Solow: It is all about physical capital accumulation

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Crecimiento Económico / Growth Theory

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Modern economic growth

- We have seen that over the last 150 years, economic growth in advanced economies can be described by exponential growth.
- To understand this phenomenon, as well as the large cross-sectional differences in GDP per capita, we need a theory.
- Our theory is guided by certain data facts.
- These data facts should be "universally" true, i.e., relatively stable over time.

Kaldor facts

Kaldor (1961) summarized six facts about income. We will consider the first five:

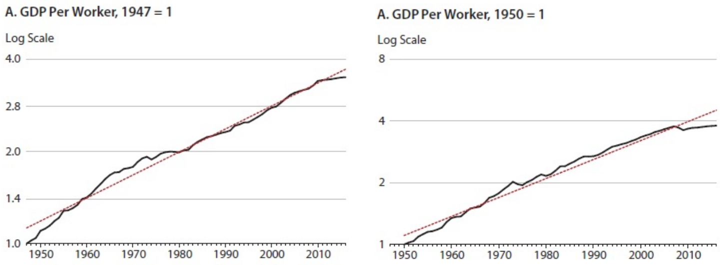
1. Labor productivity grows at a constant rate over time.
2. Capital per worker grows at a constant rate over time.
3. Capital has a constant rate of return over time.
4. The capital to output ratio is constant over time.
5. The share of income going to capital is constant over time.

Kaldor facts II

Recently, Herrendorf et al. (2019) consider these facts anew including recent data:

1. Broadly speaking, the Kaldor facts still hold.
2. However, some data moments show some time variation.
3. In this course, we will, nevertheless, use the Kaldor facts as the benchmark.

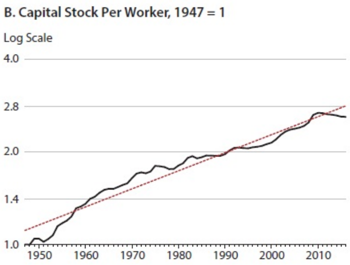
Constant growth in labor productivity



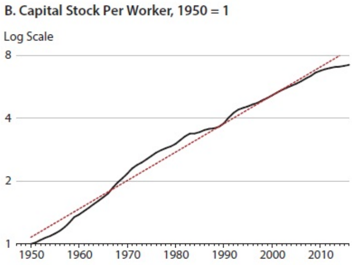
U.S. UK

- A constant labor productivity growth is a good approximation.
- However, we observe a slow-down after 1970.

Constant growth in capital per worker



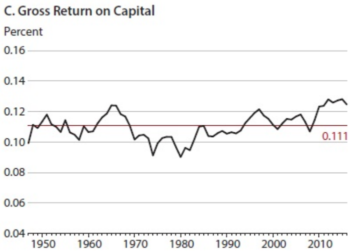
U.S.



UK

- A constant capital per worker growth is a good approximation.
- However, we observe a slow-down after 1970.

Constant return to capital



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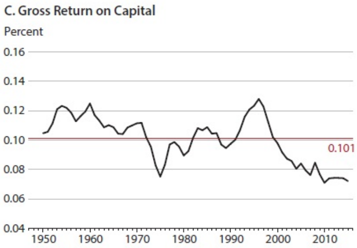
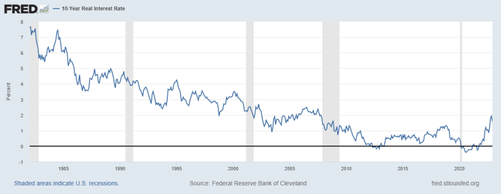


UK

- A constant return to capital is a good approximation.
- Since the 2000s, we observe some time variation that is different across countries.

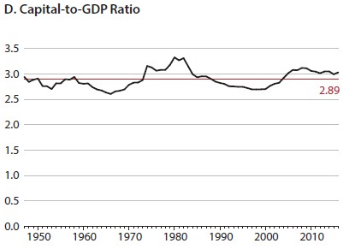
Constant return to capital: a caution



U.S.

- You sometimes read that real interest rates are falling over time.
- This claim is based on government bond returns and inflation data, not equity data.
- What seems to be going on is that the risk premium is rising over time.

