,p)\le\mu_2(M)$$

and

$$\left(|a|+\left|\frac{\partial a}{\partial u}\right|\right)(1+|p|)+\left|\frac{\partial a}{\partial x}\right|+|b|\le\mu_2(M)(1+|p|)^2.$$

The problem admits no more than a classical solution in R_T that is bounded together with its derivatives of first and second orders if the following additional conditions hold.

C5.- *For all $M>0$ there are non negative constants $\nu_1(M)$ and $\nu_2(M)$ such that*

$$\max_{\substack{(t,x)\in R_T\\|u|,|p|\le M}}\left|\frac{\partial^2 a}{\partial p\partial u},\frac{\partial^2 a}{\partial p^2},\frac{\partial A}{\partial p}\right|\le\nu_1(M),$$

$$\min_{\substack{(t,x)\in R_T\\|u|,|p|\le M}}\frac{\partial A}{\partial u}\ge-\nu_2(M),$$

where

$$A=a-\frac{\partial a}{\partial u}p-\frac{\partial a}{\partial x}.$$

Proof. The result is a consequence of Theorem 8.1 of Ladyzhenskaya *et al* (1969), the only difference being that we have set the problem in $[0,T]\times(0,\infty)$ instead of $[0,T]\times\mathbb{R}$ and that we require more smoothness. The method of proof of Theorem 8.1 in Ladyzhenskaya *et al* consists in considering truncated problems on the strip $[0,T]\times[1/n,n]$ with boundary conditions $u_n(0,x)=u_0(x)$ for all $x\in[1/n,n]$ and[12] $u_n(t,1/n)=u_0(1/n)$, $u_n(t,n)=u_0(n)$ for all $t\geq 0$. These solutions converge smoothly to a solution u of the original equation as $n\to\infty$. □

Motivated by the necessity to drop the boundedness of the data defining the game, we will consider the EL equation (6) for the function $u=\phi/f$ written in the divergence form

$$u_\tau-\frac{\partial}{\partial x}a(x,u_x)=b(x,u,u_x), \tag{28}$$

where $\tau=T-t$ and

$$\begin{aligned}
a(x,p)&=\frac{\sigma(x)^2}{2}p,\\
b(x,u,p)&=\Big(F-Nuf+(N-1)R(uf)+\frac{f'}{f}\sigma^2\Big)p\\
&\quad+\Big(F-Nuf+(N-1)R(uf)+\sigma'\sigma\Big)u\frac{f'}{f}\\
&\quad+\frac{1}{2}\sigma^2u\frac{f''}{f}-\frac{\sigma^2}{2}\frac{P(uf)}{f}(pf+uf')^2+\frac{r-F'}{f}R(uf).
\end{aligned}$$

The initial condition is

$$u(0,x)=\varphi_0(x)=\frac{\varphi(x)}{f(x)}. \tag{29}$$

Note that even if the initial condition φ_0 is now bounded by assumption, still we cannot apply the above theorem directly. The difficulties are two: (i) the function σ vanishes at $x=0$, thus it is not uniformly bounded away from zero; and (ii) the function P is in general not defined at 0 and in fact $\lim_{c\to 0^+}P(c)=\infty$ for problems with CRRA utility, where $P(c)=(1+\delta)/c$. To deal with (i) we consider the PDE (28) on bounded subintervals $I_n=[1/n,n]$, $n=1,2,\ldots$, and then we take a limit as $n\to\infty$, while for (ii) we will prove that the solutions u_n found in the subintervals above remain uniformly bounded away from 0, in the sense that there exists a lower bound $l_m>0$ such that $u_n\geq l_m$ for all $n\geq m$ for all $x\in I_m$. As a byproduct of the proof, we obtain the estimates claimed in the theorem.

- *C.1* is fulfilled, since $\varphi_0(x)=\varphi(x)/f(x)$ is bounded and smooth on $(0,\infty)$, by assumption.
- *C.2* holds, as the function a has the required smoothness. As explained above, we will prove below that u never vanishes on $(0,\infty)$, thus the term $P(uf)$ does not pose any problem at all for the smoothness of function b.

[12]The selection of the boundary conditions at $1/n$ and n can be done differently, with the only requisite of being compatible with $u_0(x)$, that is, conserving continuity and smoothness; in the proof below we will use a different set of boundary conditions still compatible.

