Optimal asset allocation for aggregated defined benefit pension funds with stochastic interest rates¹

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Abstract

In this paper we study the optimal management of an aggregated pension fund of defined benefit type, in the presence of a stochastic interest rate. We suppose that the sponsor can invest in a savings account, in a risky stock and in a bond, with the aim of minimizing deviations of the unfunded actuarial liability from zero along a finite time horizon. We solve the problem by means of optimal stochastic control techniques and analyze the influence on the optimal solution of some of the parameters involved in the model.

Mathematics Subject Classification (2000): 91B28, 93E20, 62P05, 60H10, 60J60.

JEL Classification: G23, G11, C61.

Subject and Insurance Branch Codes: E13, B81.

Keywords: Pension funds; Stochastic control; Optimal portfolio; Stochastic interest rate.

¹ Both authors gratefully acknowledge financial support from Consejería de Educación y Cultura de la Junta de Castilla y León (Spain) under project VA099/04 and Spanish Ministerio de Ciencia y Tecnología and FEDER funds under project MTM2005-06534.

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1 Introduction

Pension funds currently represent one of the most important institutions in financial markets because of their high investment capacity and because they complement the role of the Government, allowing those workers who have reached retirement age to maintain their standard of living. These two aspects justify the interest generated over recent years in the study of the optimum management of pension plans.

There are two principal alternatives in the design of pension plans in correspondence to the assignation of risk. In a defined contribution (DC) plan the risk derived from the fund management is borne by the beneficiary. However, in a defined benefit (DB) plan, where the benefits are normally related to the final salary level, the financial risk is assumed by the sponsor agent.

Our aim in this paper is to analyze a BD pension fund of aggregated type, which is a common model in the employment system. We provide here an extension of previous work of the authors, Josa–Fombellida and Rincón–Zapatero (2001, 2004, 2006, 2008), in an attempt to incorporate to the model more realistic assumptions, dropping out the hypothesis of a constant riskless rate of interest. Thus in our model there are three sources of uncertainty: i) the fund assets returns; ii) the instantaneous riskless rate of interest, and iii) the evolution of benefits, based in the behavior of salaries and/or other main components of the pension plan.

There are several previous papers dealing with the management of DC funds in the presence of a stochastic rate of interest. Some of them are Boulier et al (2001), Battocchio and Menoncin (2004), Cairns et al (2006) and Menoncin (2005), where the interest rate is assumed of Vasicek type. In Deelstra et al (2003) the interest rate has an affine structure as in Duffie and Kan (1996) which includes as special case the CIR and the Vasicek models. Other interesting papers where the interest rate is random though in a discrete time are Vigna and Haberman (2001) and Haberman and Vigna (2002). The importance of DB funds calls for completing the theory studying this case. Moreover, the differences in both type of pension plans makes not possible to translate the results from DC plans to DB plans.

The objective of the shareholder in a DC pension fund is to maximize the expected utility obtained from fund accumulation at a fixed date. The contribution rate is exogenous to this optimization process, since it is generally determined by salary. However, in a
DB plan the amortization effort is a control variable. The fund assets could be artificially increased with high contributions, borrowing if necessary. Obviously, this makes no sense, since benefits are fixed in advance. Thus the objective in a DB plan should be related with minimization of risks instead of maximization of fund assets. Of course, the main concern of the sponsor is the solvency risk, related with the security of the pension fund in attaining the comprised liabilities. Similar objectives have been considered in other works as Haberman and Sung (1994), Haberman et al (2000) and Josa–Fombellida and Rincón–Zapatero (2001, 2004. The optimal management of DB plans in the presence of a random interest rate is found, but in discrete time, in Haberman and Sung (1994), Chang (1999) and Chang et al (2003).

We make the contribution rate endogenous and dependent of the main variables of the fund, by adopting a spread method of amortization, as in Owadally and Haberman (1999). In this way, the contributions are proportional to the unfunded liabilities, requiring more amortization effort when the plan is underfunded. The pension plan is stochastic, supposing that benefits follow a geometric Brownian motion as in Josa–Fombellida and Rincón–Zapatero (2004). Then it is shown that both the stochastic actuarial liability and the normal cost are also geometric Brownian motions, and a relationship between these variables is found. The riskless rate of interest is supposed to be given by a mean-reverting process, as in Vasicek (1977). An interesting question addressed in the paper is the selection, according to a valuation criterium, of the technical rate of actualization to value the liabilities. The financial market comprises also a family of zero coupon bonds of fixed maturity and a risky stock, which are correlated with the source of uncertainty of the benefits.

The results obtained are based in the analytical solutions found by means of the dynamic programming approach. The optimal investment in the bond has four summands: i) the classical optimal one in Merton (1971); ii) the market price of risk multiplied by an expression involving diffusion coefficients of the bond and the stock and the excess expected return of the stock; iii) a positive term decreasing to zero with the terminal date of the plan, involving parameters of the riskless rate of interest; and iv) a term proportional to the actuarial liability, that vanishes if there is no correlation in the financial instruments or if the benefits are deterministic. The optimal investment in the risky asset follows a similar pattern, but now there are no the corresponding term described in i) and iii).
The paper is organized as follows. Section 2 defines the elements of the pension scheme of an employment system. We suppose the technical rate of interest is random. The actuarial functions are also introduced and we prove a relation between these functions when the benefits are given by a geometric Brownian motion. In Section 3 we expose the financial market structure. In Section 4 we find a risk–neutral valuation of the liabilities, given rise to an expression for the technical rate of actualization, that relates it with the interest rate an the parameters of correlation between the sources of uncertainty, as well as with the parameters defining the stochastic evolution of liabilities. In Section 5 we consider that the fund is invested in a riskless asset (savings account) and in two risky assets (a bond and a stock). We estate the problem of minimizing the expected value of the terminal solvency risk, and we explicitly solve it. In Section 6 the results are illustrated with a numerical analysis of the problem, analyzing the investment time evolution pattern in the bond and in the stock. Finally, Section 7 is dedicated to establish some conclusions and possible extensions. All proofs are in Appendix A.