A constant capital to output ratio



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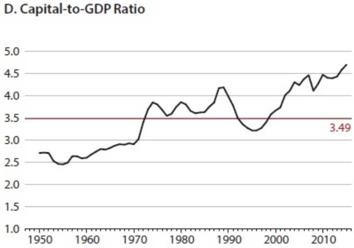
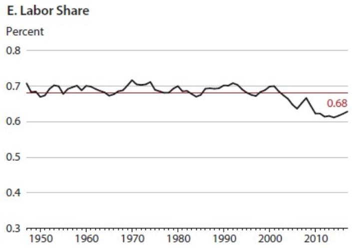


UK

- In the U.S., capital and output grow approximately at the same rate.
- In the UK, capital grows faster than output.

A constant capital share in income



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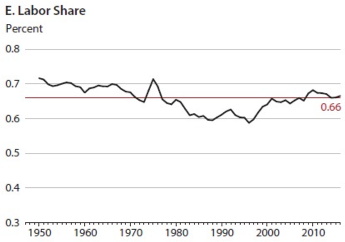


UK

- Instead of the capital share, they consider the easier to measure labor share.
- The labor share started to fall in the U.S. during the 2000s. It was falling in the UK between the 50s and 2000s.

Understanding modern economic growth

A first attempt to understand modern economic growth

- Solow (1956) presents a framework on how to understand the phenomenon of modern economic growth (Kaldor factors). For that work, he won the Nobel price.
- It is a closed economy model where production takes place by labor, capital, and technology.
- Importantly, it takes technological growth as exogenous and puts physical capital accumulation center stage.

How production takes place

The real world

- In a modern economy, production takes place mostly at firms which are often multinational corporations.

- These firms produce thousands of different goods and services relying on thousands on imports and creating thousands of exports.

- For production, they employ
  - labor of different types (education, age, sex...)
  - equipment, structures, roads, land, raw materials...

- A lot of production is also done by the government.

- Most exchange of goods and services as well as factor inputs is conducted in thousands of markets.

- People make decisions about consumption today versus the future.

Abstractions

The world is quite complicated (more so than medieval England) and we will have to make simplifications to make progress in understanding it:

- We assume a closed economy.
- We abstract from the government and treat it just as the private sector.
- There is only one output good.
- There will be a single aggregate production function combining factors of production into the output good.
- We assume that we can ignore some factor inputs and aggregate others together.

Abstractions: capital and labor

To focus on the right factor inputs, we look at data from national accounts:

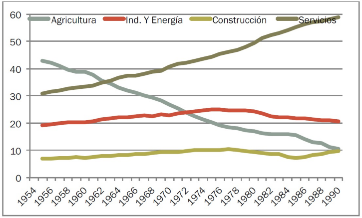
- As we have seen before, labor compensation is around 2/3 of national income, i.e., it is important. To deal with it, we will assume that we can aggregate all different labor inputs into just one input.
- Similarly, we assume that we can aggregate all physical capital inputs into just one.
- This implies we will treat land as physical capital. One may object that land is finite. However, fertilizers and tall buildings suggest otherwise.
- Even for the U.S., a major oil producer, income from natural resources is relatively small. Hence, we will ignore them. One can also interpret them as physical capital recognizing that, so far, their supply did not run out.

Abstractions: technology

We also assume a single measure on how well we use capital and labor to produce output, i.e., technology.

- This may relate to firm organization, e.g., management style.
- This may relate to logistics, e.g., just-in-time delivery.
- This may relate to new products that are better than the old product but not more expensive to produce, e.g., a faster computer algorithm.
- In fact, many of the products we consume today did not exist 50 years ago.
- Hence, we will think of improvements in A as new *ideas*, or better recipes.

More products or new products?



The type of products we produce today looks very different from the type of products we produced in 1955.

Abstractions: markets

To understand markets, we again look at national accounts:

- The profit share of national income is relatively small, around 5%.
- This suggests that product markets and input markets are close to perfectly competitive.

This has two important implications:

- All output is redistributed to households in form of factor payments:

$$Y(t) = r(t)K(t) + w(t)L(t). \tag{1}$$

- The factors of production earn their marginal products.