- *C.3.* There are constants b_1 and b_2 such that

$$b(x,u,0)u = \Big(F - Nuf + (N-1)R(uf) + \sigma'\sigma\Big)u^2\frac{f'}{f} + \frac{1}{2}\sigma^2 u^2 \frac{f''}{f} - \frac{\sigma^2}{2}\frac{P(uf)}{f}(uf')^2 u + \frac{r - F'}{f}R(uf)u \le b_1 u^2 + b_2$$

(since $a(x,0) = 0$). To see this, note that, thanks to our assumptions,

$$b(x,u,0)u \le \alpha^+ u^3 f' + \left((F + \sigma'\sigma)\frac{f'}{f} + \frac{1}{2}\sigma^2\frac{f''}{f} - \frac{b^-}{2}\sigma^2\left(\frac{f'}{f}\right)^2 + \gamma^+\right)u^2 < \alpha^+(\min_{x\in(0,\infty)} f')u^3 + \beta^+ u^2 \le \beta^+ u^2,$$

since $\alpha^+ < 0$ and the solution $u > 0$ (this will be proved below).
- *C.4*, first part, is also fulfilled, as

$$\frac{\partial a(x,p)}{\partial p} = \frac{\sigma^2(x)}{2}$$

is positive for $x > 0$ and continuous, thus it is bounded away from 0, as well as bounded above in any compact subset $[1/M, M]$ of $(0,\infty)$[13]. The second part of *C.4* is a local assumption, that also holds because function $|b|$ is quadratic in p, with continuous coefficients, and thus bounded on compact subsets of $(0,\infty)$; the same is true for

$$\left(|a| + \left|\frac{\partial a}{\partial u}\right|\right)(1 + |p|) + \left|\frac{\partial a}{\partial x}\right| = \frac{\sigma^2(x)}{2}|p|(1 + |p|) + |\sigma(x)\sigma'(x)||p|,$$

which is also quadratic in p, and since both σ and σ' are continuous and thus they are bounded in compact subsets of $[0,\infty)$.
- *C.5.* Both σ and σ' are continuous, and therefore bounded over any compact interval of the state space, achieving uniqueness of the bounded solution.

We now show that we can construct a solution $u > 0$ of the PDE (28) as limit of positive truncated solutions u_n in $[0,T] \times [1/n, n]$ as $n \to \infty$. Then, *C.2* holds, as P is smooth in $(0,\infty)$. In fact we prove more than that, as we obtain upper and lower estimates for the solution. The latter will imply in particular that the solution is positive for $x > 0$.

Let the PDE (28) with initial condition (29) and boundary conditions at the extreme points of the interval I_n given by

$$u_n(\tau, 1/n) = \frac{\beta^+ e^{\tau\beta^+}\varphi_0(1/n)}{\alpha^+\varphi_0(1/n)f'(1/n)(1 - e^{\tau\beta^+}) + \beta^+}, \qquad u_n(\tau, n) = \frac{\beta^+ e^{\tau\beta^+}\varphi_0(n)}{\alpha^+\varphi_0(n)f'(n)(1 - e^{\tau\beta^+}) + \beta^+}. \tag{30}$$

The reason for this particular selection of the boundary conditions is shown next. Formulating these approximating problems, we have eliminated the degeneration in the truncated equation.

[13]Note that what we require here is $\mu_1(M) \le \frac{\partial a(x,p)}{\partial p} \le \mu_2(M)$ for x belonging to a compact set of the kind $[1/M, M]$ and not simply to $x \le M$. This is a slight variation that is not problematic in this framework.

It is easy to check that all conditions in Theorem 8.1 of Ladyzhenskaya *et al* (1968) are fulfilled in a small neighborhood of $\tau = 0$, since $\varphi_0(x)$ is bounded away from zero in I_n and hence a solution u_n of the truncated problem exists which does not vanish. From this our aim is to extend this truncated solution to all $[0,T]$ and after this, to get the solution $\phi(t,x)$ with initial condition $\phi(0,x) = \varphi(x)$ as the smooth limit of $u_n(\tau,x)f(x)$ as $n \to \infty$. As explained above, this is the procedure used in Ladyzhenskaya *et al* (1968). We show next that the local solution can be extended in time and space. To this end we define

$$M_n(\tau) = \max_{y\in[1/n,n]} u_n(\tau,y).$$

By Danskin's Theorem, function M_n is almost everywhere differentiable, and at points of differentiability, the derivative $\dot{M}_n(\tau) = u_{n,\tau}(\tau, x_n(\tau))$, where $u_n(\tau, x_n(\tau)) = M_n(\tau)$.