2 The pension model

The pension plan we take into account is an aggregated pension fund of DB type, thus the benefits are established in advance by the manager. With the objective of delivery of retirement benefits to the workers, the plan sponsor continuously withdraws time–varying funds. The variables listed below refer to the total group of participants. The principal elements intervening in the funding process and the essential hypotheses allowing its temporary evolution to be determined are as follows.
Notation of the elements of the pension plan

\( T \): Planning horizon or date of the end of pension plan, with \( 0 < T < \infty \).

\( F(t) \): Value of fund assets at time \( t \).

\( P(t) \): Benefits promised to the participants at time \( t \). They are related with the salary at the moment of retirement.

\( C(t) \): Contribution rate made by the sponsor at time \( t \) to the funding process.

\( AL(t) \): Actuarial liability at time \( t \), that is, total liabilities of the sponsor.

\( NC(t) \): Normal cost at time \( t \); if the fund assets match the actuarial liability, and if there are no uncertain elements in the plan, the normal cost is the value of the contributions allowing equality between asset funds and obligations.

\( UAL(t) \): Unfunded actuarial liability at time \( t \), equal to \( AL(t) - F(t) \).

\( M(s) \): Percentage of the value of the future benefits accumulated until age \( s \in [a, d] \), where \( a \) is the common age of entrance in the fund and \( d \) is the common age of retirement.

\( \delta(t) \): Rate of valuation of the liabilities, which can be specified by the regulatory authorities.

\( r(t) \): Risk–free rate of interest of market.

Josa–Fombellida and Rincón–Zapatero (2004) considers there exit disturbances affecting the evolution of the benefits \( P \) and hence the evolution of the normal cost \( NC \) and the actuarial liability \( AL \), but the rate of valuation \( \delta \) of the plan is constant. In this paper we add a more general assumption: we suppose the short rate of interest \( r \) is random. This makes that the technical rate of interest \( \delta \) is random also. In order to simplify we will suppose both processes have the same source of uncertainty. As we have commented in the Introduction three sources of randomness appear in the problem: benefits, interest rate and stock. Thus to model this situation, we consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \mathcal{F} = \{ \mathcal{F}_t \}_{t \geq 0} \) is a complete and right continuous filtration generated by the three–dimensional standard Brownian motion \( \{(w(t), w_B(t), w_S(t)) : t \geq 0\} \) and \( \mathbb{P} \) is a probability measure on \( \Omega \). We assume that \( r \) and \( \delta \) satisfy stochastic differential
equations depending on $w_B$ only. The benefits randomness is due to another Brownian motion $w_P$. Given that the benefits $P$ are conditioned for the increase in salary of the sponsoring employees, we suppose the existence of correlation $q_1 \in [-1,1]$ between the Brownian motions $w_P$ and $w_B$, and $q_2 \in [-1,1]$ between Brownian motions $w_P$ and $w_S$, which can be explained by the effects of salary on inflation and the effects of the latter on the asset prices. This means that $w_P(t) = \sqrt{1 - q_1^2 - q_2^2} w(t) + q_1 w_B(t) + q_2 w_S(t)$, for all $0 \leq t \leq T$, supposing $q_1^2 + q_2^2 \leq 1$. When $q_1^2 + q_2^2 < 1$ the risk in the benefits outgo cannot be eliminated by trading in the financial market.

We consider that $r$, $\delta$ and $P$ are diffusion processes given by the stochastic differential equations

$$dr(t) = \mu_r(t, r(t))dt + \eta_r(t, r(t))dw_B(t),$$

$$d\delta(t) = \mu_\delta(t, \delta(t))dt + \eta_\delta(t, \delta(t))dw_B(t),$$

$$dP(t) = \mu_P(t, P(t))dt + \eta_P(t, P(t))dw_P(t)$$

$$= \mu_P(t, P(t))dt + \eta_P(t, P(t))\sqrt{1 - q_1^2 - q_2^2} dw(t)$$

$$+ \eta_P(t, P(t))q_1 dw_B(t) + \eta_P(t, P(t))q_2 dw_S(t),$$

for all $t \geq 0$, with $r(0) = r_0$, $\delta(0) = \delta_0$ and $P(0) = P_0$ representing the initial values of the interest rates and the benefits. However, to obtain a closed form solution we need a more concrete specification of these processes.

Following Bowers et al (1986) and Josa–Fombellida or Rincón–Zapatero (2004), we introduce the hypothesis that the benefits accumulated by the worker throughout his or her life are distributed according to the distribution function $M$, with an associated density function $m$. For age $x$, the value $M(x)$ represents the percentage of the actuarial value of the future benefits accumulated until $x$. The support of $m$ is the fixed interval $[a, d]$, hence $m(x) = 0$, if $x \leq a$ or $x \geq d$. We are supposing that all the participants enter the plan at age $a$, whereas the common age of retirement is $d$.

Along the lines of the constant rate of valuation case, the stochastic actuarial liability
and the stochastic normal cost are defined as follows:

\[
AL(t) = \int_a^d E \left( e^{-\delta(t)(d-x)} M(x) P(t+d-x) | \mathcal{F}_t \right) dx,
\]

\[
NC(t) = \int_a^d E \left( e^{-\delta(t)(d-x)} m(x) P(t+d-x) | \mathcal{F}_t \right) dx,
\]

for every \( t \geq 0 \), where \( E(\cdot | \mathcal{F}_t) \) denotes conditional expectation with respect to the filtration associated to the standard Brownian motion \( \{(w(t), w_B(t), w_S(t))\}_{t \geq 0} \). Thus, to compute the actuarial functions at time \( t \), the manager makes use of the information available up to that time, in terms of the conditional expectation. The information is resumed in the corresponding element of the filtration, \( \mathcal{F}_t \). \( AL(t) \) is the total expected value of the promised benefits accumulated according to \( M \), discounted at the rate \( \delta(t) \), that we suppose adapted to filtration. Observe that when computing total liability at time \( t \), the actuary applies the same value of the technical rate, \( \delta(t) \), in the whole interval of ages \([a, d]\). Analogous comments can be given to the normal cost \( NC(t) \). Note that benefits of retired participants is not a tradable asset and in consequence the inherent risk cannot be hedged and the market is incomplete.

