Idiosyncratic Uncertainty with Incomplete Markets

Felix Wellschmied

UC3M

Macroeconomics III

Outline

- So far, we have estimated idiosyncratic earnings risk.

  persistent shocks.

  transitory shocks.

- We want to understand what are the implications of this risk.
- We will focus on persistent shocks and neglect transitory shocks.

Why Persistent Shocks?

Consider Friedman's permanent income model with finite life, T. The household maximizes consumption starting its life with zero assets, a_0, and has to dye with zero assets. To keep the math simple, I simplify the income process:

$$\max_{c_t, a_{t+1}} \mathbb{E}_0 \Big\{ \sum_{t=0}^{T} \beta^t u(c_t) \Big\} \tag{1}$$

s.t.

$$c_t + a_{t+1} = y_t + (1+r)a_t \tag{2}$$

$$a_{T+1} = a_0 = 0 \tag{3}$$

$$y_{t+1} = z_{t+1} + \iota_{t+1} \tag{4}$$

$$z_{t+1} = z_t + \epsilon_{t+1}. \tag{5}$$

Income follows a random walk with innovations ϵ and transitory shocks ι.

Why Persistent Shocks? II

The Euler equation is given by:

$$(1+r)\beta \mathbb{E}_t u'(c_{t+1}) = u'(c_t). \tag{6}$$

Assume $\beta = \frac{1}{1+r}$ and quadratic utility: $u = c_t - \frac{b}{2}c_t^2$. Then

$$u' = 1 - bc_t \tag{7}$$

$$\mathbb{E}_t u'(c_{t+1}) = 1 - b\mathbb{E}_t c_{t+1} \tag{8}$$

$$c_t = \mathbb{E}_t c_{t+1} \tag{9}$$

Consumption is a random walk responding to income shocks.

Why Persistent Shocks? II

$$c_t = \mathbb{E}_t c_{t+1} \tag{10}$$

In period $T-1$,

$$\mathbb{E}_{T-1} c_T = \mathbb{E}_{T-1} Y_T + (1+r)a_T = Y_{T-1} + (1+r)a_T. \tag{11}$$

For notation, suppose a household enters period $T-1$ with a before shock income $\tilde{Y}_{T-1}$ with $Y_{T-1} = \tilde{Y}_{T-1} + \iota_{T-1} + \epsilon_{T-1}$. Moreover, suppose $a_{T-1} = 0$. Its remaining expected lifetime income is

$$Y_{T-1} + \mathbb{E}_{T-1} Y_T = Y_{T-1} + Y_{T-1} - \iota_{T-1} \tag{12}$$

$$= \tilde{Y}_{T-1} + \iota_{T-1} + \epsilon_{T-1} + \tilde{Y}_{T-1} + \epsilon_{T-1}. \tag{13}$$

Why Persistent Shocks? III

A permanent shock, ϵ_{T-1}, changes lifetime income by $2\epsilon_{T-1}$. Hence, the household

$$\Delta c_{T-1} = \mathbb{E}_{T-1}\Delta c_T = \epsilon_{T-1}. \tag{14}$$

Now consider a transitory shock ι_{T-1}. The flow budget constraints are:

$$\mathbb{E}_{T-1}c_T = (1+r)a_T + \mathbb{E}_{T-1}Y_T \tag{15}$$

$$c_{T-1} + a_T = \tilde{Y}_{T-1} + \iota_{T-1} \tag{16}$$

With $c_{T-1} = \mathbb{E}_{T-1}c_T$ and $\tilde{Y}_{T-1} = \mathbb{E}_{T-1}Y_T$ we have

$$c_{T-1} = \frac{1+r}{2+r}\left[\tilde{Y}_{T-1} + \iota_{T-1}\right] + \frac{1}{2+r}\mathbb{E}_{T-1}Y_T. \tag{17}$$

Hence, $\Delta c_{T-1} = \Delta\mathbb{E}_{T-1}c_T = \frac{1+r}{2+r}\iota_{T-1}$.

