

**Appendices to the paper "Detecting Big Structural Breaks in Large Factor Models" (2011)
by Chen, Dolado and Gonzalo.**

A.1: Proof of Propositions 1 and 2

The proof proceeds by showing that the errors, factors and loadings in model (6) satisfy Assumptions A to D of Bai and Ng (2002). Then, once these results are proven, Propositions 1 and 2 just follow immediately from application of Theorems 1 and 2 of Bai and Ng (2002). Define $F_t^* = [F_t' \quad G_t']'$, $\varepsilon_t = HG_2' + e_t$, and $\Gamma = [A \quad \Lambda]$.

Lemma 1. $E\|F_t^*\|^4 < \infty$ and $T^{-1} \sum_{t=1}^T F_t^* F_t^{*'} \rightarrow \Sigma_F^*$ as $T \rightarrow \infty$ for some positive matrix Σ_F^* .

Proof. $E\|F_t^*\|^4 < \infty$ follows from $E\|F_t\|^4 < \infty$ by Assumption 2 and the definition of G_t^1 .

To prove the second part, we partition the matrix $\Sigma_F (= \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T F_t F_t')$ into:

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma'_{12} & \Sigma_{22} \end{pmatrix}$$

where $\Sigma_{11} = \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T F_t^1 F_t^{1'}$, $\Sigma_{22} = \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T F_t^2 F_t^{2'}$, $\Sigma_{12} = \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T F_t^1 F_t^{2'}$, and F_t^1 is the $k_1 \times 1$ subvector of F_t that has big breaks in their loadings, F_t^2 is the $k_2 \times 1$ subvector of F_t that doesn't have big breaks in their loadings. By the definition of F_t^* and G_t^1 we have:

$$T^{-1} \sum_{t=1}^T F_t^* F_t^{*'} = \begin{pmatrix} T^{-1} \sum_{t=1}^T F_t^1 F_t^{1'} & T^{-1} \sum_{t=1}^T F_t^1 F_t^{2'} & T^{-1} \sum_{t=\tau+1}^T F_t^1 F_t^{1'} \\ T^{-1} \sum_{t=1}^T F_t^2 F_t^{1'} & T^{-1} \sum_{t=1}^T F_t^2 F_t^{2'} & T^{-1} \sum_{t=\tau+1}^T F_t^2 F_t^{1'} \\ T^{-1} \sum_{t=\tau+1}^T F_t^1 F_t^{1'} & T^{-1} \sum_{t=\tau+1}^T F_t^1 F_t^{2'} & T^{-1} \sum_{t=\tau+1}^T F_t^1 F_t^{1'} \end{pmatrix}.$$

By Assumption 2, the above matrix converges to

$$\Sigma_F^* = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & (1 - \pi^*)\Sigma_{11} \\ \Sigma'_{12} & \Sigma_{22} & (1 - \pi^*)\Sigma'_{12} \\ (1 - \pi^*)\Sigma_{11} & (1 - \pi^*)\Sigma_{12} & (1 - \pi^*)\Sigma_{11} \end{pmatrix}.$$

Moreover,

$$\det(\Sigma_F^*) = \det \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & 0 \\ \Sigma'_{12} & \Sigma_{22} & (1 - \pi^*)\Sigma'_{12} \\ 0 & 0 & \pi^*(1 - \pi^*)\Sigma_{11} \end{pmatrix} = \det(\Sigma_F) \det(\pi^*(1 - \pi^*)\Sigma_{11}) > 0$$

because Σ_F is positive definite by assumption. This completes the proof. \square

Lemma 2. $E\|\Gamma_i\|^4 < \infty$, and $N^{-1} \Gamma' \Gamma \rightarrow \Sigma_\Gamma$ as $N \rightarrow \infty$ for some positive definite matrix Σ_Γ .

Proof. This follows directly from Assumptions 1.a and 3. \square

The following lemmatae involve the new errors ε_t . Let M and M^* denote some positive constants.

Lemma 3. $E(\varepsilon_{it}) = 0$, $E|\varepsilon_{it}|^8 \leq M^*$.

Proof. This follows easily from Assumptions 1.b, 2 and 4. \square

Lemma 4. $E(\boldsymbol{\varepsilon}'_s \boldsymbol{\varepsilon}_t / N) = E(N^{-1} \sum_{i=1}^N \boldsymbol{\varepsilon}_{is} \boldsymbol{\varepsilon}_{it}) = \gamma_N^*(s, t)$, $|\gamma_N^*(s, s)| \leq M^*$ for all s , and $\sum_{s=1}^T \gamma_N^*(s, t)^2 \leq M^*$ for all t and T .

Proof.

$$\begin{aligned}
\gamma_N^*(s, t) &= N^{-1} \sum_{i=1}^N E(\boldsymbol{\varepsilon}_{is} \boldsymbol{\varepsilon}_{it}) \\
&= N^{-1} \sum_{i=1}^N E(e_{is} + \boldsymbol{\eta}'_i G_s^2) E(e_{it} + \boldsymbol{\eta}'_i G_t^2) \\
&= N^{-1} \sum_{i=1}^N [E(e_{is} e_{it}) + E(\boldsymbol{\eta}'_i G_s^2 \boldsymbol{\eta}'_i G_t^2)] \\
&\leq N^{-1} \sum_{i=1}^N E(e_{is} e_{it}) + N^{-1} \sum_{i=1}^N \sqrt{E(\boldsymbol{\eta}'_i G_s^2)^2 E(\boldsymbol{\eta}'_i G_t^2)^2}.
\end{aligned}$$

Since $N^{-1} \sum_{i=1}^N E(e_{is} e_{it}) = \gamma_N(s, t)$ by Assumption 4, and $E(\boldsymbol{\eta}'_i G_t^2)^2 \leq E\|\boldsymbol{\eta}_i\|^2 E\|F_t\|^2 = O(\frac{1}{NT})$ for all t by Assumptions 1.b and 2, we have $\gamma_N^*(s, t) \leq \gamma_N(s, t) + O(\frac{1}{NT})$. Then

$$|\gamma_N^*(s, s)| \leq |\gamma_N(s, s)| + O(\frac{1}{NT}) \leq M^*$$

by Assumption 4. Moreover,

$$\begin{aligned}
\sum_{s=1}^T \gamma_N^*(s, t)^2 &\leq \sum_{s=1}^T (\gamma_N(s, t) + O(\frac{1}{NT}))^2 \\
&= \sum_{s=1}^T \gamma_N(s, t)^2 + O(\frac{1}{N}) \\
&\leq M + O(\frac{1}{N}) \leq M^*
\end{aligned}$$

also by Assumption 4. Thus, the proof is complete. \square

Lemma 5. $E(\boldsymbol{\varepsilon}_{it} \boldsymbol{\varepsilon}_{jt}) = \boldsymbol{\tau}_{ij,t}^*$ with $|\boldsymbol{\tau}_{ij,t}^*| \leq |\boldsymbol{\tau}_{ij}^*|$ for some $\boldsymbol{\tau}_{ij}^*$ and for all t ; and $N^{-1} \sum_{i=1}^N \sum_{j=1}^N |\boldsymbol{\tau}_{ij}^*| \leq M^*$.