Abstractions: The aggregate production function

Our assumptions imply that firms operate an aggregate production function that combines a single labor input, L, a single capital input, K, and some technology level, A, into a single output good: $Y = F(K, L, A)$. Our model should be consistent with the Kaldor facts. We now use the fact of constant income shares to figure out how F should look like:

$$\frac{r(t)K(t)}{Y(t)} = \alpha, \tag{2}$$

$$\frac{w(t)L(t)}{Y(t)} = 1 - \alpha. \tag{3}$$

Given the assumption of competitive markets, we have:

$$\frac{\frac{\partial Y(t)}{\partial K(t)}K(t)}{Y(t)} = \alpha, \tag{4}$$

$$\frac{\frac{\partial Y(t)}{\partial L(t)}L(t)}{Y(t)} = 1 - \alpha. \tag{5}$$

Abstractions: The aggregate production function II

This holds for, among others, the Cobb-Douglas production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} \tag{6}$$

Note, the place of $A(t)$ in the production function is not particularly important:

$$Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} = A(t)^{1-\alpha} K(t)^{\alpha} L(t)^{1-\alpha} = E(t)K(t)^{\alpha} L(t)^{1-\alpha}, \tag{7}$$

with $E(t) = A(t)^{1-\alpha}$. The way I have written it above will make the math easier.

Abstractions: Household decisions

- One of the most important questions in modern macroeconomics is how households trade-off consumption today against tomorrow.
- The Solow model abstracts from this and assumes that households save a constant fraction of their income each period.
- As aggregate income equals aggregate production, aggregate savings are $S(t) = sY(t)$.

Solving the model

Capital accumulation

The Solow model assumes that every period a fraction δ of the capital stock depreciates. Working against this, households invest $I(t) = S(t)$:

$$\dot{K}(t) = S(t) - \delta K(t) \tag{8}$$

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{9}$$

$$\dot{K}(t) = sK(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} - \delta K(t). \tag{10}$$

Dynamics of population and technology

The Solow model assumes that the population and technology grow at exogenous rates. To be consistent with the Kaldor facts on labor productivity, it assumes they grow exponentially:

$$L(t) = L(0)\exp(nt) \Rightarrow \frac{\dot{L}(t)}{L(t)} = n \tag{11}$$

$$A(t) = A(0)\exp(gt) \Rightarrow \frac{\dot{A}(t)}{A(t)} = g. \tag{12}$$

Steady state

As before, we will start our analysis with the steady state of the model. For this, we need to find a variable that has a steady state. It turns out, in the Solow model, these are output and capital per efficient worker:

$$\tilde{k}(t) = \frac{K(t)}{A(t)L(t)} \tag{13}$$

$$\tilde{y}(t) = \frac{Y(t)}{A(t)L(t)} = \tilde{k}(t)^{\alpha}. \tag{14}$$

Rewriting the capital accumulation equation

$$\dot{K}(t) = sK(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha} - \delta K(t) \tag{15}$$

$$\frac{\dot{K}(t)}{K(t)} = s\tilde{k}(t)^{\alpha-1} - \delta \tag{16}$$

We need to rewrite the left-hand-side in terms of efficient workers. For this, we need to find an expression for $\frac{\dot{K}(t)}{K(t)}$.

Rewriting the capital accumulation equation II

Given the definition:

$$\tilde{k}(t) = \frac{K(t)}{A(t)L(t)} \tag{17}$$

$$\ln \tilde{k} = \ln K(t) - \ln A(t) - \ln L(t). \tag{18}$$

Now take the derivative with respect to time and use the fact that the derivative of a variable in logs with respect to time is the growth rate of that variable:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{19}$$

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - g - n \tag{20}$$

Rewriting the capital accumulation equation III

Combining the equations yields:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} + n + g = s\tilde{k}(t)^{\alpha-1} - \delta \tag{21}$$

$$\dot{\tilde{k}}(t) = s\tilde{k}(t)^{\alpha} - (n + g + \delta)\tilde{k}(t). \tag{22}$$

Capital per efficient worker grows over time because of savings per efficient worker, $s\tilde{k}(t)^{\alpha}$. It shrinks because of population growth, technological progress, and capital depreciation.