Why Persistent Shocks? IV

We always have a full consumption response to a persistent shock and

$$c_{T-1} = \frac{1+r}{2+r}\left[\tilde{Y}_{T-1} + \iota_{T-1}\right] + \frac{1}{2+r}\mathbb{E}_{T-1}Y_T. \tag{18}$$

In general, one can show that

$$\Delta c_t = \epsilon_t + \frac{r}{1+r}\frac{1}{1-((1+r))^{-(T-t+1)}}\iota_t \tag{19}$$

which becomes small as $t << T$. The earlier in the life cycle a transitory shock occurs, the more periods the household has to smooth this shock.

Hence, persistent shocks lead to much larger consumption responses than transitory shocks making them more interesting to study.

Idiosyncratic Uncertainty with Incomplete Markets

Idiosyncratic Uncertainty with Incomplete Markets

We look for a simple and tractable way to introduce heterogeneity.

- Bewley/Huggett/Aiyagari world.
- Agents face **idiosyncratic shocks** s to income. The shocks are independent across agents.
- There are market incompletenesses: No state contingent claims on future income and a borrowing constrained.
- Agents can trade a risk-free asset.

Huggett (1993): exchange economy

- A continuum of agents get an endowment s with a time-invariant transition matrix Ω.
- There is no production or storage.
- Agents can trade one-period claims $a \in [\underline{a}, \infty)$ of consumption goods with price q.
- Focus on steady states: the price q is constant.

Household problem

The household's problem can be written as:

$$V(s,a) = \max_{c,a'} \left\{ u(c) + \beta \sum_{s'} \Omega(s,s') V(s',a') \right\}$$

subject to

$$a'q + c = s + a$$
$$c \geq 0 \text{ and } a' \geq \underline{a}$$

We are looking for decision rules $a' = g^a(s,a)$ and $c = g^c(s,a)$.
A stationary distribution of households over productivities and asset holdings: μ.

Deterministic case

In the **deterministic case** (we get for sure $s' = E[s]$) the FOC is:

$$u_c(c) = \frac{\beta}{q} u_c(c')$$

An equilibrium with positive consumption will only exist for $\frac{\beta}{q} = 1$

- if $\frac{\beta}{q} > 1 \Rightarrow u_c(c) > u_c(c') \Rightarrow c < c'$ for all periods, so eventually we will have infinite consumption. This is sustainable only if a goes to infinity too.
- if $\frac{\beta}{q} < 1 \Rightarrow u_c(c) < u_c(c') \Rightarrow c > c'$ for all periods, so eventually we will have zero consumption.

Stochastic case

Let's turn now to the **stochastic problem**. The FOC is:

$$u_c(c) = \frac{\beta}{q} \sum_{s'} \Omega(s, s') u_c(c')$$

▷ Then we concentrate on $\frac{\beta}{q} < 1$,

- Returns from savings are lower than the discount rate. In the absence of uncertainty, households would like to bring consumption from the future to the present.
- However, with uncertainty there is a force that compensates for $\frac{\beta}{q} < 1$: precautionary savings. Uncertainty gives assets an extra return, that of insuring against bad shocks.

Stochastic case II

- Huggett (1997) contains a proof for $\frac{\beta}{q} < 1$ with very general utility functions.
- The proof is simple when consumers are prudent, i.e., $u_{ccc} > 0$.
  Marginal utilities are a convex function.
- if $\frac{\beta}{q} \geq 1 \Rightarrow u_c(c) \geq \sum_{s'} \Omega(s,s')u_c(c')$. Then,

$$u_{ccc} > 0 \Rightarrow \sum_{s'} \Omega(s,s')u_c(c') > u_c(\sum_{s'} \Omega(s,s')c')$$
$$\Rightarrow u_c(c) > u_c(\sum_{s'} \Omega(s,s')c')$$

  which implies that $c < \sum_{s'} \Omega(s,s')c'$.
  So consumption grows to infinity and so does accumulation.
- Hence the equilibrium must be characterized by $\frac{\beta}{q} < 1$.

An upper bound for assets

- The return on the *insurance part* decreases with wealth.
  This is because, with higher wealth, income fluctuations translate less into consumption fluctuations
- Accumulation dominates depletion for low levels of assets. As the household gets wealth-richer he is becoming better self-insured and the intertemporal motive for savings gets more important.
- There is a point in which depletion (low $\frac{\beta}{q}$) starts dominating. We call this point the *endogenous upper bound* for assets $\bar{a}$.
  Hence, $a \in \mathbf{a} \equiv [\underline{a}, \bar{a}]$
- Huggett (1993) provides a formal proof.