We prove that for any τ

$$M_n(\tau) \le \frac{\beta^+ e^{\tau\beta^+} M_n(0)}{\alpha^+(\min_{I_n} f')M_n(0)(1-e^{\tau\beta^+})+\beta^+}. \tag{31}$$

where $M_n(0) = \sup_{x\in I_n} \varphi_0(x)$. Suppose, by way of contradiction, than $M_n(\tau_0)$ is greater that the right hand side of (31) for some $\tau_0 > 0$. Let $\overline{\tau}_0$ be the inferior of all the τ_0s satisfying this property, and hence, by continuity of M_n, $M_n(\overline{\tau}_0)$ equals the right hand side of (31). Then $x_n(\tau)$ is interior to I_n for every $\tau \in [\overline{\tau}_0, \tau_0]$, due to the boundary conditions (30). In consequence, $u_{n,x}(\tau, x_n(\tau)) = 0$ and $u_{n,xx}(\tau, x_n(\tau)) \le 0$ for any $\tau \in [\overline{\tau}_0, \tau_0]$. Hence, $(\partial/\partial x)(\sigma^2 u_{n,x})|_{(\tau,x_n(\tau))} \le 0$. This information, used in the equation (28) for ν_n, provides the following chain of inequalities

$$\begin{aligned}
\dot{M}_n(\tau) &\le b(x_n(\tau), M_n(\tau), 0) \\
&= \Big(F - NfM_n(\tau) + (N-1)R(fM_n(\tau)) + \sigma'\sigma\Big)M_n(\tau)\frac{f'}{f} \\
&\quad + \frac{1}{2}\sigma^2 M_n(\tau)\frac{f''}{f} - \frac{1}{2}\sigma^2\frac{P(fM_n(\tau))}{f}(M_n(\tau)f')^2 + \frac{r-F'}{f}R(fM_n(\tau)) \\
&\le \alpha^+ M_n^2(\tau)f' + \left((F+\sigma'\sigma)\frac{f'}{f} + \frac{1}{2}\sigma^2\frac{f''}{f} - \frac{b^-}{2}\sigma^2\left(\frac{f'}{f}\right)^2 + \gamma^+\right)M_n(\tau) + \\
&< \alpha^+(\min_{I_n} f')M_n^2(\tau) + \beta^+ M_n(\tau), \quad \text{a.e. } \tau.
\end{aligned} \tag{32}$$

We have used the definitions of α^+ and β^+ done in Assumption (A3) and (A5), respectively, and the fact that $\alpha^+ < 0$. From this differential inequality of Ricatti we get the estimate

$$M_n(\tau) \le \frac{\beta^+ e^{(\tau-\overline{\tau}_0)\beta^+} M_n(\overline{\tau}_0)}{\alpha^+(\min_{I_n} f')M_n(\overline{\tau}_0)(1-e^{(\tau-\overline{\tau}_0)\beta^+})+\beta^+}.$$

Once the expression for $M_n(\overline{\tau}_0) = \frac{\beta^+ e^{\overline{\tau}_0\beta^+} M_n(0)}{\alpha^+(\min_{I_n} f')M_n(0)(1-e^{\overline{\tau}_0\beta^+})+\beta^+}$, which holds by our contradiction argument is substituted into (C.1), inequality (31) easily follows.

Following the same technique we get the lower estimate. Consider the solution v_n of the PDE with the boundary conditions

$$v_n(\tau, 1/n) = \frac{\beta^- e^{\tau\beta^-} \varphi_0(1/n)}{\alpha^- \varphi_0(1/n) f'(1/n)(1 - e^{\tau\beta^-}) + \beta^-},$$
$$v_n(\tau, n) = \frac{\beta^- e^{\tau\beta^-} \varphi_0(n)}{\alpha^- \varphi_0(n) f'(n)(1 - e^{\tau\beta^-}) + \beta^-}.$$

and

$$m_n(\tau) = \min_{y\in[1/n,n]} v_n(\tau, y),$$

with the minimum attained at some $y_n(\tau)$. Similar arguments as done above for the maximum lead to the estimate

$$m_n(\tau) \geq \frac{\beta^- e^{\tau\beta^-} m_n(0)}{\alpha^- m_n(0)(\max_{I_n} f')(1 - e^{\tau\beta^-}) + \beta^-}. \tag{33}$$

as follows. By way of contradiction one finds

$$\dot{m}_n(\tau) \geq g(y_n(\tau), m_n(\tau), 0) \geq \alpha^-(\max_{I_n} f') m_n^2(\tau) + \beta^- m_n(\tau), \quad \text{a.e. } \tau.$$