Solving for the steady state

Conjecture that in steady state, $\dot{\tilde{k}}(t) = 0$:

$$0 = s(\tilde{k}^*)^\alpha - (n + g + \delta)\tilde{k}^* \tag{23}$$

$$\tilde{k}^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{1}{1-\alpha}}. \tag{24}$$

Note, we have found indeed a steady state. Our variable, $\tilde{k}^*$, depends only on time-invariant parameters. The steady state capital per efficient worker increases in the savings rate and decreases in the population growth rate, technological progress, and capital depreciation rate.

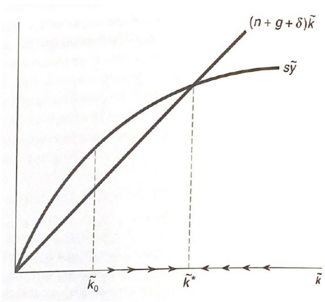
Solving for the steady state II

Once we know $\tilde{k}^*$, it is straight forward to compute the other endogenous variables in steady state:

$$\tilde{y}^* = \frac{K(t)^\alpha \left(A(t)L(t)\right)^{1-\alpha}}{A(t)L(t)} = (\tilde{k}^*)^\alpha \tag{25}$$

$$\tilde{c}^* = (1-s)\tilde{y}^* = (1-s)(\tilde{k}^*)^\alpha. \tag{26}$$

The steady state graphically



- Note, there exist one steady state with $\tilde{k}^* > 0$.
- Key for this is that $s(\tilde{k}^*)^\alpha = s\tilde{y}^*$ is concave. For this, we require diminishing marginal returns to capital.

Output per capita in steady state

$$\tilde{y}^* = \left(\frac{s}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha}} \tag{27}$$

$$\left(\frac{Y(t)}{L(t)} \right)^* = A(t) \left(\frac{s}{n+g+\delta} \right)^{\frac{\alpha}{1-\alpha}} \tag{28}$$

Different from the Malthus model, the Solow model can explain long-run differences in income per capita:

- A higher technology level increases output per capita.
- A higher savings rate increases output per capita.
- A higher population growth rate or capital depreciation rate decreases output per capita.

Factor payments in steady state

The rental price of capital is given by

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha K(t)^{\alpha-1} (A(t)L(t))^{1-\alpha} = \alpha \tilde{k}(t)^{\alpha-1}, \quad (29)$$

which is a constant in the long run. Hence, the Solow model is consistent with the Kaldor fact on constant returns to capital.

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)K(t)^{\alpha} A(t) (A(t)L(t))^{-\alpha} = (1-\alpha)A(t)\tilde{k}(t)^{\alpha}, \quad (30)$$

which is growing with technology. Kaldor did not study wages but this fact is also born out by the data.

Growth in steady state

In steady state, by definition, capital (and output) per efficient worker, $\tilde{k}(t)$, is constant. This does not mean, however, that capital or capital per capita, $k(t) = \frac{K(t)}{L(t)}$, are constant. In fact, we already know that:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{31}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^{*} = n + g \tag{32}$$

$$\left(\frac{\dot{k}(t)}{k(t)}\right)^{*} = g. \tag{33}$$

That is, in steady state, capital per capita grows at the rate of technological progress. A constant growth rate of capital per capita is one of the Kaldor facts.

Growth in steady state II

Similarly for output,

$$\frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} = \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{34}$$

$$\left(\frac{\dot{Y}(t)}{Y(t)}\right)^* = n + g \tag{35}$$

$$\left(\frac{\dot{y}(t)}{y(t)}\right)^* = g. \tag{36}$$

Hence, output per capita in steady state also growth at the rate of technological progress. Moreover, as Y and K both grow at rate $n + g$, there ratio is constant which completes the Kaldor facts.

Growth in steady state III

Finally, for consumption,

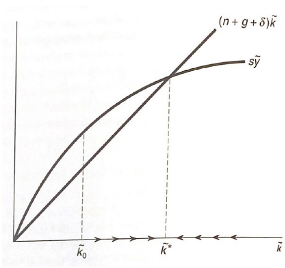
$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{\dot{C}(t)}{C(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{37}$$

$$\left(\frac{\dot{C}(t)}{C(t)}\right)^* = n + g \tag{38}$$

$$\left(\frac{\dot{c}(t)}{c(t)}\right)^* = g. \tag{39}$$

Hence, consumption per capita in steady state also growth at the rate of technological progress. A steady state in which all endogenous variables grow at the same rate is referred to as a *balanced growth path*.