Comparison to Friedman

Before continuing, lets compare this idea to Friedman's life cycle model that we started out with. In that case, we had with $\beta = \frac{1}{1+r}$

$$u'(c_{t+1}) = \mathbb{E}_t u'(c_{t+1}) \tag{20}$$
$$c_t = \mathbb{E}_t c_{t+1}, \tag{21}$$

i.e., despite risk aversion, people do not engage in precautionary savings which simplified the problem significantly. The reason is the particular utility function that we had used:

$$u = c_t - \frac{b}{2} c_t^2 \tag{22}$$
$$u'''(c_t) = 0. \tag{23}$$

Characterizing heterogeneity

- Ex-ante identical households will differ in equilibrium in their earnings position and asset holdings, $\{s, a\} \in S \times \mathbf{a}$
- Let $\mathbf{B} \equiv S \times \mathbf{a}$ and let $\mathcal{B}$ be the σ-algebra generated in $\mathbf{B}$ by its open intervals (think of $\mathcal{B}$ as a very comprehensive family of subsets of $\mathbf{B}$)
  In particular, $B \in \mathcal{B} \Rightarrow B \subset \mathbf{B}$
- Then, $\mu : \mathcal{B} \to [0, 1]$ is a probability measure over $\mathcal{B}$ that exhaustively describes the economy by stating how many households are of each type.
  Intuitively, μ gives the size of any subset of $\mathbf{B}$

Evolution of the probability measure

- Let $b = \{s, a\} \in \mathbf{B}$ and $B \subset \mathbf{B}$ such that $B \in \mathcal{B}$.
  Then, the transition function $Q(b, B)$ denotes the probability of an agent of type b of becoming of any type $b' \in B$.
  The function Q describes how the economy evolves:

$$\mu'(B) = \int_{\mathbf{B}} Q(b, B)\, d\mu$$

- It can be shown that, if the Markov process for the idiosyncratic shocks is *well-behaved* (monotone mixing condition), there is a unique stationary distribution for this economy.
- Being at any point b does not exclude me from ending up at any other b' somewhere in the future.

Characterizing heterogeneity: an example

- Imagine $S = \{s_1, s_2\}$ and $\mathbf{a} = [0.0, 30.0]$; $\mathbf{B} \equiv \{s_1, s_2\} \times [0.0, 30.0]$
- Then, a particular example of $B \subset \mathbf{B}$ can be given by,

$$B \equiv \{s, a \mid s = s_1 \text{ and } a \in [25.0, 30.0]\}$$

- And $\mu(B)$ tells us how many households there are in the set B
- If we define $b = \{s_2, 3.45\}$, then $Q(b, B)$ tells us the probability that an individual with the shock s_2 and assets equal to 3.45 becomes of a type in B in the next period.

Stationary Equilibrium

A stationary equilibrium is a set of functions $\{g^c, g^a\}$, price q, and a probability measure μ such that:

1. Given the price, $\{g^c, g^a\}$ solve households' optimization problem.
2. Asset and goods markets clear:

$$\int_{\mathbf{B}} g^a(s,a)d\mu = 0$$

$$\int_{\mathbf{B}} g^c(s,a)d\mu = \int_{\mathbf{B}} sd\mu$$

3. The measure of households is stationary:

$$\mu(B) = \int_{\mathbf{B}} Q(b,B)\,d\mu \quad \forall B \in \mathcal{B}.$$

Stationary Equilibrium II

- It can be shown that the stationary distribution μ is unique.
- This implies that in the steady state there is a unique price q.

Aiyagari (1994): Production Economy

Now we are ready to see the economy with an aggregate production technology. We will analyze it only in the steady state (aggregate allocations and prices are constant).

- The production side is standard.
- There is a continuum of households facing uninsurable idiosyncratic shocks to their labor endowment $s \in S$ that follows a Markov process with transition matrix Ω.
- For simplicity, households do not value leisure and therefore work their whole endowment of market time.
- Households can save (and possibly borrow) by means of a risk free asset $k \in \mathbf{k} \equiv \left[\underline{k}, \bar{k}\right]$ paying return $R = 1 + r$.