Reasoning much as for the case of M_n, one gets the estimate (33) easily. Thus, we have shown that the local solution is strictly uniformly bounded away from zero in the intervals I_n, and that it is also bounded above. Since this fact is independent of τ, as well as the upper bound obtained above, the solution can be extended up to the whole $[0, T]$, for any T.

Hence, since $v_n \to v$, we have that v is bounded away from zero, and hence v is a solution of the Cauchy problem , since all the conditions of the Theorem of Ladyzhenskaya *et al* (1968) are fulfilled. Taking limits as $n \to \infty$ one has

$$\frac{\phi(t,x)}{f(x)} = u(T-t, x) \geq m(T-t) \geq \frac{\beta^- e^{(T-t)\beta^-} m(0)}{\alpha^- m(0)(\sup_{(0,\infty)} f')(1 - e^{(T-t)\beta^-}) + \beta^-}.$$

By the above estimates the limit $\phi(t,x) = u(T-t,x)f(x)$ is a solution of the Cauchy problem (6) satisfying

$$\frac{\phi(t,x)}{f(x)} = v(T-t, x) \leq M(T-t) \leq \frac{\beta^+ e^{(T-t)\beta^+} M}{\alpha^+(\inf_{(0,\infty)} f') M (1 - e^{(T-t)\beta^+}) + \beta^+},$$

since $M_n(0) \to M$ as $n \to \infty$, and $\inf_{(0,\infty)} f' \leq \min_{I_n} f'$.

□

C.2. **Proof of Theorem 5.1.** With CRRA preferences, $a^+ = a^- = a = 1/\delta$, $\alpha^+ = \alpha^- = \alpha$ and $b^+ = b^- = b = 1 + \delta$. See the definition of these constants in Section 2, assumption (A3) and in (8). See also the computations done after Definition 2.1. We will follow the proof of Theorem 3.1, using the same notation. Retaking inequality (32) in the aforementioned proof, in the case of CRRA preferences we have

$$\begin{aligned}\dot{M}_n(\tau) &\leq b(x_n(\tau), M_n(\tau), 0) \\ &= \Big(F - NfM_n(\tau) + (N-1)afM_n(\tau)) + \sigma'\sigma\Big) M_n(\tau)\frac{f'}{f} \\ &\quad + \frac{1}{2}\sigma^2 M_n(\tau)\frac{f''}{f} - \frac{1}{2}\sigma^2 \frac{b}{f^2 M_n(\tau)}(M_n(\tau)f')^2 + \frac{r-F'}{f}M_n(\tau)a \\ &\leq \alpha f' M_n^2(\tau) + \Big((F+\sigma'\sigma)f' + \frac{1}{2}\sigma^2 f'' - \frac{1}{2}\sigma^2\frac{b}{f}f'^2 + a(r-F')f\Big)\frac{M_n(\tau)}{f}\end{aligned}$$

Since f is solution of the stationary EL equation (6) with CRRA preferences, then

$$(F+\sigma'\sigma)f' + \frac{1}{2}\sigma^2 f'' - \frac{1}{2}\sigma^2\frac{b}{f}f'^2 + (r-F')af = -\alpha f f'.$$

Plugging this into the inequality above it simplifies to

$$\dot{M}_n(\tau) \leq \alpha f' M_n^2(\tau) - \alpha f' M_n(\tau) \leq \alpha(\inf_{I_n} f')M_n^2(\tau) - \alpha(\sup_{I_n} f')M_n(\tau).$$

Hence

$$M_n(\tau) \leq \frac{M_n(0)e^{-\tau\alpha(\sup_{I_n} f')}}{-M_n(0)(1-e^{-\tau\alpha(\sup_{I_n} f')})+1}.$$