Proof. By Assumption 4, $|\boldsymbol{\tau}_{ij,t}^*| \leq |\boldsymbol{\tau}_{ij}^*|$ for some $\boldsymbol{\tau}_{ij}^*$ and all t , where $\boldsymbol{\tau}_{ij,t}^* = E(e_{it} e_{jt})$. Then:

$$\begin{aligned}
|\boldsymbol{\tau}_{ij,t}^*| &= |E(\boldsymbol{\varepsilon}_{it} \boldsymbol{\varepsilon}_{jt})| \\
&= |E(e_{it} + \boldsymbol{\eta}'_i G_t^2)(e_{jt} + \boldsymbol{\eta}'_j G_t^2)| \\
&\leq |E(e_{it} e_{jt})| + \sqrt{E(\boldsymbol{\eta}'_i G_t^2)^2 E(\boldsymbol{\eta}'_j G_t^2)^2} \\
&\leq |\boldsymbol{\tau}_{ij}^*| + O(\frac{1}{NT})
\end{aligned}$$

for all t . Therefore

$$\begin{aligned} N^{-1} \sum_{i=1}^N \sum_{j=1}^N |\tau_{ij}^*| &\leq N^{-1} \sum_{i=1}^N \sum_{j=1}^N \left(|\tau_{ij}| + O\left(\frac{1}{NT}\right) \right) \\ &\leq M + O\left(\frac{1}{T}\right) \\ &\leq M^* \end{aligned}$$

by Assumption 4. □

Lemma 6. $E(\varepsilon_{it}\varepsilon_{js}) = \tau_{ij,ts}^*$ and $(NT)^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{t=1}^T \sum_{s=1}^T |\tau_{ij,ts}^*| \leq M^*$.

Proof. By Assumption 4, $(NT)^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{t=1}^T \sum_{s=1}^T |\tau_{ij,ts}| \leq M$, where $E(e_{it}e_{js}) = \tau_{ij,ts}$. Then:

$$E(\varepsilon_{it}\varepsilon_{js}) = \tau_{ij,ts}^* = E(e_{it}e_{js}) + E(\eta_i' G_t^2 \eta_j' G_s^2) = \tau_{ij,ts} + E(\eta_i' G_t^2 \eta_j' G_s^2)$$

and we have

$$\begin{aligned} (NT)^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{t=1}^T \sum_{s=1}^T |\tau_{ij,ts}^*| &\leq (NT)^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{t=1}^T \sum_{s=1}^T |\tau_{ij,ts}| + (NT)^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{t=1}^T \sum_{s=1}^T |E(\eta_i' G_t^2 \eta_j' G_s^2)| \\ &\leq M + O(1) \\ &\leq M^* \end{aligned}$$

following the same arguments as above. □

Lemma 7. For every (t, s) , $E|N^{-1/2} \sum_{i=1}^N [\varepsilon_{is}\varepsilon_{it} - E(\varepsilon_{is}\varepsilon_{it})]|^4 \leq M^*$.

Proof. Since $\varepsilon_{it} = e_{it} + \eta_i' G_t^2 = e_{it} + O_p\left(\frac{1}{\sqrt{NT}}\right)$, we have:

$$\begin{aligned} E|N^{-1/2} \sum_{i=1}^N [\varepsilon_{it}\varepsilon_{is} - E(\varepsilon_{it}\varepsilon_{is})]|^4 &= E|N^{-1/2} \sum_{i=1}^N [e_{it}e_{is} - E(e_{it}e_{is}) + O_p\left(\frac{1}{\sqrt{NT}}\right) + O\left(\frac{1}{\sqrt{NT}}\right)]|^4 \\ &= E|N^{-1/2} \sum_{i=1}^N [e_{it}e_{is} - E(e_{it}e_{is})] + O_p\left(\frac{1}{\sqrt{T}}\right) + O\left(\frac{1}{\sqrt{T}}\right)|^4 \\ &\leq M + O\left(\frac{1}{\sqrt{T}}\right) \\ &\leq M^*. \end{aligned}$$

□

Lemma 8. $E\left(\frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t^* \varepsilon_{it} \right\|^2\right) \leq M^*$.

Proof. By the definition of ε_{it} we have:

$$E\left(\frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t^* \varepsilon_{it} \right\|^2\right) \leq E\left(\frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t^* e_{it} \right\|^2\right) + E\left(\frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t^* \eta_i' G_t^2 \right\|^2\right)$$

then by the definition of F_t^* and G_t^2 ,

$$\begin{aligned}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^T F_t^* e_{it}\right\|^2 &= \left\|\frac{1}{\sqrt{T}}\sum_{t=1}^T F_t e_{it}\right\|^2 + \left\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^T F_t^1 e_{it}\right\|^2, \\ \left\|\frac{1}{\sqrt{T}}\sum_{t=1}^T F_t^* \eta'_t G_t^2\right\|^2 &= \left\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^T F_t \eta'_t F_t^2\right\|^2.\end{aligned}$$

Therefore, by Assumptions 1.b, 2, 4 and 5, it follows easily that the first part of the right hand side of the last inequality is $O(1)$. For the second part, notice that

$$\begin{aligned}& E\left\|\frac{1}{\sqrt{T}}\sum_{t=\tau+1}^T F_t \eta'_t F_t^2\right\|^2 \\ &= \frac{1}{T}\sum_{p=1}^r E\left(\sum_{t=\tau+1}^T F_{tp} \eta'_t F_t^2\right) \\ &= \frac{1}{T}\sum_{p=1}^r \sum_{t=\tau+1}^T \sum_{s=\tau+1}^T E\left(F_{tp} F_{sp} (\eta'_t F_t) (\eta'_s F_s)\right)\end{aligned}$$

and that

$$\begin{aligned}& E\left(F_{tp} F_{sp} (\eta'_t F_t) (\eta'_s F_s)\right) \\ &\leq E(F_{tp})^2 E(\eta'_t F_t)^2 \\ &\leq O(1) E\|\eta_t\|^2 E\|F_t\|^2 \\ &= O\left(\frac{1}{NT}\right).\end{aligned}$$