Since \( P \) is a diffusion process, it satisfies the Markov property (see Øksendal (2003)), hence conditional expectation with respect to the filtration equals conditional expectation with respect the current values of \( P \) at time \( t \). It is plausible to think that in the task of computing the ideal values of the fund, the information given by the evolution of the random source will be used. Taking into account that \( \delta(t) \) is \( \mathcal{F}_t \)-measurable (see for
instanØksendal (2003), p. 30), the previous definitions can be rewritten:

\[ AL(t) = \int_0^d e^{-\delta(t)(d-x)} M(x) \mathbb{E}(P(t + d - x) | \mathcal{F}_t) \, dx, \]

\[ NC(t) = \int_0^d e^{-\delta(t)(d-x)} m(x) \mathbb{E}(P(t + d - x) | \mathcal{F}_t) \, dx. \]

For analytical tractability, we will need a more concrete specification for \( P \). A typical way of modelling \( P \) in the certain case is to postulate exponential growth, see Bowers et al (1986). The stochastic counterpart is to consider the benefits outgo as a geometric Brownian motion. This is the content of the following hypothesis.

**Assumption A.** The benefits \( P \) satisfies

\[ dP(t) = \mu P(t) \, dt + \eta P(t) \, dw_P(t), \quad t \geq 0, \]

where \( \mu \in \mathbb{R} \) and \( \eta \in \mathbb{R}_+ \). The initial condition \( P(0) = P_0 \) is a random variable that represents the initial liabilities.

Hence we are supposing that the benefits increases or decreases on average at a constant exponential rate. The behavior of the actuarial functions \( AL \) and \( NC \) is then given in the following proposition.

**Proposition 2.1** Under Assumption A there are processes \( \psi_{AL}(t) \) and \( \psi_{NC}(t) \) such that \( AL = \psi_{AL}P \) and \( NC = \psi_{NC}P \). Furthermore, \( \psi_{NC}(t) = 1 + (\mu - \delta)\psi_{AL}(t) \) and the identity \( (\delta(t) - \mu)AL(t) + NC(t) - P(t) = 0 \) holds for every \( t \geq 0 \).

From the proposition we deduce the stochastic differential equation that \( AL \) verifies:

\[ dAL(t) = \mu AL(t) \, dt + \eta AL(t) \, dw_P(t), \quad AL(0) = AL_0 = \psi_{AL}P_0. \quad (1) \]

Now we will use a spread method of fund amortization, as mentioned in the Introduction. Thus we will assume that the supplementary contribution rate (difference between contribution rate and normal cost) is proportional to the unfunded actuarial liability, that is

\[ C(t) = NC(t) + k(AL(t) - F(t)), \]

\[ 4 \text{Given integrable processes } Y, Z, \text{ with } Y(t) \mathcal{F}_t-\text{measurable, the identity } \mathbb{E}(Y(t)Z(s)|\mathcal{F}_t) = Y(t)\mathbb{E}(Z(s)|\mathcal{F}_t) \text{ holds.} \]
with \( k \) being a constant selected by the employer, representing the rate at which surplus or deficit is amortized. Though actuarial practice takes \( 1/k \) equal to a continuous annuity with amortization over \( m \) years, we consider more flexibility in the selection of \( k \) than actuarial practice suggests, as in Haberman and Sung (1994) or Josa–Fombellida and Rincón–Zapatero (2001, 2004).

3 The financial market

Now we establish the financial market. The plan sponsor manages the fund by means of a portfolio formed by a riskless \( R \), a coupon zero bond \( B \) and a stock \( S \).

First we assume the following hypothesis.

**Assumption B.** The instantaneous riskless interest rate \( r(t) \) satisfies the stochastic differential equation:

\[
    dr(t) = \alpha(\beta - r(t))dt + \sigma dw_B(t), \quad r(0) = r_0, \tag{3}
\]

where \( \alpha, \beta \) and \( \sigma \) are strictly positive constants.

This process of type mean–reverting and known as Orstein–Uhlenbeck process, has been introduced in Vasicek (1977) to explain the interest rate behavior.

We assume the price process of the riskless asset \( R \) is given by

\[
    dR(t) = r(t)R(t)dt, \quad R(0) = R_0, \tag{4}
\]

where the evolution of \( r(t) \) is given by (3). This asset can be interpreted as a bank account paying the instantaneous interest rate \( r(t) \) without any risk.

Given \( r \) we assume that there exits a market for zero coupon bonds with a fixed maturity \( T_1 > T \). Following Vasicek (1977) (see also Battocchio and Menoncin (2004)) the price at instant \( t \) of a zero coupon bond with maturity \( T_1 \), with \( t < T < T_1 \), is given by

\[
    B(t, T_1) = e^{c(t,T_1)−b(t,T_1)r(t)},
\]

where

\[
    b(t, T_1) = \frac{1}{\alpha}(1 - e^{-\alpha(T_1-t)}),
\]

\[
    c(t, T_1) = - R(\infty)(T_1-t) + b(t, T_1) \left(R(\infty) - \frac{\sigma^2}{2\alpha^2}\right) + \frac{\sigma^2}{4\alpha^3}(1 - e^{-2\alpha(T_1-t)}),
\]
and \( R(\infty) = \beta + \sigma \zeta \alpha - \sigma^2/(2\alpha^2) \) represents the return of a zero coupon bond with maturity equal to infinite, and \( \zeta \) is the constant market price of risk. Applying the Itô’s formula, the price of the bond process verifies the stochastic differential equation

\[
\frac{dB(t, T_1)}{B(t, T_1)} = \left( r(t) + \sigma \zeta b(t, T_1) \right) dt - \sigma b(t, T_1) dw_B(t), \quad B(T_1, T_1) = 0. \tag{5}
\]

Finally, we consider a stock whose dynamics is given by the stochastic differential equation

\[
dS(t) = S(t) \left( \mu_S(r(t)) dt + \sigma_r dw_B(t) + \sigma_S dw_S(t) \right), \quad S(0) = s_0, \tag{6}
\]

where the \( \sigma_r, \sigma_S \) are positive constants defining the stock volatility, that is \( \sqrt{\sigma_r^2 + \sigma_S^2} \), and the drift parameter \( \mu_S(r) \) is the instantaneous mean having the form \( \mu_S(r) = r + m_S \), with \( m_S \) a constant representing the expected excess return from investing in the stock, as in Deelstra et al (2003), Battocchio and Menoncin (2004) or Menoncin (2005).