A similar computation for the minimum shows

$$m_n(\tau) \geq \frac{m_n(0)e^{-\tau\alpha(\inf_{I_n} f')}}{-m_n(0)(1-e^{-\tau\alpha(\inf_{I_n} f')})+1}.$$

Notice that in the former case f' is evaluated at the point where u_n attains a maximum in $I_n = [1/n, n]$, say x_n, whereas in the latter case it is at the point where u_n attains a minimum, say y_n. Thus,

$$\frac{m_n(0)e^{-\tau\alpha(\inf_{I_n} f')}}{-m_n(0)(1-e^{-\tau\alpha(\inf_{I_n} f')})+1} \leq u_n(t,x) \leq \frac{M_n(0)e^{-\tau\alpha(\sup_{I_n} f')}}{-M_n(0)(1-e^{-\tau\alpha(\sup_{I_n} f')})+1}.$$

Taking the limit as n tends to ∞ and since $u_n(t,x) \to \phi(t,x)/f(x)$ as $n \to \infty$, we find

$$\frac{me^{-\tau\alpha(\inf_{I_n} f')}}{-m(1-e^{-\tau\alpha(\inf_{I_n} f')})+1} \leq \frac{\phi(\tau,x)}{f(x)} \leq \frac{Me^{-\tau\alpha(\sup_{I_n} f')}}{-M(1-e^{-\tau\alpha(\sup_{I_n} f')})+1}.$$

As $T \to \infty$, $\tau \to \infty$ and one finally find that $\phi(t,x)$ converges to $f(x)$. □

C.3. **Proof of Theorem 5.2.** We follow the same scheme of proof as in theorems 3.1 and 5.1. Notice that the hypotheses of the theorem imply $\varphi'(x) = S''(x)/L''(S'(x)) \geq 0$. Now, derive the EL Equation (6) with respect to x and let $v = \phi_x$. Then w solves the Cauchy problem

$$\begin{aligned}v_\tau - \frac{1}{2}\frac{\partial}{\partial x}(\sigma(x)^2 v_x) &= g_x(x,\phi,v) + vg_c(x,\phi,v) + v_x g_v(x,\phi,v) \\ &\quad + v\sigma'(x)^2 + v\sigma(x)\sigma''(x) + v_x\sigma(x)\sigma'(x). \\ v(0,x) &= \varphi'(x) \geq 0.\end{aligned}$$

Function g is

$$g(x,c,v) = (F(x) - Nc + (N-1)R(c))v - \frac{1}{2}\sigma(x)^2 P(c)v^2 + (r - F'(x))R(c).$$

We will follow a similar strategy of proof as in Theorem 3.1, considering truncated intervals for the variable x, $I_n = [1/n, n]$ and the solution $v_n(\tau, x)$ in $[0,T] \times I_n$ satisfying the boundary conditions

$$v_n(\tau, 1/n) = \varphi'(1/n) > 0, \quad v_n(\tau, n) = \varphi'(n) > 0.$$

Let $\nu_n(\tau) = \min_{x \in I_n} v_n(\tau, x)$. A reasoning by contradiction, assuming the existence of τ_0 satisfying $\nu_n(\tau_0) < \nu_n(0)$, will lead to a contradiction as follows. Let $\overline{\tau}_0$ be the inferior of all the τs satisfying this property; by continuity of ν_n, $\nu_n(\overline{\tau}_0) = \nu_n(0)$. Then we obtain the inequality

$$\dot{\nu}_n \geq g_x(x, \phi, \nu_n) + \nu_n \big(g_c(x, \phi_n, \nu_n) + \sigma'(x)^2 + \sigma(x)\sigma''(x) \big).$$

We have used the same arguments as those used in the proof of Theorem 3.1, hence we do not repeat it here. The notation ϕ_n is used for the solution of the EL equation (6) in $[0,T] \times I_n$. The term $\phi_n(\tau, \zeta(\tau))$, where $\zeta(\tau)$ minimizes v_n over I_n, does not pose any problem at all. Given that $g_x(x,c,v) = v\big(F'(x) - v\sigma(x)\sigma'(x)P(c)\big) - F''(x)R(c)$, we have

$$\begin{aligned} \dot{\nu}_n \geq & -F''(x)R(\phi_n) \\ & + \nu_n \big(F'(x) - \nu_n \sigma(x)\sigma'(x)P(\phi_n) + g_c(x, \phi_n, \nu_n) + \sigma'(x)^2 + \sigma(x)\sigma''(x) \big), \end{aligned}$$