Thus we have $E\left(\frac{1}{N}\sum_{i=1}^N \left\|\frac{1}{\sqrt{T}}\sum_{t=1}^T F_t^* \eta'_t G_t^2\right\|^2\right) = O\left(\frac{1}{N}\right)$, and the proof is complete. \square

As mentioned before, once it has been shown that the new factors: F_t^* , the new loadings: Γ and the new errors: ε_t all satisfy the necessary conditions of Bai and Ng (2002), Propositions 1 and 2 just follow directly from their Theorems 1 and 2, with r replaced by $r + k_1$ and F_t replaced by F_t^* .

A.2: Proof of Theorem 1

We only derive the limiting distributions for the two versions of the LM test, since the proof for the Wald tests is very similar. Let \hat{F}_t define the $r \times 1$ vector of estimated factors. Under the null: $k_1 = 0$, when $\bar{r} = r$ we have

$$\hat{F}_t = DF_t + o_p(1).$$

Let $D_{(i)}$ denote the i th row of D , and $D_{(\cdot, j)}$ denote the j th column of D . Define $\hat{\mathcal{F}}_t = DF_t$, and $\hat{\mathcal{F}}_{kt} = D_{(k, \cdot)} \times F_t$ as the k th element of $\hat{\mathcal{F}}_t$. Let \hat{F}_{1t} be the first element of \hat{F}_t , and $\hat{F}_{-1t} = [\hat{F}_{2t} \cdots \hat{F}_{rt}]$, while $\hat{\mathcal{F}}_{1t}$ and $\hat{\mathcal{F}}_{-1t}$ can be defined in the same way. Note that $\hat{\mathcal{F}}_t$ depends on N and T because $D = (\hat{F}F/T)(A'A/N)$ (see Bai and Ng, 2002). For simplicity, let $T\pi$ denote $[T\pi]$.

Note that under H_0 , we allow for the existence of small breaks, so that the model can be written as $X_{it} = A_i F_t + e_{it} + \eta_i G_t^2$. However, since $\eta_i G_t^2$ is $O_p(1/\sqrt{NT})$ by Assumption 1, we can

use similar methods as in Appendix A.1 to show that an error term of this order can be ignored and that the asymptotic properties of \hat{F}_t will not be affected (See Remark 5 of Bai, 2009). Therefore, for simplicity in the presentation below, we eliminate the last term and consider instead the model $X_{it} = A_i F_t + e_{it}$ in the following lemmata (9 to 13) required to prove Lemma 14 which is the key one in the proof of Theorem 1.

Lemma 9.

$$\sup_{\pi \in [0,1]} \left\| \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - \hat{\mathcal{F}}_t) F_t' \right\| = O_p(\delta_{N,T}^{-2}).$$

Proof. Following Bai (2003) we have:

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - \hat{\mathcal{F}}_t) F_t' &= T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T \hat{F}_s F_t' \gamma_N(s,t) + T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T \hat{F}_s F_t' \zeta_{st} + T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T \hat{F}_s F_t' \kappa_{st} + T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T \hat{F}_s F_t' \xi_{st} \\ &= I + II + III + IV \end{aligned}$$

where

$$\begin{aligned} \zeta_{st} &= \frac{e_s' e_t}{N} - \gamma_N(s,t). \\ \kappa_{st} &= F_s' A' e_t / N. \\ \xi_{st} &= F_t' A' e_s / N. \end{aligned}$$

First, note that:

$$I = T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T (\hat{F}_s - DF_s) F_t' \gamma_N(s,t) + T^{-2} D \sum_{t=1}^{T\pi} \sum_{s=1}^T F_s F_t' \gamma_N(s,t).$$

Consider the first part of the right hand side, we have

$$\begin{aligned} &\left\| T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T (\hat{F}_s - DF_s) F_t' \gamma_N(s,t) \right\| \\ &= \left\| T^{-2} \sum_{s=1}^T \left((\hat{F}_s - DF_s) \sum_{t=1}^{T\pi} F_t' \gamma_N(s,t) \right) \right\| \\ &\leq T^{-1/2} \sqrt{\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s - DF_s\|^2} \sqrt{\frac{1}{T} \sum_{t=1}^{T\pi} \|F_t\|^2} \sqrt{\frac{1}{T} \sum_{s=1}^T \sum_{t=1}^{T\pi} \gamma_N(s,t)^2}. \end{aligned}$$

$\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s - DF_s\|^2$ is $O_p(\delta_{N,T}^{-2})$ by Theorem 1 of Bai and Ng (2002), $\sup_{\pi \in [0,1]} \frac{1}{T} \sum_{t=1}^{T\pi} \|F_t\|^2 \leq \frac{1}{T} \sum_{t=1}^T \|F_t\|^2 = O_p(1)$ by Assumption 2, and $\sup_{\pi \in [0,1]} \frac{1}{T} \sum_{s=1}^T \sum_{t=1}^{T\pi} \gamma_N(s,t)^2 \leq \frac{1}{T} \sum_{s=1}^T \sum_{t=1}^T \gamma_N(s,t)^2 = O_p(1)$ by Lemma 1(i) of Bai and Ng (2002). Therefore:

$$\sup_{\pi \in [0,1]} \left\| T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T (\hat{F}_s - DF_s) F_t' \gamma_N(s,t) \right\| = O_p(\delta_{N,T}^{-1} T^{-1/2}).$$

For the second part, note that:

$$\begin{aligned}
& \sup_{\pi \in [0,1]} \left\| T^{-2} D \sum_{t=1}^{T\pi} \sum_{s=1}^T F_s F_t' \gamma_N(s,t) \right\| \\
& \leq \sup_{\pi \in [0,1]} T^{-2} \|D\| \sum_{t=1}^{T\pi} \sum_{s=1}^T \|F_s F_t'\| |\gamma_N(s,t)| \\
& \leq O_p(1) T^{-1} \left(\frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T |\gamma_N(s,t)| \right) \\
& = O_p(T^{-1})
\end{aligned}$$

because $\|D\|$, $\|F_s F_t'\|$ and $\frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T |\gamma_N(s,t)|$ are all $O_p(1)$ from Bai and Ng (2002) and Assumptions 2 and 4. Therefore, we have

$$\sup_{\pi \in [0,1]} \|I\| = O_p\left(\frac{1}{\delta_{N,T} \sqrt{T}}\right). \quad (\text{A.1})$$