4 Risk neutral valuation of the technical rate of actualization

The adequate valuation of liabilities is of the utmost importance for the sponsor. We address here the question of how to perform a fair valuation, having in mind that the promised benefits to participants is not a tradeable asset. To circumvent this problem we resort to a valuation based on a concept of equilibrium, see e.g. Constantinides (1978). The risk–neutral valuation of liabilities offers a univocally defined value for the technical rate of actualization, \( \delta \). This value is a modification of the short rate of interest, \( r \), to take into account the drift and diffusion components of the financial instrument and benefits as well as the several correlations existing between them.

For age \( x \), let \( Y^x(t, P) \) be the asset, valued at time \( t \), consisting in a payment of \( P \) monetary units at the age of retirement, \( d \), to a participant with current age \( x \). \( P \) is a geometric Brownian motion according to Assumption A. As \( P \) is not tradeable, it cannot be used to form a portfolio to hedge the risk. Thus we form a portfolio formed by \( Y^x, B \) and \( S \), with two tradeable assets and three independent Brownian motions. To obtain a risk neutral valuation it is assumed that the risk uncorrelated with the two freely traded financial instruments, i.e. the stock and the bond, is not priced. Notice that the actuarial
liability of the fund is
\[ AL(t) = \int_a^d Y^x(t, P(t))M(x) \, dx. \]
Once \( Y^x \) is found, and after matching this expression of \( AL \) with that given in Section 2, the value of \( \delta \) is determined. To this end, consider the asset at any intermediate time, \( Y^x(t + \tau, P) \), for \( 0 \leq \tau \leq d - x \). Forming a portfolio \( \Pi = Y^x + \pi_B B + \pi_S S \) with one unit of asset \( Y^x \), \( \pi_B \) units of \( B \) and \( \pi_S \) units of \( S \), and applying Itô’s Lemma, we have
\[
d\Pi = dY^x + \pi_B dB + \pi_S dS
\]
\[
= \left( Y^x_p \mu P + \frac{1}{2} Y^x_{pp} \eta^2 P^2 + Y^x_r \right) dt + Y^x_p \eta P \, dw_P
\]
\[
+ \pi_B \left( (r + \sigma \zeta b) B dt - \sigma b B \, dw_B \right) + \pi_S \left( \mu_S(r) dt + \sigma_S db + \sigma_S S \, dw_S \right)
\]
\[
= \left( Y^x_p \mu P + \frac{1}{2} Y^x_{pp} \eta^2 P^2 + Y^x_r + \pi_B (r + \sigma \zeta b) B + \pi_S \mu_S(r) S \right) dt
\]
\[
+ Y^x_p \eta \sqrt{1 - q_1^2 - q_2^2} P \, dw + \left( Y^x_r \eta P q_1 - \pi_B \sigma b B + \pi_S \sigma S S \right) \, dw_B + \left( Y^x_p \eta P q_2 + \pi_S \sigma S S \right) \, dw_S.
\]
The first equality is due to the self-financing property of the strategies \( \pi_B, \pi_S \). In the following equalities we use Assumption A, (5) and (6).

Now we select \( \pi_B \) and \( \pi_S \) in order to eliminate the risks related with \( w_B \) and \( w_S \), that is \( \pi_B = (Y^x_p \eta P q_1 + \sigma \pi_S)/\sigma_b \) and \( \pi_S S = -Y^x_r \eta P q_2/\sigma_S \). We also disregard the risk orthogonal to them, that is, the risk related with \( w \) is not priced. The total return of the hedge portfolio must be equal to the risk free rate of interest at time \( t, r(t) \). Thus we obtain
\[ r(t)(Y^x + \pi_B B + \pi_S S) = Y^x_r + Y^x_p \mu P + \frac{1}{2} Y^x_{pp} \eta^2 P^2 + \pi_B (r + \sigma \zeta b) B + \pi_S \mu S(r) S, \]
that, with the of the expressions for \( \pi_B \) and \( \pi_S \) found above and using \( \mu_S(r) = r + m_S \), transform in the pricing PDE
\[
r(t)Y^x = Y^x_r + \omega PY^x_p + \frac{1}{2} \eta^2 P^2 Y^x_{pp}
\]
with boundary conditions \( Y^x(t + d - x, P) = P, Y^x(t + \tau, 0) = 0 \), and where
\[ \omega = \mu + \zeta \eta q_1 - \frac{m_S + \zeta \sigma r}{\sigma_S} \eta q_2. \]
The solution is \( Y^x(t + \tau, P) = Pe^{-(r(t) - \omega)(d-x-\tau)} \), hence for \( \tau = 0 \), \( Y^x(t, P) = Pe^{-(r(t) - \omega)(d-x)} \).

In consequence,
\[
AL(t) = \int_a^d Y^x(t, P(t))M(x) \, dx = P(t) \int_a^d e^{-(r(t) - \omega)(d-x)} M(x) \, dx.
\]
On the other hand, the actuarial liability was defined in Section 2 as

\[ AL(t) = \int_a^d e^{-\delta(t)(d-x)} \mathbb{E}(P(t + d - x)|\mathcal{F}_t) M(x) \, dx = P(t) \int_a^d e^{(\mu - \delta(t))(d-x)} M(x) \, dx, \]

where the second equality follows from the proof of Proposition 2.1 in the Appendix. Comparing both values \( \delta(t) \) must be chosen equal to \( r(t) + \mu - \omega \) in order to attain a risk–neutral valuation.

Thus we will assume throughout the paper, in a similar way as in Josa–Fombellida and Rincón–Zapatero (2004), that the technical interest rate coincides with the rate of return of the bond modified to get rid of the sources of uncertainty, that is:

**Assumption C.** The technical rate of actualization satisfies for every \( t \)

\[ \delta(t) = r(t) - \zeta \eta q_1 + \frac{m_S + \zeta \sigma_r}{\sigma_S} \eta q_2. \]

Besides of the risk–neutral valuation it provides, this selection of \( \delta \) allows us to solve explicitly the problem of the following section.