with $\nu_n(\overline{\tau}) = 0$. Since F'' is concave and R is non-negative, $-F''(x)R(c) \geq 0$, thus

$$\nu_n(\tau) \geq \nu_n(\overline{\tau}_0) e^{\int_{\overline{\tau}_0}^{\tau} \{\cdots\}\, d\tau} = 0 \quad \forall \tau \in [\overline{\tau}_0, \tau_0],$$

in contradiction with $\nu_n(\tau_0) < 0$. Hence $0 \leq \nu_n(\tau) \leq v_n(\tau, x)$ for all τ, $x \in I_n$, and then the limit function $\phi_x(\tau, x) \geq 0$. □

C.4. **Proof of Theorem 5.3.** We first establish a lemma about the concavity of the consumption rate at the final time T. It is established for general L, not only in the class CRRA. In the lemma, $\rho_{\{\}}$ stands for the elasticity of the marginal utility and $\pi_{\{\}}$ for the relative prudence index of a given utility function. For the linear game, the lemma implies that φ is concave if and only if the bequest function S satisfies

$$\frac{S'(x)S'''(x)}{S''^2(x)} \geq 1 + \frac{1}{\delta}.$$

Lemma C.1. $\varphi'' \leq 0$ *if and only if for all* $x > 0$

$$\rho_S(x)\pi_S(x) \geq \rho_L(\varphi(x))\pi_L(\varphi(x)).$$

Proof. Deriving twice in $L'(\varphi(x)) = S'(x)$ we get

$$\begin{aligned} & L''(\varphi(x))\varphi'(x) = S''(x), \\ & L'''(\varphi(x))\varphi'(x)^2 + L''(\varphi(x))\varphi''(x) = S'''(x). \end{aligned}$$

Solving for $\varphi''(x)$ and imposing $\varphi''(x) \leq 0$ we obtain the inequality (we eliminate arguments)

$$\frac{S'''}{S''^2} \geq \frac{L'''}{L''^2}$$

or equivalently, multiplying both sides of the inequality by $S' > 0$

$$\left(\frac{-xS'''}{S''}\right)\left(\frac{-S'}{xS''}\right) \geq \left(\frac{-\varphi L'''}{L''}\right)\left(\frac{-S'}{\varphi L''}\right).$$

Noting that $S' = L'(\varphi)$, and plugging this equality into the right hand side of the inequality above, we obtain the claim of the lemma. □

We now proceed with the proof of Theorem 5.3 with the same techniques as those used in the proofs of Theorems 3.1 and 5.2, deriving twice in the EL equation (6) to find a PDE for ϕ_{xx}. We refer the reader to the proofs of those theorems for filling in the missing details. Deriving twice in (6) we get

$$(\phi_{xx})_\tau - \frac{\partial}{\partial x}\left(\frac{\sigma^2 x^2}{2}\phi_{xxx}\right) = j(x, \phi, \phi_x, \phi_{xx}, \phi_{xxx})$$

where the function j is defined as

$$\begin{aligned} j(x,c,v,w,z) = {} & w\left(3\sigma^2 + 2F'(x) + 3\alpha v + \frac{r - F'(x)}{\delta} - 4\sigma^2 x\frac{1+\delta}{c}v + \frac{5}{2}\sigma^2 x^2\frac{1+\delta}{c^2}v^2 - \sigma^2 x^2\frac{1+\delta}{c}w\right) \\ & - \sigma^2 v^2 \frac{1+\delta}{c}\left(1 - \frac{xv}{c}\right)^2 \\ & - F'''(x)\frac{c}{\delta} + F''(x)\left(1 - \frac{2}{\delta}\right)v \\ & + \left(2\sigma^2 x + F(x) + g_N(c) - \sigma^2 x^2\frac{1+\delta}{c}v\right)z. \end{aligned}$$

Recall that in the CRRA case, $\alpha = -N + (N-1)/\delta$. Following the same method of proof as in the above referenced theorems, and defining w_n as the solution of the PDE above in the interval I_n, we get

$$w_{n,\tau} - \frac{\partial}{\partial x}\left(\frac{\sigma^2 x^2}{2}w_{n,x}\right) = j(x, \phi_n, \phi_{n,x}, w_n, w_{n,x}). \tag{34}$$