Household Problem

For given prices w and $R = R_t = R_{t+1}$ the household problem is:

$$V(s,k) = \max_{c,k'} \left\{ u(c) + \beta \sum_{s'} \Omega(s,s') V\left(s',k'\right) \right\}$$

subject to

$$k' + c = sw + Rk$$
$$c \geq 0 \quad \text{and} \quad k' \geq \underline{k}$$

which yields decision rules $k' = g^k(s,k)$ and $c = g^c(s,k)$.
The Euler equation is:

$$u_c(c) = \beta R \sum_{s'} \Omega(s,s') u_c(c')$$

Equilibrium Interest Rate

- if $\beta R \geq 1 \Rightarrow u_c(c) \geq \sum_{s'} \Omega(s,s')u_c(c')$. Then,

$$u_{ccc} > 0 \Rightarrow \sum_{s'} \Omega(s,s')u_c(c') > u_c(\sum_{s'} \Omega(s,s')c')$$

  which implies that $c < \sum_{s'} \Omega(s,s')c'$.
  So consumption grows to infinity and so does accumulation.
- Hence the equilibrium must be characterized by $\beta R < 1$.
- Instead, in the deterministic case we have,

$$u_c\left(c\right) = \beta R\, u_c\left(c'\right)$$

  and to obtain $c = c'$ we need $\beta R = 1$.

Stationary Equilibrium

A stationary equilibrium is a set of functions $\{g^c, g^k\}$, allocations $\{K, L\}$, prices $\{w, R\}$, and a probability measure μ such that:

1. Given the prices, $\{g^c, g^k\}$ solve households' optimization problem.
2. Given the prices, $\{K, L\}$, solve firms' optimization problem:

$$F_K(K, L) = R - 1 + \delta,\ F_L(K, L) = w$$

3. Capital, labor and goods markets clear:

$$\int_{\mathbf{B}} g^k(k, s) d\mu = K,\ \int_{\mathbf{B}} s d\mu = L$$

$$\int_{\mathbf{B}} g^c(k, s) d\mu + \delta K = F(K, L)$$

4. The measure of households is stationary:

$$\mu(B) = \int_{\mathbf{B}} Q(b, B)\, d\mu \quad \forall B \in \mathcal{B}$$

Stationary Equilibrium II

The distribution of households in this economy evolves as follows,

$$\mu'(B) = \int_{\mathbf{B}} Q(b, B)\, d\mu$$

where $\mathbf{B} \equiv S \times \mathbf{k}$. The transition function Q defines a mapping T from the space of probability measures into itself such that

$$\mu' = T(\mu)$$

It can be proved that:

1. there is unique fixed point μ^*, i.e., there exists a unique μ^* such that $\mu^* = T(\mu^*)$
2. any initial distribution μ^0 converges to μ^*, i.e.,

$$\forall \mu^0, \quad \lim_{n \to \infty} T^n(\mu^0) = \mu^*$$

Solving the Aiyagari model

Solving the Household Problem

$$V(s,k) = \max_{c,k'} \left\{ u(c) + \beta \sum_{s'} \Omega(s,s') V\left(s',k'\right) \right\}$$

subject to

$$k' + c = sw + Rk$$
$$c \geq 0 \quad \text{and} \quad k' \geq \underline{k}$$

The Euler equation is:

$$u_c(c) = \beta R \sum_{s'} \Omega(s,s') u_c(c')$$

Given prices, we can solve this model by using the value function, or the first-order condition.

Risk Transition

We need to parametrize the transition matrix on the discrete grid $\Omega(s, s')$.

- We estimate risk in continuous form: $z_{ih} = \rho z_{ih-1} + \epsilon_{ih}$.
- What is called the *Tauchen Algorithm* does exactly that.
- Idea: Use a first-order Markov chain to approximate the continuous $AR(1)$ process.

Markov Chains

- A Markov-chain is characterized by a discrete grid s_i , $i = 1 : N$ and a transition probability matrix Ω giving the probability to move from point i to j, p_{ij}.

- Hence, $S_t = \Omega S_{t-1}$ gives the probability distribution over states in recursive form.

- The ergodic distribution of a Markov-chain occurs when it runs to infinity: $(I - \Omega)\pi = 0$.