Next, II can be written as:

$$II = T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T (\hat{F}_s - DF_s) F_t' \zeta_{st} + T^{-2} D \sum_{t=1}^{T\pi} \sum_{s=1}^T F_s F_t' \zeta_{st}.$$

Similarly, we have

$$\begin{aligned}
& \sup_{\pi \in [0,1]} \left\| T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T (\hat{F}_s - DF_s) F_t' \zeta_{st} \right\| \\
& \leq \sup_{\pi \in [0,1]} \sqrt{\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s - DF_s\|^2} \sqrt{\frac{1}{T} \sum_{t=1}^{T\pi} \|F_t\|^2} \sqrt{\frac{1}{T^2} \sum_{s=1}^T \sum_{t=1}^{T\pi} \zeta_{st}^2} \\
& \leq \sqrt{\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s - DF_s\|^2} \sqrt{\frac{1}{T} \sum_{t=1}^T \|F_t\|^2} \sqrt{\frac{1}{T^2} \sum_{s=1}^T \sum_{t=1}^T \zeta_{st}^2} \\
& = O_p\left(\frac{1}{\delta_{N,T} \sqrt{N}}\right)
\end{aligned}$$

because $\zeta_{st} = N^{-1} \sum_{i=1}^N [e_{it} e_{is} - E(e_{it} e_{is})]$ is $O_p(1/\sqrt{N})$ by Assumption 4. As for the second term of II, we can write:

$$T^{-2} D \sum_{t=1}^{T\pi} \sum_{s=1}^T F_s F_t' \zeta_{st} = \frac{1}{\sqrt{NT}} \frac{1}{T} D \sum_{t=1}^{T\pi} q_t F_t'$$

where

$$q_t = \frac{1}{\sqrt{NT}} \sum_{s=1}^T \sum_{i=1}^N [e_{it} e_{is} - E(e_{it} e_{is})] F_s.$$

Since $E\|q_t\|^2 < M$ by Assumption 4, we have

$$\begin{aligned}
& \sup_{\pi \in [0,1]} \left\| T^{-2} D \sum_{t=1}^{T\pi} \sum_{s=1}^T F_s F_t' \zeta_{st} \right\| \\
&= \frac{1}{\sqrt{NT}} \sup_{\pi \in [0,1]} \left\| T^{-1} D \sum_{t=1}^{T\pi} q_t F_t' \right\| \\
&\leq \frac{1}{\sqrt{NT}} \|D\| \sup_{\pi \in [0,1]} \left\| \sqrt{\frac{1}{T} \sum_{t=1}^{T\pi} \|q_t\|^2} \sqrt{\frac{1}{T} \sum_{t=1}^{T\pi} \|F_t\|^2} \right\| \\
&\leq O_p(1) \frac{1}{\sqrt{NT}} \left\| \sqrt{\frac{1}{T} \sum_{t=1}^T \|q_t\|^2} \sqrt{\frac{1}{T} \sum_{t=1}^T \|F_t\|^2} \right\| \\
&= O_p\left(\frac{1}{\sqrt{NT}}\right).
\end{aligned}$$

Then it follows that

$$\sup_{\pi \in [0,1]} \|II\| = O_p\left(\frac{1}{\delta_{N,T} \sqrt{N}}\right). \quad (\text{A.2})$$

Regarding III, it can be written as:

$$III = T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T (\hat{F}_s - DF_s) F_t' \kappa_{st} + T^{-2} D \sum_{t=1}^{T\pi} \sum_{s=1}^T F_s F_t' \kappa_{st}$$

and the second part on the right hand side can be written as

$$D \left(\frac{1}{T} \sum_{s=1}^T F_s F_s' \right) \frac{1}{NT} \sum_{t=1}^{T\pi} \sum_{i=1}^N \alpha_i F_t' e_{it}.$$

Therefore:

$$\begin{aligned}
& \sup_{\pi \in [0,1]} \left\| T^{-2} D \sum_{t=1}^{T\pi} \sum_{s=1}^T F_s F_t' \kappa_{st} \right\| \\
&\leq \frac{1}{\sqrt{NT}} \|D\| \left\| \frac{1}{T} \sum_{s=1}^T F_s F_s' \right\| \sup_{\pi \in [0,1]} \left\| \frac{1}{\sqrt{NT}} \sum_{t=1}^{T\pi} \sum_{i=1}^N \alpha_i F_t' e_{it} \right\| \\
&= O_p\left(\frac{1}{\sqrt{NT}}\right)
\end{aligned}$$

by Assumption 8.

As for the first part on the right hand side of III, we have

$$\begin{aligned}
& \sup_{\pi \in [0,1]} \left\| T^{-2} \sum_{t=1}^{T\pi} \sum_{s=1}^T (\hat{F}_s - DF_s) F_t' \kappa_{st} \right\| \\
& \leq \sqrt{\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s - DF_s\|^2} \sup_{\pi \in [0,1]} \sqrt{\frac{1}{T} \sum_{s=1}^T \left\| \frac{1}{T} \sum_{t=1}^{T\pi} F_t' \kappa_{st} \right\|^2} \\
& = O_p(\delta_{N,T}^{-1}) \frac{1}{\sqrt{NT}} \sup_{\pi \in [0,1]} \sqrt{\frac{1}{T} \sum_{s=1}^T \left\| F_s' \frac{1}{\sqrt{NT}} \sum_{t=1}^{T\pi} \sum_{i=1}^N \alpha_i F_t' e_{it} \right\|^2} \\
& \leq O_p(\delta_{N,T}^{-1}) \frac{1}{\sqrt{NT}} \sqrt{\frac{1}{T} \sum_{s=1}^T \|F_s\|^2} \sup_{\pi \in [0,1]} \sqrt{\left\| \frac{1}{\sqrt{NT}} \sum_{t=1}^{T\pi} \sum_{i=1}^N \alpha_i F_t' e_{it} \right\|^2} \\
& = O_p\left(\frac{1}{\delta_{N,T}} \frac{1}{\sqrt{NT}}\right)
\end{aligned}$$

by Assumption 8. Thus,

$$\sup_{\pi \in [0,1]} \|III\| = O_p\left(\frac{1}{\sqrt{NT}}\right). \quad (\text{A.3})$$