Here, ϕ_n denotes the restriction of ϕ to $[0,T] \times I_n$. Let $\omega_n(\tau) = \max_{I_n} w_n(\tau, x)$. Reasoning by contradiction supposing that $\omega_n(\tau) > 0$ at some $\overline{\tau} > 0$, one has that $w_{n,x}(\overline{\tau}) = 0$ and $\frac{\partial}{\partial x}\left(\frac{\sigma^2 x^2}{2}w_{n,x}\right) \leq 0$, thus plugging this into equation (34) we get

$$\dot{\omega}_n \leq \omega_n\{\cdots\} - \sigma^2\phi_{n,x}^2\frac{1+\delta}{\phi_n}\left(1 - \frac{x\phi_{n,x}}{\phi_n}\right)^2 - F'''(x)\frac{\phi_n}{\delta} + F''(x)\left(1 - \frac{2}{\delta}\right)\phi_{n,x}.$$

In the linear game $F'' = F''' = 0$, hence the second summand in the above expression is non positive. Given that $\omega_n(0) = \sup_{[0,\infty)} \varphi''(x) \leq 0$ is also non–positive, we arrive to a contradiction, because it is never possible to have $\omega_n(\overline{\tau}) > 0$ from the above estimate for $\dot{\omega}_n$. □

C.5. **Proof of Theorem 5.4.** Let us show that if $\sigma_i \in \Sigma$, $i = 1,2$ and $\sigma_1 \le \sigma_2$, then $Y^{\sigma_1} \le Y^{\sigma_2}$ a.s. Clearly, for $\sigma \in \Sigma$, $\mu(Y(s))$ satisfies the Novikov condition and then

$$M(s) = \exp\left\{\int_t^s \mu(Y(a))d(a) - \frac{1}{2}\int_t^s \mu^2(Y(a))da\right\}, \quad s \in [t,T]$$

is a $\mathbf{P}$-martingale, where $\mathbf{P}$ is the objective probability measure. Define now the probability measure $\widetilde{\mathbf{P}}$ by

$$\frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}} = M(T).$$

It is known that $\widetilde{\mathbf{P}}$ is absolutely continuous with respect to $\mathbf{P}$. By Girsanov's Theorem, $\widetilde{\omega}(s) = \omega(s) - \int_t^s \mu(Y(a))d(a)$ is a $\widetilde{\mathbf{P}}$-Brownian motion and, in the new measure, Y satisfies

$$dY(s) = \frac{1}{2}\sigma(Y(s))\sigma'(Y(s))ds + \sigma(Y(s))dw(s).$$

Now, given that $0 < \sigma_1 \le \sigma_2$, it holds that $\int_x^y \frac{dz}{\sigma_1(z)} \ge \int_x^y \frac{dz}{\sigma_2(z)}$ for all $y \ge x$, which is the sufficient condition of Example 2 of Zhiyuan (1984) assuring $Y^{\sigma_1} \le Y^{\sigma_2}$ for every $s \in [t,T]$, $\widetilde{\mathbf{P}}$-a.s., whence $\mathbf{P}$-a.s. Then $\varphi(Y^{\sigma_1}(T)) \le \varphi(Y^{\sigma_2}(T))$ $\mathbf{P}$-a.s., since φ is non-decreasing by hyphotesis.

Now, let us define $f^i(\omega,s,C,Z)$ as the drift term in the backward SDE for dC in (21) when the forward process is $Y^{\sigma_i}(s)$, $i=1,2$, that is

$$f^i(\omega,s,C,Z) = C\left(\rho(C)(F'(Y^{\sigma_i}) - r) + \frac{1}{2}\pi(C)Z^2 - \frac{g_N(C)Z}{\sigma(Y^{\sigma_i})}\right),$$

as well as $\varphi^i(T) = \varphi(Y^{\sigma_i}(T))$. In the above, ω denotes an element of the sample space Ω where the probability $\mathbf{P}$ is defined, in order to stress the dependence of the drift term with respect to the stochastic process Y^{σ_i}. Let also (C^1,Z^1), (C^2,Z^2) be solutions of the BSDE (21) when $\sigma = \sigma_i$, $i=1,2$. We can write (21) in integral form

$$C^i(s) = \varphi^i - \int_s^T f^i(\omega,v,C(v),Z(v))dv - \int_s^T Z^i(v)dw(v).$$