## 5 The bond and stock portfolio problem

In this section we analyze how the sponsor may selects in an optimal way the proportion of fund assets put on a savings account, or invested in a bond and in a risky stock. So the sponsor faces three elements of randomness: one due to the benefits, which is inherent to the pension plan, and the remainder two due to the financial market, concretely the stochastic interest rate and a risky stock.

The plan sponsor invests the fund in a portfolio formed by the savings account \( R \), given by (4), the zero–coupon bond \( B \) with maturity \( T_1 > T \), given by (5), and the stock \( S \) which dynamic is given by (6).

The amounts invested in the bond \( B \) and in the stock \( S \) are denoted by \( \lambda_B \) and \( \lambda_S \), respectively. The remainder, \( F - \lambda_B - \lambda_S \), goes to the savings account \( R \). Borrowing and shortselling is allowed. A negative value of \( \lambda_B \) (resp. \( \lambda_S \)) means that the sponsor sells shares of the \( B \) (resp. \( S \)) short, while, if \( \lambda_B + \lambda_S \) is greater than \( F \), then he or she gets into debt to purchase the stocks, borrowing money at the interest rate \( r \).
We suppose \( \{(\lambda_B(t), \lambda_S(t)) : t \geq 0\} \) is a markovian control process adapted to filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) and satisfying
\[
E \int_0^T (\lambda_B^2(t) + \lambda_S^2(t)) dt < \infty.
\] (7)

Therefore, the fund dynamic evolution under the investment policy \((\lambda_B, \lambda_S)\) is:
\[
dF(t) = \frac{\lambda_B(t)}{B_t} dB(t) + \frac{\lambda_S(t)}{S_t} dS(t) + \frac{(F(t) - \lambda_B(t) - \lambda_S(t))}{R(t)} dR(t) + (C(t) - P(t)) dt.
\] (8)

By substituting (4), (5) and (6) in (8), and taking into account Proposition 2.1 and (2) we obtain:
\[
dF(t) = (b(t) \sigma \zeta \lambda_B(t) + \lambda_S(t)(\mu_S(r(t)) - r(t)) + (r(t) - k) F(t) + (k + \mu - \delta(t)) AL(t)) dt
\]
\[
+ (-b(t) \sigma \lambda_B(t) + \sigma_r \lambda_S(t)) dw_B(t) + \sigma_S \lambda_S(t) dw_S(t),
\] (9)

with initial condition \( F(0) = F_0 > 0 \). By (1), in terms of \( X = F - AL \) equation (9) is
\[
dx(t) = (b(t) \sigma \zeta \lambda_B(t) + m_S \lambda_S(t) + (r(t) - k) X(t) + (r(t) - \delta(t)) AL(t)) dt
\]
\[
- \eta \sqrt{1 - q_1^2} - q_2^2 AL(t) dw(t) - (b(t) \sigma \lambda_B(t) - \sigma_r \lambda_S(t) + \eta q_1 AL(t)) dw_B(t)
\]
\[
+ (\sigma_S \lambda_S(t) - \eta q_2 AL(t)) dw_S(t),
\] (10)

with the initial condition \( X(0) = X_0 \).

Let us now turn to the preferences of the controller. We assume that he or she wishes to minimize the terminal solvency risk. Thus, the objective functional to be minimized over the class of admissible controls \( \mathcal{A}_{X_0, AL_0, r_0} \), is given by
\[
J((X_0, AL_0, r_0); \lambda) = \mathbb{E}_{X_0, AL_0, r_0} X^2(T).
\] (11)

Here, \( \mathcal{A}_{X_0, AL_0, r_0} \) is the set of measurable processes \( \{(\lambda_B(t), \lambda_S(t))\}_{t \geq 0} \) where \((\lambda_B, \lambda_S)\) satisfies (7) and where \( X, AL \) and \( r \) satisfy (10), (1) and (3), respectively. In the above, \( \mathbb{E}_{X_0, AL_0, r_0} \) denotes conditional expectation with respect to the initial conditions \((X_0, AL_0, r_0)\).

In the following developments we will suppose only the underfunded case where \( X_0 < 0 \), so we will refer to \( X \) as debt.

The dynamic programming approach is used to solve the problem. To make the process work, some properties of the value function need to be established. The value function is
defined as
\[
\hat{V}(t, X, AL, r) = \min_{(\lambda_B, \lambda_S) \in A_t, F, AL, r} \left\{ J(t, (F, AL, r); (\lambda_B, \lambda_S)) : \text{s.t. } (10), (1), (3) \right\}.
\] (12)

The connection between value functions in optimal control theory (deterministic or stochastic) and optimal feedback controls is accomplished by the HJB equation, see Fleming and Soner (1993).

We have the following result.

**Theorem 5.1** Suppose that Assumptions A, B and C hold. Then the optimal investments are given by

\[
\lambda_B^*(t, X, AL) = \frac{-\alpha}{\sigma(1 - e^{-\alpha(T_1 - t)})} \left( \zeta + \frac{m_s + \zeta \sigma_r}{\sigma_S} - 2\frac{\sigma}{\alpha} (1 - e^{-\alpha(T-t)}) \right) X
\]

\[
+ \left( q_1 - \frac{\sigma_r}{\sigma_S} q_2 \right) \eta AL \right)
\] (13)

\[
\lambda_S^*(t, X, AL) = -\frac{m_S + \zeta \sigma_r}{\sigma^2_S} X + \frac{q_2}{\sigma_S} \eta AL.
\] (14)

**Remark 5.1** From (13) in Theorem 5.1, the optimal investments do not depend on \( r \) and the investment in the bond is of the form

\[
\lambda_B^*(t, X, AL) = \frac{1}{1 - e^{-\alpha(T_1 - t)}} \left( \frac{-\alpha \zeta}{\sigma} X + 2(1 - e^{-\alpha(T-t)}) X
\]

\[
- \frac{\alpha \sigma_r}{\sigma^2_S} (m_S + \sigma_r \zeta) X - \frac{\alpha \eta}{\sigma_S} (\sigma_S q_1 - \sigma_r q_2) AL \right).
\]