- An ergodic distribution exists iff being at any point i does not exclude me ending up in any j somewhere in the future.

Markov Approximation of $AR(1)$

Consider the generalized $AR(1)$ process:

$$z_{ih} = (1-\rho)\mu + \rho z_{ih-1} + \epsilon_{ih} \quad \epsilon_{ih} \sim N(0, \sigma^2)$$

- The process has a mean μ.
- We impose normality for the shock distribution!
- Ergodic distribution is $N(\mu, \sigma^2_{AR})$ with $\sigma^2_{AR} = \frac{\sigma^2}{1-\rho^2}$.

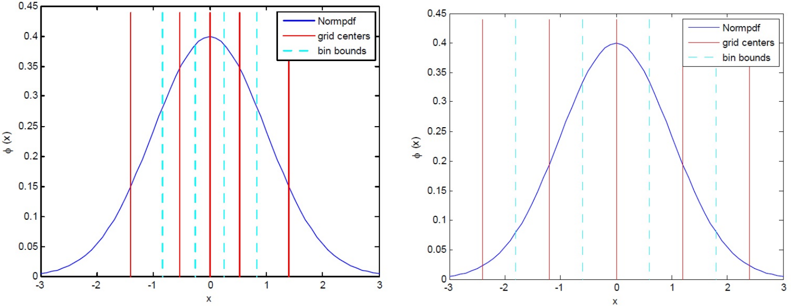
Tauchen (1986) Algorithm

- Idea: Partition ergodic distribution in N bins and choose points in bins *representing* those bins.
- Two natural ways to choose:

  Equilikely: Choose N bins such that each is equally likely.

  Equidistant: Choose N equally distant points.
- In most cases, first option provides better approximation.
- If interested in tail behavior, second option may be preferable.

Graphical Representation



Create Bins

Choose boundaries, b_i , of bins, S_i , according to:

$$\Omega(s \in S_i) = \Phi\left(\frac{b_{i+1} - \mu}{\sigma_{AR}}\right) - \Phi\left(\frac{b_i - \mu}{\sigma_{AR}}\right) = \frac{1}{N}.$$

Hence,

$$\Phi\left(\frac{b_{i+1} - \mu}{\sigma_{AR}}\right) = \frac{i}{N}.$$

or

$$b_{i+1} = \sigma_{AR}\Phi^{-1}\left(\frac{i}{N}\right) + \mu.$$

Centers of Bins

Next is to choose a representative element, s_i , for each bin:

$$s_i = (s | s \in S_i).$$

One can show that with a normal distribution this is:

$$s_i = N\sigma_{AR}\left[\phi\left(\frac{b_i - \mu}{\sigma_{AR}}\right) - \phi\left(\frac{b_{i+1} - \mu}{\sigma_{AR}}\right)\right] + \mu.$$

Transition Probabilities

We need to know the transition matrix. E.g., what is the probability for $s \in S_i$ to move to $s' \in S_j$?

We need

$$b_j \leq \rho s + (1-\rho)\mu + \epsilon$$
$$b_{j+1} \geq \rho s + (1-\rho)\mu + \epsilon$$

Thus

$$\epsilon \in [b_j - \rho s - (1-\rho)\mu, b_{j+1} - \rho s - (1-\rho)\mu].$$

Simplified Tauchen Algorithm

$$p_{i,j} = \Omega(s' \in S_j | s \in S_i) = \Phi\left(\frac{b_{j+1} - \rho s_i - (1-\rho)\mu}{\sigma}\right) - \Phi\left(\frac{b_j - \rho s_i - (1-\rho)\mu}{\sigma}\right).$$

- There is a more accurate formulation where all points in S_i are taken into account, not only s_i.
- This requires integrating over the relevant part of the distribution and weighting by the probability of each occurrence.

Value Function Iteration for Household Problem

- Construct a grid for capital $k_i = \{k_1, k_2, ...k_{N_k}\}$.
- Construct a grid for earnings $s_j = \{s_1, s_2, ...s_{N_s}\}$ and corresponding transition matrix Ω.

1. Guess a continuous/increasing value function $V^0(k_i, s_j)$ of dimension $N_k \ X \ N_s$.
2. Solve $V^n(k,z) = \max_{c,k'} \Big\{ u(c) + \beta\Omega(s,s')V^{n-1}(k',z') \Big\}$.
3. Replace last iteration guess by new solution $V^{n-1} = V^n$.
4. Iterate until $|V^n - V^{n-1}| < crit$.