Let us check that $-f^1(\omega,s,C^2,Z^2) \le -f^2(\omega,s,C^1,Z^1)$. Note that there are two terms that depend on i in the definition of f^i: one is $-\frac{g_N(C)Z}{\sigma(Y^{\sigma_i})}$; we have already proved that $\sigma(Y^{\sigma_1}) \le \sigma(Y^{\sigma_2})$. Since σ is increasing, that $Z \ge 0$ for all s $\mathbf{P}$-a.s., and that $g_N(C) \le 0$ (see Section 4), then $-\frac{g_N(C)}{\sigma(Y^{\sigma_2})} \le -\frac{g_N(C)}{\sigma(Y^{\sigma_1})}$. The other term is $F'(Y^{\sigma_i})$; since F is concave, F' is non-increasing, thus $-F'(Y^{\sigma_1}) \le -F'(Y^{\sigma_2})$. Hence, $-f^1(\omega,s,C^2,Z^2) \le -f^2(\omega,s,C^1,Z^1)$, as claimed. Now, the Comparison Theorem 2.2 in El Karoui et al (1997) ensure that $C^1(s) \le C^2(s)$ for all s, $\mathbf{P}$-a.s. Since the process C^i is deterministic at (t,x) and $C^i(t) = \phi^{\sigma^i}(t,x)$, we have that $\phi^{\sigma_1}(t,x) \le \phi^{\sigma_2}(t,x)$ and the proof is finished. □

C.6. **Proof of Theorem 5.5.** To show the result we will use a representation of the MPNE by means of an FBSDE, alternative to the KR rule introduced in Section 4, more amenable for our purposes. Let the process Y that satisfies

$$dY = (F(Y) + (\sigma\sigma')(Y))\,ds + \sigma(Y)\,d\omega, \qquad Y(t) = x,$$

where we have omitted the argument s to shorten notation. Then, for $C(s) = \phi(s, Y(s))$ and the square–integrable adapted process Z (=$\sigma(Y(s))\phi_x(s, Y(s))$), where ϕ is a solution of the EL equation (6)) we have, as in Section 5.4

$$dC(s) = \left(R(C(s))(F'(Y(s)) - r) - \frac{1}{2}P(C(s))Z^2(s) + \frac{g_N(C(s))}{\sigma(Y(s))}Z(s) \right) ds + Z(s)dw(s),$$

and $C(T) = \varphi(Y(T))$. Note that the process Z does not depend on the number of players, N. Let

$$f^N(\omega, s, C, Z) = R(C)(F'(Y(s)) - r) - \frac{1}{2}P(C)Z^2 + \frac{g_N(C)}{\sigma(Y(s))}Z$$

be the drift term of dC when the number of players is N and let (C^N, Z^N) be the corresponding solution. We have included $\omega \in \Omega$ into the notation to stress the dependence with respect to the process Y. We know that g_N is negative thanks to assumption (A3) (a) (see also (8)). On the other hand, it is easy to see that g_N is monotonous increasing if $\rho(c) > 1$ and decreasing if $\rho(c) < 1$. Let us suppose first that $\rho(c) > 1$. Given that $Z \geq 0$ a.s. by Theorem 4.1, then $\frac{g_{N_1}(C)}{\sigma}Z \leq \frac{g_{N_2}(C)}{\sigma}Z$ a.s. if $N_1 \leq N_2$. Hence $-f^{N_1}(\omega, s, C^{N_2}, Z^{N_2}) \leq -f^{N_2}(\omega, s, C^{N_2}, \xi^{N_2})$, as well as $C^{N_1}(T) = C^{N_2}(T)$. According to the Comparison Theorem 2.2 of El Karoui et al (1997), $C^{N_1}(s) \leq C^{N_2}(s)$ for all $t \leq s \leq T$, a.s. The case $\rho(c) < 1$ is analyzed analogously. Since $\phi^{N_i}(t, x) = C^{N_i}(t)$ is deterministic, the theorem is proved. □

C.7. **Proof of Theorem 5.6.** We follow the same steps as in the proof of Theorem 5.5, using the same FBSDE representation. The process Y is independent of the preference rate and the drift term of dC in (C.6), f^r, is decreasing in the preference rate. Hence $r_1 \leq r_2$ implies $-f^{r_1} \leq -f^{r_2}$ and, by the Comparison Theorem 2.2 of El Karoui et al (1997), we have the result. □

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