Letting aside the common factor, it is the sum of four terms. The first term coincides with the classical optimal one in Merton (1971) when the coefficients are deterministic. The second term is proportional to \( X \) and on the time horizon planned, and vanishes at the terminal date \( T \). The third term is also proportional to the debt, with a coefficient that depends on several of the elements defining the prices of the bond and the stock. The fourth term is quite different, as it involves the random liability instead of debt. The summand now is proportional to \( AL \), with coefficient that depends on the volatilities of the processes \( AL, B \) and \( S \) and their respective correlations. Thus, this last term cares about the random evolution of liabilities. In fact it vanishes when benefits are deterministic or when the relation: \( q_1 \sigma_S = q_2 \sigma_r \), between variances and covariances is verified. In
both cases the optimal investment in the bond is proportional to the unfunded liability, $UA = -X$.

The optimal investment in the stock given by (14) is simpler. It is the sum of two terms, one proportional to debt $X$ and the other proportional to the actuarial liability $AL$. The latter is zero that is, $\lambda_S^* \propto X$, either if benefits are deterministic or when there does not exist correlation between stock and benefits.

Substituting (13) and (14) in (10) we obtain that the optimal debt satisfies the stochastic differential equation:

$$dX(t) = \left( -\zeta^2 - \frac{(m_S + \zeta \sigma_r)^2}{\sigma_S^2} + \frac{2}{\alpha} (1 - e^{-\alpha(T-t)} \zeta \sigma + r(t) - k) \right) X(t) dt$$

$$- \eta \sqrt{1 - q_1^2 - q_2^2} AL(t) dw(t) + \left( \zeta - \frac{2}{\alpha} (1 - e^{-\alpha(T-t)} \sigma) \right) X(t) dw_B(t)$$

$$- \frac{m_S + \zeta \sigma_r}{\sigma_S} X(t) dw_S(t),$$

with the initial condition $X(0) = X_0$ and where $AL$ is given by (1) and $r$ by (3). In the following section we will integrate numerically the linear system of SDEs formed by (1), (3) and (15) to illustrate the results.

6 A numerical illustration

In this section we consider a numerical application in order to illustrate the dynamic behavior of the debt and its expected value, and the optimal portfolio strategy. The parameters defining the financial market have been taken from Boulier et al (2001). Thus, the initial value for the interest rate $r_0 = 0.05$ coincides with its equilibrium value $\beta$, the maturity is $T_1 = 10$ and the market price of risk is $\zeta = 0.15$. These and the remainder parameter values are shown in Table 1.

We consider a contribution period before retirement of $T = 6$ years and that benefits are random with $\mu = 0.04$ and $\eta = 0.08$. The effort of amortization is $k = 0.06$. The initial values for the actuarial liability and the fund wealth are taken to be $AL_0 = 100$ and $F_0 = 80$ respectively, so $X_0 = -20$, that is the fund is 20% underfunded. Initial benefits are supposed 1% of $AL_0$, that is, $P_0 = 1$. The value of the parameters is listed in Table 1.
Table 1
Values of parameters

<table>
<thead>
<tr>
<th>Interest rate</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean reversion, $\alpha$</td>
<td>0.2</td>
</tr>
<tr>
<td>Mean rate, $\beta$</td>
<td>0.05</td>
</tr>
<tr>
<td>Volatility, $\sigma$</td>
<td>0.02</td>
</tr>
<tr>
<td>Initial rate, $r_0$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Maturity bond

| Maturity, $T_1$                        | 10    |
| Market price of risk, $\zeta$          | 0.15  |

Stock

| Risk premium, $m_S$                    | 0.06  |
| Interest rate source risk, $\sigma_r$  | 0.06  |
| Stock own volatility, $\sigma_S$       | 0.19  |

The correlation between benefits and short rate is selected as $q_1 = 0.2$ and the correlation between benefits and stock as $q_2 = 0.2$. Figure 1 shows the evolution of debt, fund assets, actuarial liability and its expected values along the planning interval. To obtain the graph we have used the Euler method to find the solution of the system (1), (3) and (15); see e.g. Kloeden and Platen (1999). The calculations have been done with MATLAB®. The particular sample path drawn for the debt, takes values in the range $-22$ to $-2$ and it attains the value of $-16$ at instant $t = 6$, from an initial value of $-20$ at $t = 0$. The next graph in Figure 2 shows the evolution of $F$ and $AL$, for the same simulation. Obviously, growth along time of fund assets and liabilities is expected, since benefits presents a positive mean increase. This trend is better seen in the next graph, where the expected values of debt, fund and actuarial liability is shown. These curves have been computed with Monte Carlo simulation. The expected value of $X$ is increasing, that is, the debt decreases at every time. In our example debt reduces in mean from $-20$ to $-9$, that suppose 45% of its value. This fact is better appreciated in the fourth graph, where $\mathbb{E}F(t)$ gets closer to $\mathbb{E}AL(t)$ as $t$ increases.
Figure 2: Debt, fund, actuarial liability and its expected values.

Figure 3 represents the proportions of fund invested in the portfolio in order to minimize the terminal solvency risk. The paths correspond to the same sample that Figure 2. These functions depend on the individual values of correlations $q_1$ and $q_2$, see (13) and (14), whereas the processes in Figure 2 depend on the aggregate value $q_1^2 + q_2^2$, see (15). Thus we consider four possible scenes: $(q_1, q_2) = (-0.2, -0.2), (-0.2, 0.2), (0.2, -0.2), (0.2, 0.2)$.

The four graphs in Figure 3 show a similar pattern. In the first years where debt is large, the optimal strategy is to take more risk, borrowing money to invest in the bond and in the stock. The higher mean returns they provide compared with the bank account is
the factor may explain this behavior. In fact, at the time where debt takes the maximum value, that is, when $X$ is minimum, is just when $\lambda_B^* + \lambda_S^*$ attains also the maximum value. At this point the strategy is quite aggressive indeed, demanding borrowing money for the

![Figure 3: Investment proportions for four different cases of correlations.](image)

amount of approximately 125% fund’s wealth, or $1.25F$ to invest with risk (first graph in Figure 3, with negative value of both correlations).