Compute the Distribution μ

Goal is to find μ^* that solves: $\mu^* = T(\mu^*)$. Two possibilities:

1. Monte Carlo simulation of N households for P (large) periods:

   Provide each household with initial asset k_0 and productivity s_0.

   Compute next periods productivity through law of motion and next periods capital through $g^k(k, s)$.

2. More accurate, iterate on a probability measure (distribution function iteration).

   Initialize μ_0 on the grid N_k X N_s.

   Compute the operator T.

      For productivity using Ω.

      For capital using $g^k(k, s)$.

   Apply the operator until $|\mu^n - \mu^{n-1}| < crit$.

Equilibrium Prices

- So far, we have found μ for assumed prices r, w.
- This implies $K = \int_{\mathbf{B}} g^k(k, s) d\mu$ a supply of capital.
- Nothing assures that this supply matches demand:

$$F_K(K, L) = R - 1 + \delta, \; F_L(K, L) = w$$

.

- With CRS production, we only need to worry about correct interest rate:

$$w = F_L\left(F_K^{-1}(r + \delta)\right).$$

- We know an open interval in which equilibrium must lye:

$$r^* \in \left(-\delta, \frac{1}{\beta} - 1\right).$$

Algorithm to Find Equilibrium

1. Guess a feasible r.
2. Compute implied wages.
3. Solve the household problem.
4. Solve for the stationary distribution μ^*.
5. Compute implied interest rate $r^{new} = F_K - \delta$.
6. Stop if $|r^{new} - r| < crit$.
7. Update interest rate $r = (1 - \lambda)r + \lambda r^{new}$.

Comparison to the RBC model

Key Insights from Aiyagary Model

- Households save too much.
- The capital stock is higher than in social planner solution.
- Output is higher than in social planner solution.

Key Insights from Aiyagary Model II

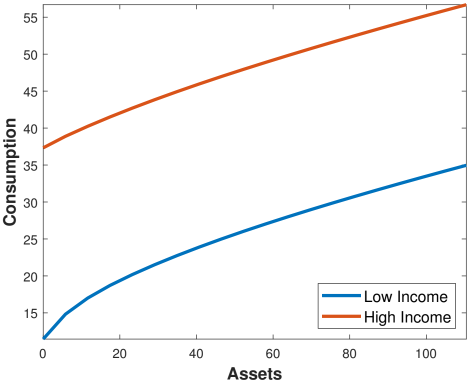
Optimal consumption is given by

$$\begin{cases} u_c(c_t) = \beta R \sum_{s'} \Omega(s,s') u_c(c_{t+1}) & \text{if } k_{t+1} > \bar{k} \\ c_t = R a_t + s_t - \bar{k} & \text{else.} \end{cases}$$

- Constrained household: $u_c(c_t) > \beta R \sum_{s'} \Omega(s,s') u_c(c_{t+1})$.
- MPC of constrained household is one.
- $\frac{\Delta c_t}{\Delta a_t}$ is high close to the constraint.
- With mean reversion, particularly true for temporarily unproductive.

Key Insights from Aiyagary Model II

Figure: Policy functions



Life-Cycle model

So Far

We have studied economies of the form

$$\max_{\mathbf{y}_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(\mathbf{y}_t, \mathbf{x}_t)$$

$$\mathbf{y}_t = f(\mathbf{x}_t, \mathbf{x}_{t+1})$$

- We have seen conditions when we could write it recursively and value function is stationary:

$$V(\mathbf{x}) = \max_{\mathbf{y}} \{u(\mathbf{y}, \mathbf{x}) + \beta \mathbb{E} V(\mathbf{x}')\}$$

- Solution is the stationary value, $V(\mathbf{x})$, and policy function, $\mathbf{x}' = \phi(\mathbf{x})$.

Life-Cycle

We may also be interested in problems of the form

$$\max_{\mathbf{y}_t} \mathbb{E}_0 \sum_{t=0}^{T} \beta^t u(\mathbf{y}_t, \mathbf{x}_t)$$
$$\mathbf{y}_t = f(\mathbf{x}_t, \mathbf{x}_{t+1})$$

- Wealth accumulation over the life-cycle.
- Longevity risk and bequests.
- Earnings inequality over the life-cycle.
- Age and labor market risk.
- ...