It can also be proved in the similar way that

$$\sup_{\pi \in [0,1]} \|IV\| = O_p\left(\frac{1}{\sqrt{NT}}\right). \quad (\text{A.4})$$

Finally we have:

$$\begin{aligned}
& \sup_{\pi \in [0,1]} \left\| \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - \hat{\mathcal{F}}_t) F_t' \right\| \leq \sup_{\pi \in [0,1]} \|I\| + \sup_{\pi \in [0,1]} \|II\| + \sup_{\pi \in [0,1]} \|III\| + \sup_{\pi \in [0,1]} \|IV\| \\
& = O_p\left(\frac{1}{\sqrt{T}\delta_{N,T}}\right) + O_p\left(\frac{1}{\sqrt{N}\delta_{N,T}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) = O_p\left(\frac{1}{\delta_{N,T}^2}\right).
\end{aligned}$$

□

Lemma 10.

$$\sup_{\pi \in [0,1]} \left\| \frac{1}{T} \sum_{t=1}^{T\pi} \hat{F}_t \hat{F}_t' - \frac{1}{T} \sum_{t=1}^{T\pi} \hat{\mathcal{F}}_t \hat{\mathcal{F}}_t' \right\| = O_p(\delta_{N,T}^{-2}).$$

Proof. Note that:

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^{T\pi} \hat{F}_t \hat{F}_t' - \frac{1}{T} \sum_{t=1}^{T\pi} \hat{\mathcal{F}}_t \hat{\mathcal{F}}_t' \\
& = \frac{1}{T} \sum_{t=1}^{T\pi} \hat{F}_t \hat{F}_t' - \frac{1}{T} \sum_{t=1}^{T\pi} (DF_t)(F_t' D') \\
& = \frac{1}{T} \sum_{t=1}^{T\pi} \hat{F}_t (\hat{F}_t' - F_t' D') + \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - DF_t)(F_t' D') \\
& = \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - DF_t)(\hat{F}_t - DF_t)' + \frac{1}{T} D \sum_{t=1}^{T\pi} F_t (\hat{F}_t - DF_t)' + \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - DF_t)(F_t' D').
\end{aligned}$$

Thus,

$$\begin{aligned}
& \sup_{\pi \in [0,1]} \left\| \frac{1}{T} \sum_{t=1}^{T\pi} \hat{F}_t \hat{F}_t' - \frac{1}{T} \sum_{t=1}^{T\pi} \hat{\mathcal{F}}_t \hat{\mathcal{F}}_t' \right\| \\
& \leq \sup_{\pi \in [0,1]} \left\| \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - DF_t)(\hat{F}_t - DF_t)' \right\| + 2\|D\| \sup_{\pi \in [0,1]} \left\| \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - DF_t) F_t' \right\| \\
& \leq \frac{1}{T} \sum_{t=1}^T \|\hat{F}_t - DF_t\|^2 + 2\|D\| \sup_{\pi \in [0,1]} \left\| \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - DF_t) F_t' \right\|
\end{aligned}$$

since $\frac{1}{T} \sum_{t=1}^T \|\hat{F}_t - DF_t\|^2 = O_p(\delta_{N,T}^{-2})$ and $\sup_{\pi \in [0,1]} \left\| \frac{1}{T} \sum_{t=1}^{T\pi} (\hat{F}_t - DF_t) F_t' \right\|$ is $O_p(\delta_{N,T}^{-2})$ by Lemma 9, the proof is complete. \square

The next two lemmae follow from Lemma 10 and Assumption 6:

Lemma 11.

$$\sup_{\pi \in [0,1]} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \hat{F}_{-1t} \hat{F}_{1t}' - \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \hat{\mathcal{F}}_{-1t} \hat{\mathcal{F}}_{1t}' \right\| = o_p(1).$$

Proof. See Lemma 10 and Assumption 6. \square

Lemma 12.

$$\left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T \hat{\mathcal{F}}_{-1t} \hat{\mathcal{F}}_{1t}' \right\| = o_p(1).$$

Proof. By construction we have $\frac{1}{T} \sum_{t=1}^T \hat{F}_{-1t} \hat{F}_{1t}' = 0$, and then the result follows from Lemma 11. \square

Let \Rightarrow denote *weak convergence*, $D^* = Q\Sigma_A$, where $Q = \lim \frac{\hat{F}'F}{T}$ (see proposition 1 of Bai (2003)), $\Sigma_A = \lim \frac{A'A}{N}$, and define $\mathcal{F}_t = D^*F_t$, $S = \lim \text{Var} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \mathcal{F}_{-1t} \mathcal{F}_{1t}' \right)$. Then:

Lemma 13.

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} (\mathcal{F}_{-1t} \mathcal{F}_{1t}' - E(\mathcal{F}_{-1t} \mathcal{F}_{1t}')) \Rightarrow S^{1/2} \mathcal{W}_{r-1}(\pi)$$

for $\pi \in [0,1]$, where $\mathcal{W}_{r-1}(\cdot)$ is a $r-1$ vector of independent Brownian motions on $[0,1]$.

Proof. $\mathcal{F}_{-1t} \mathcal{F}_{1t}'$ is stationary and ergodic because F_t is stationary and ergodic by Assumption 7. First, we show that $\{\mathcal{F}_{kt} \mathcal{F}_{1t}' - E(\mathcal{F}_{kt} \mathcal{F}_{1t}'), \Omega_t\}$ is an adapted mixingale of size -1 for $k = 2, \dots, r$. By definition, we have $\mathcal{F}_{kt} \mathcal{F}_{1t}' = (D_{(k)}^* F_t)(D_{(1)}^* F_t)' = (\sum_{p=1}^r D_{kp}^* F_{pt}) (\sum_{p=1}^r D_{1p}^* F_{pt})' = \sum_{h=1}^r \sum_{p=1}^r D_{kp}^* D_{1h}^* F_{pt} F_{ht}'$, and $\mathcal{F}_{kt} \mathcal{F}_{1t}' - E(\mathcal{F}_{kt} \mathcal{F}_{1t}') = \sum_{h=1}^r \sum_{p=1}^r D_{kp}^* D_{1h}^* (F_{pt} F_{ht}' - E(F_{pt} F_{ht}')) =$