In the final part of the time interval the amount held in cash increases and the amount investment in the bond and in the stock diminishes, reducing considerably the risky composition of the portfolio. The behavior described is similar in the four cases of correlations considered. However, the relative weight of the bond and the stock in the portfolio is
highly influenced by the signs of correlations, at least in the sample shown. In the first two graphs where $q_1 > 0$, the bond participates in the portfolio in a larger proportion than the stock, independently of the sign of $q_2$. When $q_1 < 0$ the situation is reversed. Thus, the feature observed in the model studied in Menoncin (2005), where bond’s share in the portfolio is larger than the share of the stock is not maintained in our model. This different behavior may be due on the one hand to the existence of correlations and on the other hand, to the aiming of the sponsor of minimizing the expected square of debt, instead of maximizing expected utility from surplus.

7 Conclusions

We have analyzed the management of a pension funding process of an DB pension plan when the short interest rate is the Vasicek model. The problem of the minimization of the terminal solvency risk have been solved analytically when the benefits process is a geometric Brownian motion under a suitable selection of the technical rate of interest. The components of the optimal portfolio (investments in the bond, in the stock and in the cash) are the sum of two terms, one proportional to the unfunded actuarial liability, extending to the Merton model, and another to the actuarial liability, depending on parameters of randomness of benefits and its correlations with the interest rate and the stock.

We have done a numerical simulation showing some properties of the model. Though there are three sources of randomness, the debt is reduced by means of risky investment along the first years and with a more conservative investment policy in the last years of the period planned.

Further research should be directed to include other dynamics for the interest rate processes, as the Cox–Ingersoll–Ross (CIR) model, the Ho–Lee model or affine models in general. From the side of benefits, it would also interesting to consider the possibility of jumps as in Ngwira and Gerrard (2006) or some more general Lévy process.
A Appendix

Proof of Proposition 2.1. If Assumption A holds, following the proof of Proposition 2.1 in Josa–Fombellida and Rincón–Zapatero (2004), we obtain that the equality holds:

$$
\mathbb{E}(P(t + d - x) | \mathcal{F}_t) = e^{\mu(d-x)} P(t),
$$

for every $x \in [a, d]$ and $t \geq 0$. Therefore, defining

$$
\psi_{AL}(t) = \int_a^d e^{(\mu - \delta(t))(d-x)} M(x) \, dx, \quad \psi_{NC}(t) = \int_a^d e^{(\mu - \delta(t))(d-x)} m(x) \, dx,
$$

the statement of the proposition follows. Finally, the relationship between $\psi_{AL}$ and $\psi_{NC}$ is obtained integrating by parts in the second integral and taking into account the properties of $M$ and that $M' = m$. □

Proof of Theorem 5.1. Consider the value function (12) of the control problem (1)– (3)–(10)–(11). This function so defined is non–negative and strictly convex. It is well known $\hat{V}$ is solution of the HJB equation, see Fleming and Soner (1993):

$$
\dot{V}_t + \min_{\lambda_B, \lambda_S} \left\{ \left( b \sigma \lambda_B + m_S \lambda_S + (r - k)X + (r - \delta)AL \right) V_X + \mu AL V_{AL} + \alpha (\beta - r) V_r \right. \\
+ \frac{1}{2} \left( (1 - q_1^2 - q_2^2) \eta^2 AL^2 + (b \sigma \lambda_B - \sigma_r \lambda_S + \eta q_1 AL)^2 + (\sigma_S \lambda_S - \eta q_2 AL)^2 \right) V_{XX} \\
+ \frac{1}{2} \eta^2 AL^2 V_{AL,AL} + (-\eta^2 AL^2 + \eta q_1 (\sigma_r \lambda_S - b \sigma \lambda_B) AL + \eta q_2 \sigma_S \lambda_S AL) V_{X,AL} \\
+ \frac{1}{2} \sigma^2 V_{rr} + \eta q_1 AL V_{r,AL} + \sigma (-b \sigma \lambda_B + \sigma_r \lambda_S - \eta q_1 AL) V_{rX} \right\} = 0,
$$

$$
V(T, X, AL, r) = X^2.
$$

If there exists a smooth solution $V$ of this equation, strictly convex with respect to $X$, then the minimizers values of the investments are given by

$$
\hat{\lambda}_B(V_X, V_{XX}, V_{X,AL}, V_{rX}) = \frac{1}{b \sigma^2 \sigma_X^2 V_{XX}} \left( - \left( \zeta (\sigma_r^2 + \sigma_S^2) + m_S \sigma_S \right) V_X + \sigma \sigma_X^2 V_{rX} \\
+ \eta \sigma_S (\sigma_r q_2 - \sigma_S q_1) AL (V_{XX} - V_{X,AL}) \right),
$$