Stochastic Death

We may formulate the problem with stochastic death (aging)

$$\max_{\mathbf{y}_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1-\xi) u(\mathbf{y}_t, \mathbf{x}_t)$$

$$\mathbf{y}_t = f(\mathbf{x}_t, \mathbf{x}_{t+1})$$

- Agent dies with probability ξ receiving utility 0 from there onwards.
- Problem still stationary, **x** contains now state of living.
- We can simulate the resulting agents from period 0, and call this age.

Back to Finite Life

- We may also solve the problem with finite life.
- Value function, and policy function depend now on t.

  Have to solve for T value and policy functions.
- Importantly, with standard assumptions, the problem is still recursive!

Recursive Formulation

$$V(\mathbf{x}, t) = \max_{\mathbf{y}} \{u(\mathbf{y}, \mathbf{x}) + \beta \mathbb{E}_t V(\mathbf{x}', t+1)\}$$

$$V_t(\mathbf{x}) = \max_{\mathbf{y}} \{u(\mathbf{y}, \mathbf{x}) + \beta \mathbb{E}_t V_{t+1}(\mathbf{x}')\}$$

- This is a sequence of static optimization problems:

$$V_T = \max_{\mathbf{y}} \{u(\mathbf{y}, \mathbf{x})\}$$

$$V_{T-1} = \max_{\mathbf{y}} \{u(\mathbf{y}, \mathbf{x}) + \beta \mathbb{E}_{T-1} V_T(\mathbf{x}')\}$$

- Assume a decision period is 1 year and agent lives 90 years. Need to solve the maximization problem 90 times.
- This may well be quicker than solving for a stationary policy rule with infinite life.

Example: Kaplan and Violante (2010)

- Individuals work for T_1 periods and live in retirement for T_2 periods.
- When working, earnings follow Markov process.
- When retired, earnings are fixed, $\bar{E}$.
- They discount the future with rate β.
- Self insurance with asset paying $R = 1 + r$.
- Two reasons for asset accumulation:

  Self-insurance.

  Consumption in old age.

A Simplified Model

Assume log earnings have a deterministic age-varying component and a stochastic component:

$$e_{ih} = H_h + u_{ih}$$
$$u_{ih} = \rho u_{ih-1} + \epsilon_{ih}.$$

We discretize u_{ih} using the Tauchen method yielding a transition matrix $\Omega(s, s')$.

In the US, retirement earnings depend on average life-time earnings. To avoid an additional state, let us simplify the problem and assume it depends only on the last realization of s:

$$SS_i = SS(s_{T_1})$$

Appendix

Comparison to the RBC Model

Consider the social planner solution to a simplefied RBC model:

$$V(z,K) = \max_{C,K'} \left\{ \ln(C) + \beta \sum_{z'} \Omega(z,z') V\left(z',K'\right) \right\}$$

subject to

$$K' + c = Y + (1-\delta)K$$
$$Y = \exp(z) K^{\alpha}$$

Comparison to the RBC Model

- In the RBC model, we also solve the value function for a state space, K, z. However, these are points in the state space where the representative household can be at any point in time.
- Hence, there is an ergodic distribution of the representative household over the state space in time. In the Aiyagary model, there is a distribution across different households.
- The representative household faces no borrowing constraint. Moreover, because productivity shocks are small, the household would never run into the borrowing constraint.
- We know the equilibrium real interest rate $r = MPK - \delta$ at each grid point. In the Aiyagary model, we have to solve for the real interest rate.

Solving the RBC model

1. Discretize a grid for the state K and z.
2. Guess the (continuous and concave) value function $V^0(K, z)$.
3. Solve $V^n(K, z) = \max_{C,K'} \left\{ ln(C) + \beta \mathbb{E} V^{n-1}(K', z') \right\}$.
4. Replace last iteration guess by new solution $V^{n-1} = V^n$.
5. Iterate until $|V^n - V^{n-1}| < crit$.
6. Given the optimal policy, simulate the economy for T periods by drawing a random sequence of shocks.

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