$\sum_{h=1}^r \sum_{p=1}^r D_{kp}^* D_{1h}^* Y_{hp,t}$. Thus:

$$\begin{aligned}
& \sqrt{E\left(E\left(\mathcal{F}_{kt}\mathcal{F}_{1t} - E\left(\mathcal{F}_{kt}\mathcal{F}_{1t}\right)\middle|\Omega_{t-m}\right)\right)^2} \\
&= \sqrt{E\left(\sum_{h=1}^r \sum_{p=1}^r D_{kp}^* D_{1h}^* E\left(Y_{hp,t}\middle|\Omega_{t-m}\right)\right)^2} \\
&\leq \sum_{h=1}^r \sum_{p=1}^r |D_{kp}^* D_{1h}^*| \sqrt{E\left(E\left(Y_{hp,t}\middle|\Omega_{t-m}\right)\right)^2} \\
&\leq \Delta \sum_{h=1}^r \sum_{p=1}^r c_t^{hp} \gamma_m^{hp} \\
&\leq \Delta r^2 \max(c_t^{hp}) \max(\gamma_m^{hp})
\end{aligned}$$

since $\max(\gamma_m^{hp})$ is $O(m^{-1-\delta})$ for some $\delta > 0$ by Assumption 7, we conclude that $\{\mathcal{F}_{kt}\mathcal{F}_{1t} - E(\mathcal{F}_{kt}\mathcal{F}_{1t}), \Omega_t\}$ is an adapted mixingale of size -1 for $k = 2, \dots, r$.

Next, we prove the weak convergence using the Crame-Rao device. Define

$$z_t = a' S^{-1/2} (\mathcal{F}_{-1t} \mathcal{F}_{1t} - E(\mathcal{F}_{-1t} \mathcal{F}_{1t}))$$

where $a \in \mathbb{R}^{r-1}$, and $a'a = 1$. Note that

$$z_t = \sum_{k=2}^r \tilde{a}_k [\mathcal{F}_{kt} \mathcal{F}_{1t} - E(\mathcal{F}_{kt} \mathcal{F}_{1t})]$$

where \tilde{a}_k is the $k - 1$ th element of $a' S^{-1/2}$.

$$\begin{aligned}
E(z_t^2) &\leq \left(\sum_{k=2}^r \sqrt{E\left(\tilde{a}_k [\mathcal{F}_{kt} \mathcal{F}_{1t} - E(\mathcal{F}_{kt} \mathcal{F}_{1t})]\right)^2} \right)^2 \\
&\leq \Delta \left(\sum_{k=2}^r \sqrt{E\left(\mathcal{F}_{kt} \mathcal{F}_{1t}\right)^2 - \left(E\left(\mathcal{F}_{kt} \mathcal{F}_{1t}\right)\right)^2} \right)^2 \leq M
\end{aligned}$$

because $E\|F_t\|^4 < \infty$ and $\mathcal{F}_{kt} = D_k^* F_t$. Moreover, z_t is stationary and ergodic, and we can show $\{z_t, \Omega_t\}$ is an adapted mixingale sequence of size -1 because:

$$\begin{aligned}
\sqrt{E\left(E(z_t|\Omega_{t-m})\right)^2} &= \sqrt{E\left(\sum_{k=2}^r \tilde{a}_k E\left(\mathcal{F}_{kt}\mathcal{F}_{1t} - E\left(\mathcal{F}_{kt}\mathcal{F}_{1t}\right)\middle|\Omega_{t-m}\right)\right)^2} \\
&\leq \sum_{k=2}^r |\tilde{a}_k| \sqrt{E\left(E\left(\mathcal{F}_{kt}\mathcal{F}_{1t} - E\left(\mathcal{F}_{kt}\mathcal{F}_{1t}\right)\middle|\Omega_{t-m}\right)\right)^2} \\
&\leq \max(|\tilde{a}_k|) \sum_{k=2}^r \tilde{c}_t^k \tilde{\gamma}_m^k.
\end{aligned}$$

By the results above we know that $\tilde{\gamma}_m^k$ is $O(m^{-1-\delta})$ for $k = 2, \dots, r$. Hence it follows that $\{z_t, \Omega_t\}$ is an adapted mixingale sequence of size -1 . Then it follows from Theorem 7.17 of White (2001) that:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} z_t = a' S^{-1/2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} (\mathcal{F}_{-1t} \mathcal{F}_{1t} - E(\mathcal{F}_{-1t} \mathcal{F}_{1t})) \Rightarrow \mathcal{W}(\pi).$$

Moreover, it can be shown that:

$$a_1' \frac{1}{\sqrt{T}} \sum_{t=T\pi_1}^{T\pi_2} (\mathcal{F}_{-1t} \mathcal{F}_{1t} - E(\mathcal{F}_{-1t} \mathcal{F}_{1t})) + a_2' \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi_0} (\mathcal{F}_{-1t} \mathcal{F}_{1t} - E(\mathcal{F}_{-1t} \mathcal{F}_{1t})) \rightsquigarrow \mathcal{N}(0, (\pi_2 - \pi_1) a_1' S a_1 + \pi_0 a_2' S a_2)$$

by using Corollary 3.1 of Woodridge and White (1988). The proof is complete by using Lemma A.4 of Andrews (1993). \square

Lemma 14.

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \hat{F}_{-1t} \hat{F}_{1t} \Rightarrow S^{1/2} \mathcal{B}_{r-1}^0(\pi)$$

for $\pi \in [0, 1]$, where the process $\mathcal{B}_{r-1}^0(\pi) = \mathcal{W}_{r-1}(\pi) - \pi \mathcal{W}_{r-1}(1)$ indexed by π is a vector of Brownian Bridge on $[0, 1]$.

Proof. If we show that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \left[\mathcal{F}_{-1t} \mathcal{F}_{1t} - T^{-1} \sum_{s=1}^T \mathcal{F}_{-1s} \mathcal{F}_{1s} \right] \Rightarrow S^{1/2} \mathcal{B}_{r-1}^0(\pi) \quad (\text{A.5})$$

for $\pi \in [0, 1]$ and

$$\sup_{\pi \in [0, 1]} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \hat{\mathcal{F}}_{-1t} \hat{\mathcal{F}}_{1t} - \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \left[\mathcal{F}_{-1t} \mathcal{F}_{1t} - T^{-1} \sum_{s=1}^T \mathcal{F}_{-1s} \mathcal{F}_{1s} \right] \right\| = o_p(1), \quad (\text{A.6})$$

then the result follows from Lemma 11.