Convergence to steady state



Recall the capital per efficient worker accumulation equation:

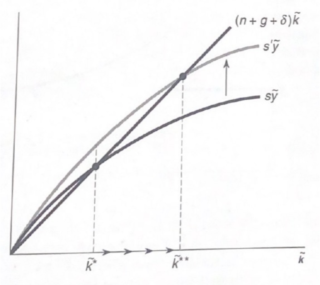
$$\dot{\tilde{k}}(t) = s\tilde{k}(t)^{\alpha} - (n+g+\delta)\tilde{k}(t). \tag{40}$$

$$s\tilde{k}(t)^{\alpha} > (n+g+\delta)\tilde{k}(t) \text{ if } \tilde{k}(t) < \tilde{k}^* \tag{41}$$

$$s\tilde{k}(t)^{\alpha} < (n+g+\delta)\tilde{k}(t) \text{ if } \tilde{k}(t) > \tilde{k}^*. \tag{42}$$

Hence, we converge to steady state from any starting point $\tilde{k}(0) > 0$.

Comparative statics: An increase in the savings rate



Consider an increase in the savings rate. For any level of $\tilde{k}(t)$, $s\tilde{y}(t) = s\tilde{k}(t)^{\alpha}$ increases. The new steady state is associated with a higher $\tilde{k}^*$ and, hence, a higher $\tilde{y}^*$. It directly follows that output per capita is also higher in the new steady state.

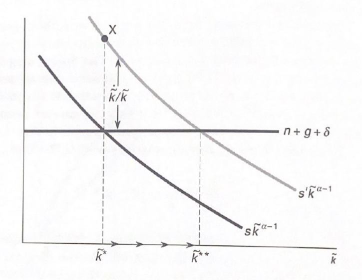
Comparative statics: An increase in the savings rate II

Note, in the old and new steady state, $\tilde{y}^*$ are constant and, hence, output per capita grows in each case at rate g. That is, the savings rate changes the level of output per capita in steady state but not its growth rate. However, on the transition path to the new steady state, $\tilde{y}$ is not constant. To see this, consider again the capital accumulation equation

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = s\tilde{k}(t)^{\alpha-1} - (n + g + \delta). \tag{43}$$

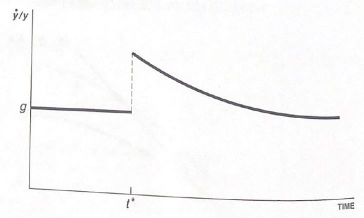
During the transition, $s\tilde{k}(t)^{\alpha-1} > n + g + \delta$ and, hence, capital per efficient worker grows.

Comparative statics: An increase in the savings rate III



Note, $\tilde{k}(t)^{\alpha-1}$ is a downward-sloping convex function. Hence, the distance between $s\tilde{k}(t)^{\alpha-1}$ and $(n+g+\delta)$ is largest in the first period of the adjustment. This is a result from the diminishing marginal returns to capital.

Comparative statics: An increase in the savings rate IV

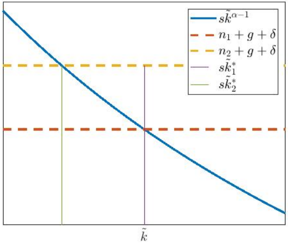


As the economy accumulates capital rapidly initially, output per capita is growing rapidly initially:

$$\frac{\dot{y}(t)}{y(t)} = g + \frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} = g + \alpha \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)}. \quad (44)$$

As capital accumulation slows down, so does output per capita growth.

Comparative statics: An increase in the population growth rate



The new steady state is associated with a lower $\tilde{k}^*$ and, hence, a lower $\tilde{y}^*$. It directly follows that output per capita is also lower in the new steady state.

Comparative statics: An increase in the population growth rate II

Consider again the capital accumulation equation

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = s\tilde{k}(t)^{\alpha-1} - (n + g + \delta). \tag{45}$$

During the transition, $s\tilde{k}(t)^{\alpha-1} < n + g + \delta$ and, hence, capital per efficient worker shrinks. As a consequence, output per capita is growing at a rate less than g during the transition phase.