$$
\hat{\lambda}_S(V_X, V_{XX}, V_{X,AL}) = \frac{1}{\sigma_X^2 V_{XX}} \left( -(m_S + \zeta \sigma_r) V_X + \eta \sigma_S q_2 AL (V_{XX} - V_{X,AL}) \right).
$$
After substitution of these values in (A) we obtain $\hat{V}$ satisfies
\[
V_t + \left((r - k)X + (r - \delta)AL + \frac{\eta}{\sigma_S}(q_2\sigma_r + q_1\sigma_S) + m_Sq_2\right) V_X + \mu AL V_{AL} + \alpha(\beta - r)V_r \\
+ \frac{1}{2}(1 - q_1^2 - q_2^2)\eta^2 AL^2 V_{XX} + \frac{1}{2}\eta^2 AL^2 V_{AL,AL} + \frac{1}{2}\sigma^2 V_{rr} - (1 - q_1^2 - q_2^2)\eta^2 AL^2 V_{X,AL} \\
+ \eta q_1\sigma AL V_{R,AL} - \frac{1}{2}\sigma^2 ((m_S + \zeta\sigma_r)^2 + \zeta^2\sigma_S^2)V_X^2 + \zeta\eta q_1 AL \frac{V_X V_{X,AL}}{V_{XX}} + \zeta \sigma \frac{V_{X} V_{r,AL}}{V_{XX}} \\
- \frac{1}{2}\sigma^2 \frac{V_{r,AL}^2}{V_{XX}} - \frac{1}{2}\eta^2 (q_1^2 + q_2^2) AL^2 \frac{V_{X,AL}^2}{V_{XX}} - \eta q_1\sigma AL \frac{V_{r,XX} V_{X,AL}}{V_{XX}} = 0,
\]
with the final condition (A). We try in (A) a quadratic solution of the form
\[
\hat{V}(t, X, AL, r) = f_{XX}(t, r)X^2 + f_{AL,AL}(t, r)AL^2 + f_{X,AL}(t, r)X AL
\]
and the following ordinary differential equations are obtained for the above coefficients:
\[
(f_{XX})_t + \left(-\zeta^2 - \frac{(m_S + \zeta\sigma_r)^2}{\sigma_S^2} + 2(r - k)\right) f_{XX} + (2\zeta\sigma + \alpha(\beta - r))(f_{XX})_r \\
- \sigma^2 \frac{(f_{XX})_r^2}{f_{XX}} + \sigma^2 \frac{(f_{XX})_{rr}}{2} = 0, \quad f_{XX}(T, r) = 1,
\]
\[
(f_{AL,AL})_t - \frac{1}{4} \left(\zeta^2 + \frac{(m_S + \zeta\sigma_r)^2}{\sigma_S^2} - 2\zeta q_1 + \eta^2 (q_1^2 + q_2^2)\right) \frac{f_{XX}}{f_{XX}} + 2\mu f_{AL,AL} \\
+ \left(r - \delta - \zeta q_1 + \eta q_2 \frac{m_S + \zeta\sigma_r}{\sigma_S} - (1 - (q_1^2 + q_2^2))\eta^2\right) f_{AL,AL} + \frac{\sigma^2}{2} \left(\zeta - \eta q_1\right) \frac{f_{X,AL}(f_{X,AL})_r}{f_{XX}} \\
+ \alpha(\beta - r)(f_{AL,AL})_r + (1 - q_1^2 - q_2^2)\eta^2 f_{XX} - \frac{\sigma^2}{4} \frac{(f_{X,AL})_r^2}{f_{XX}} + \eta^2 f_{AL,AL} + \frac{\sigma^2}{2} (f_{AL,AL})_{rr} \\
+ 2\eta q_1\sigma (f_{AL,AL})_r = 0, \quad f_{AL,AL}(T, r) = 0,
\]
\[
(f_{X,AL})_t + \left(-\zeta^2 - \frac{(m_S + \zeta\sigma_r)^2}{\sigma_S^2} + \zeta q_1 + r - k + \mu\right) f_{X,AL} + \sigma^2 \frac{(f_{X,AL})_r}{2} = 0 \\
+ 2 \left(r - \delta - \zeta q_1 + \eta q_2 \frac{m_S + \zeta\sigma_r}{\sigma_S}\right) f_{XX} + (\zeta\sigma + \eta q_1\sigma + \alpha(\beta - r))(f_{X,AL})_r \\
+ (\zeta - \eta q_1)\sigma \frac{f_{X,AL}(f_{XX})_r}{f_{XX}} - \sigma^2 \frac{(f_{X,AL})_r (f_{XX})_r}{f_{XX}} = 0, \quad f_{X,AL}(T, r) = 0.
\]
In order to solve (A) we try $f_{XX}(t, r) = g(t)e^{\gamma(t)r}$, with the final conditions $g(T) = 1$ and $\gamma(T) = 0$, and after simplification we obtain

$$
\dot{g} + (\dot{\gamma} - \alpha \gamma + 2) rg + \left( -\frac{\sigma^2}{2} \gamma^2 + (2\zeta \sigma + \alpha \beta) \gamma - 2k - \zeta^2 - \frac{(m_S + \zeta \sigma_T)^2}{\sigma_S^2} \right) g = 0.
$$

Choosing $\gamma$ such that $\dot{\gamma} - \alpha \gamma + 2 = 0$, function $g$ is given by $\dot{g} + hg = 0$, where

$$
h(t) = -(\sigma^2/2)\gamma^2(t) + (2\zeta \sigma + \alpha \beta) \gamma(t) - 2k - \zeta^2 - (m_S + \zeta \sigma_T)^2/\sigma_S^2.
$$

With the final conditions we obtain

$$
\gamma(t) = \frac{2}{\alpha} (1 - e^{-\alpha(T-t)})
$$

and $g(t) = e^{H(T)-H(t)}$ with $H$ a primitive of $h$. Hence we obtain

$$
f_{XX}(t, r) = \exp \left\{ H(T) - H(t) + \frac{2}{\alpha} (1 - e^{-\alpha(T-t)}r) \right\}.
$$

Using Assumption C it is easy to prove that function $f_{X,AL}$ verifying (A) is $f_{X,AL} = 0$.

Inserting (A) into (A)–(A) we obtain that the optimal investments are given by

$$
\lambda^*_B(t, X, AL, r) = \frac{1}{b} \left( -\frac{\zeta (\sigma_r^2 + \sigma_S^2) + m_S \sigma_r}{\sigma \sigma_S} + \frac{(f_{XX})_r}{f_{XX}} \right) X
$$

$$
+ \frac{1}{2b f_{XX}} \left( -\frac{\zeta (\sigma_r^2 + \sigma_S^2) + m_S \sigma_r}{\sigma \sigma_S^2} f_{X,AL} + (f_{X,AL})_r \right) AL
$$

$$
+ \frac{\eta (\sigma_r q_2 - \sigma_S q_1)}{b \sigma \sigma_S} \left( 1 - \frac{f_{X,AL}}{2f_{XX}} \right) AL,
$$

$$
\lambda^*_S(t, X, AL, r) = -\frac{m_S + \sigma_r \zeta}{\sigma_S^2} X + \frac{\eta q_2}{\sigma_S} AL,
$$

that is to say, (13) and (14), respectively. \qed
References