First note that

$$\begin{aligned} & \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \left[\mathcal{F}_{-1t} \mathcal{F}_{1t} - T^{-1} \sum_{s=1}^T \mathcal{F}_{-1s} \mathcal{F}_{1s} \right] \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} (\mathcal{F}_{-1t} \mathcal{F}_{1t} - E(\mathcal{F}_{-1t} \mathcal{F}_{1t})) + \frac{1}{T} \sum_{t=1}^{T\pi} \left(\frac{1}{\sqrt{T}} \sum_{s=1}^T (\mathcal{F}_{-1s} \mathcal{F}_{1s} - E(\mathcal{F}_{-1s} \mathcal{F}_{1s})) \right), \end{aligned}$$

hence A.5 can be verified by applying Lemma 13.

To prove A.6, we first define D_{-1} as the second to last rows of D , and D_1 as the first row of D . Then we have

$$\hat{\mathcal{F}}_{-1t} \hat{\mathcal{F}}_{1t} = D_{-1} F_t F_t' D_1'$$

and

$$\mathcal{F}_{-1t} \mathcal{F}_{1t} = D_{-1}^* F_t F_t' D_1^{*'}.$$

It follows that:

$$\begin{aligned} & \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} (\hat{\mathcal{F}}_{-1t} \hat{\mathcal{F}}_{1t} - \mathcal{F}_{-1t} \mathcal{F}_{1t}) \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \left(D_{-1} F_t F_t' D_1' - D_{-1} F_t F_t' D_1^{*'} + D_{-1} F_t F_t' D_1^{*'} - D_{-1}^* F_t F_t' D_1^{*'} \right) \\ &= D_{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} F_t F_t' \right) (D_1' - D_1^{*'}) + (D_{-1} - D_{-1}^*) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} F_t F_t' \right) D_1^{*'} \end{aligned}$$

Next, define $\overline{\mathcal{F}_{-1t}\mathcal{F}_{1t}} = \frac{1}{T} \sum_{s=1}^T \mathcal{F}_{-1s}\mathcal{F}_{1s}$, and $\overline{F_{-1t}F_{1t}} = \frac{1}{T} \sum_{s=1}^T F_{-1s}F_{1s}$, then:

$$\begin{aligned}
& \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \left(T^{-1} \sum_{s=1}^s \mathcal{F}_{-1s}\mathcal{F}_{1s} \right) \\
= & D_{-1}^* \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \overline{F_{-1t}F_{1t}} \right) D_1^* \\
= & D_{-1}^* \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \overline{F_{-1t}F_{1t}} \right) D_1^* - D_{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \overline{F_{-1t}F_{1t}} \right) D_1^* + D_{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \overline{F_{-1t}F_{1t}} \right) D_1^* \\
& - D_{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \overline{F_{-1t}F_{1t}} \right) D_1' + D_{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \overline{F_{-1t}F_{1t}} \right) D_1' \\
= & (D_{-1}^* - D_{-1}) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \overline{F_{-1t}F_{1t}} \right) D_1^* + D_{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \overline{F_{-1t}F_{1t}} \right) (D_1^* - D_1') + \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \left(\frac{1}{T} \sum_{s=1}^T \hat{\mathcal{F}}_{-1s}\hat{\mathcal{F}}_{1s} \right).
\end{aligned}$$

Combining the above results gives:

$$\begin{aligned}
& \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \hat{\mathcal{F}}_{-1t}\hat{\mathcal{F}}_{1t} - \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \left[\mathcal{F}_{-1t}\mathcal{F}_{1t} - T^{-1} \sum_{s=1}^T \mathcal{F}_{-1s}\mathcal{F}_{1s} \right] \\
= & \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} (\hat{\mathcal{F}}_{-1t}\hat{\mathcal{F}}_{1t} - \mathcal{F}_{-1t}\mathcal{F}_{1t}) + \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \left(T^{-1} \sum_{s=1}^T \mathcal{F}_{-1s}\mathcal{F}_{1s} \right) \\
= & D_{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} (F_t F_t' - \overline{F_{-1t}F_{1t}}) \right) (D_1' - D_1^*) + (D_{-1} - D_{-1}^*) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} (F_t F_t' - \overline{F_{-1t}F_{1t}}) \right) D_1^* \\
& + \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} \left(\frac{1}{T} \sum_{s=1}^T \hat{\mathcal{F}}_{-1s}\hat{\mathcal{F}}_{1s} \right).
\end{aligned}$$

Following the similar arguments of Lemma 13, we can prove that

$$\sup_{\pi \in [0,1]} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T\pi} (F_t F_t' - \overline{F_{-1t}F_{1t}}) \right\| = O_p(1).$$

Moreover, it is easy to check that $\|D\| = O_p(1)$ and $\|D - D^*\| = o_p(1)$. Finally, $\left\| \frac{1}{\sqrt{T}} \sum_{s=1}^T \hat{\mathcal{F}}_{-1s}\hat{\mathcal{F}}_{1s} \right\|$ is $o_p(1)$ by Lemma 12. Then A.6 has been shown and the proof is complete. \square

Theorem 1:

Proof. The results for LM and Sup-LM tests follow from Assumption 9, Lemma 14, and Continuous Mapping Theorem.

For the Wald and Sup-Wald tests, notice that:

$$\begin{aligned}
\sqrt{T}(\hat{c}_1(\pi) - \hat{c}_2(\pi)) &= (1/T \sum_{t=1}^{\tau} \hat{F}_{-1t}\hat{F}'_{-1t})^{-1} (1/\sqrt{T} \sum_{t=1}^{\tau} \hat{F}_{-1t}\hat{F}_{1t}) - (1/T \sum_{t=\tau+1}^T \hat{F}_{-1t}\hat{F}'_{-1t})^{-1} (1/\sqrt{T} \sum_{t=\tau+1}^T \hat{F}_{-1t}\hat{F}_{1t}) \\
&= \left[(1/T \sum_{t=1}^{\tau} \hat{F}_{-1t}\hat{F}'_{-1t})^{-1} + (I - 1/T \sum_{t=1}^{\tau} \hat{F}_{-1t}\hat{F}'_{-1t})^{-1} \right] (1/\sqrt{T} \sum_{t=1}^{\tau} \hat{F}_{-1t}\hat{F}_{1t}).
\end{aligned}$$

By Lemma 10 and that $D - D^* = o_p(1)$, we have:

$$1/T \sum_{t=1}^{\tau} \hat{F}_{-1t} \hat{F}'_{-1t} = \pi \frac{1}{\tau} \sum_{t=1}^{\tau} \hat{F}_{-1t} \hat{F}'_{-1t} = \pi \frac{1}{\tau} \sum_{t=1}^{\tau} \mathcal{F}_{-1t} \mathcal{F}'_{-1t} + o_p(1). \quad (\text{A.7})$$

when $\tau = T$ ($\pi = \tau/T = 1$), this implies

$$I_{r-1} = 1/T \sum_{t=1}^T \hat{F}_{-1t} \hat{F}'_{-1t} = \frac{1}{T} \sum_{t=1}^T \mathcal{F}_{-1t} \mathcal{F}'_{-1t} + o_p(1).$$

By LLN we have $E(\mathcal{F}_{-1t} \mathcal{F}'_{-1t}) = I_{r-1}$. Applying LLN again to A.7 gives:

$$1/T \sum_{t=1}^{\tau} \hat{F}_{-1t} \hat{F}'_{-1t} \xrightarrow{p} \pi I_{r-1}$$

as N and T go to infinity. It then follows from Lemma 14 that:

$$\sqrt{T}(\hat{c}_1(\boldsymbol{\pi}) - \hat{c}_2(\boldsymbol{\pi})) \Rightarrow \frac{S^{1/2} \mathcal{B}_{r-1}^0(\boldsymbol{\pi})}{\boldsymbol{\pi}(1-\boldsymbol{\pi})}$$

and the limit distributions of the Sup-Wald and Wald tests follow easily. \square

A.3: Consistent Estimator of S

We now discuss the consistent estimator of S using the HAC estimator of Newey and West (1987).

Recall that $S = \lim \text{Var}\left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \mathcal{F}_{-1t} \mathcal{F}_{1t}\right)$.

Notice that $E(\mathcal{F}_{-1t} \mathcal{F}_{1t}) = 0$, because $\frac{1}{T} \sum_{t=1}^T \mathcal{F}_{-1t} \mathcal{F}_{1t} = \frac{1}{T} \sum_{t=1}^T \hat{F}_{-1t} \hat{F}_{1t} + o_p(1)$, and $\frac{1}{T} \sum_{t=1}^T \mathcal{F}_{-1t} \mathcal{F}_{1t} \rightarrow E(\mathcal{F}_{-1t} \mathcal{F}_{1t})$ by Assumption 7 and $\frac{1}{T} \sum_{t=1}^T \hat{F}_{-1t} \hat{F}_{1t} = 0$.

First, we define the infeasible estimator of S :

$$\hat{S}(\mathcal{F}) = \hat{\Gamma}_0(\mathcal{F}) + \sum_{j=1}^m w(j, m) [\hat{\Gamma}_j(\mathcal{F}) + \hat{\Gamma}_j(\mathcal{F})']$$

where $m = O_p(T^{\frac{1}{5}})$, $w(j, m) = 1 - \frac{j}{m+1}$ is the Bartlett kernel, and

$$\hat{\Gamma}_j(\mathcal{F}) = \frac{1}{T} \sum_{t=j+1}^T \mathcal{F}_{-1t} \mathcal{F}_{1t} \mathcal{F}_{1t-j} \mathcal{F}'_{-1t-j}.$$

Since the above estimator is a HAC, it is natural to make the following assumption:

Assumption 11: $\|\hat{S}(\mathcal{F}) - S\| = o_p(1)$.

Next we consider a feasible estimator of S where \mathcal{F}_t is replaced by \hat{F}_t :

$$\hat{S}(\hat{F}) = \hat{\Gamma}_0(\hat{F}) + \sum_{j=1}^m w(j, m) [\hat{\Gamma}_j(\hat{F}) + \hat{\Gamma}_j(\hat{F})']$$

where

$$\hat{\Gamma}_j(\hat{F}) = \frac{1}{T} \sum_{t=j+1}^T \hat{F}_{-1t} \hat{F}_{1t} \hat{F}_{1t-j} \hat{F}'_{-1t-j}.$$

If we further assume:

Assumption 12: $\|D^* - D\| = o_p(T^{-\frac{1}{5}})$.

then we have the following results:

Proposition 3: *Assume that Assumptions 1 to 12 hold, under the null $H_0 : k_1 = 0$, we have*

$$\|\hat{S}(\hat{F}) - S\| = o_p(1).$$

Proof. Given Assumption 10, it suffices to show that

$$\|\hat{S}(\hat{F}) - \hat{S}(\mathcal{F})\| = o_p(1).$$

It is easy to see that:

$$\|\hat{S}(\hat{F}) - \hat{S}(\mathcal{F})\| \leq 2 \sum_{j=0}^m \|\hat{\Gamma}_j(\hat{F}) - \hat{\Gamma}_j(\mathcal{F})\|,$$

and

$$\begin{aligned} & \|\hat{\Gamma}_j(\hat{F}) - \hat{\Gamma}_j(\mathcal{F})\| \\ & \leq \frac{1}{T} \sum_{t=1}^T \left\| \hat{F}_{-1t} \hat{F}_{1t} \hat{F}_{1t-j} \hat{F}'_{-1t-j} - \mathcal{F}_{-1t} \mathcal{F}_{1t} \mathcal{F}_{1t-j} \mathcal{F}'_{-1t-j} \right\| \\ & = O_p(\|\hat{F}_t - \mathcal{F}_t\|) \\ & = O_p(\delta_{N,T}^{-1}) + O_p(\|D^* - D\|). \end{aligned}$$

Notice that the result in the last line follows from the fact that

$$\|\hat{F}_t - \mathcal{F}_t\| \leq \|\hat{F}_t - DF_t\| + \|DF_t - D^*F_t\|$$

and $\|\hat{F}_t - DF_t\| = O_p(\delta_{N,T}^{-1})$.

Finally we have

$$\|\hat{S}(\hat{F}) - \hat{S}(\mathcal{F})\| = O_p(T^{\frac{1}{5}} \delta_{N,T}^{-1}) + O_p(\|D^* - D\|) O_p(T^{\frac{1}{5}}) = o_p(1)$$

by Assumptions 6 and 11. □