The empirics of the Solow model

Results and predictions

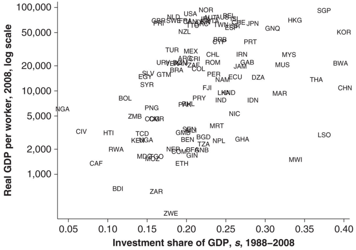
We have seen that the Solow model is consistent with all Kaldor facts. Beyond that, it makes some key predictions about the level of output per worker across countries:

$$\left(\frac{Y(t)}{L(t)}\right)^{*} = A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{46}$$

- Output per capita is increasing in the savings rate: $s = \frac{I}{Y}$.
- Output per capita is decreasing in the population growth rate.
- Output per capita is increasing in productivity (TFP).

GDP per capita and the savings rate

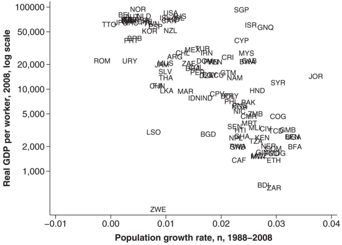
FIGURE 2.6 **GDP PER WORKER VERSUS THE INVESTMENT RATE**



Indeed, we find a positive correlation between the investment rate and GDP per capita.

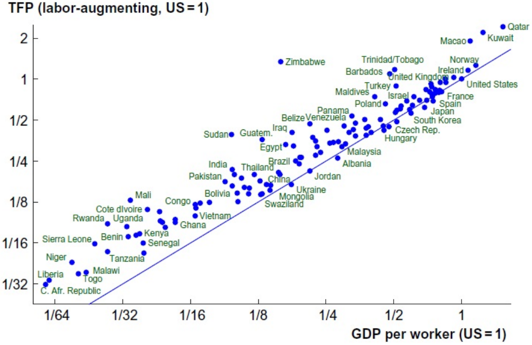
GDP per capita and the population growth rate

FIGURE 2.7 GDP PER WORKER VERSUS POPULATION GROWTH RATES



Indeed, we find a negative correlation between the population growth rate and GDP per capita.

GDP per capita and productivity



Indeed, we find a positive correlation between productivity and GDP per capita. Note, poor countries are "too productive", i.e., have "too little" capital.

A quantitative assessment

Instead of just looking qualitatively at the data, Mankiw et al. (1992) evaluate the quantitative performance of the model. Assuming that all countries are in steady state, they start from:

$$y(t) = A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{47}$$

$$y(t) = A(0)\exp(gt)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{48}$$

$$\ln y(t) = \ln A(0) + gt + \frac{\alpha}{1-\alpha}s - \frac{\alpha}{1-\alpha}(n+g+\delta). \tag{49}$$

Assuming that $\ln A(0) + gt = \beta_0 + \epsilon$, i.e., the level of technology is random across countries, and $g+\delta = 0.05$ this can be estimated by linear OLS:

$$\ln y(t) = \beta_0 + \beta_1 s - \beta_2(n+0.05) + \epsilon(t). \tag{50}$$

A quantitative assessment II

TABLE I
ESTIMATION OF THE TEXTBOOK SOLOW MODEL

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	5.48	5.36	7.97
	(1.59)	(1.55)	(2.48)
ln(I/GDP)	1.42	1.31	0.50
	(0.14)	(0.17)	(0.43)
$\ln(n+g+\delta)$	−1.97	−2.01	−0.76
	(0.56)	(0.53)	(0.84)
$\bar{R}^2$	0.59	0.59	0.01

- Savings rates and population growth explain almost 60% in cross-country variation in GDP per capita.
- The coefficients have the expected signs, and we cannot reject $\beta_1 = -\beta_2$.
- However, the implied α from the regressions is too high (0.59).

How much should we save

So far, we take the savings rate as given. One may ask whether there is an optimal savings rate. One possibility to define optimal is the savings rate that maximizes long-run consumption per worker:

$$\left(\frac{C(t)}{L(t)}\right)^* = (1-s)\left(\frac{Y(t)}{L(t)}\right)^* = (1-s)A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{51}$$

As productivity is exogenous, this is equivalent to maximize consumption per efficient worker:

$$\tilde{c}^* = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}. \tag{52}$$

The resulting savings rate is referred to as the golden rule s_{Gold}.

The golden rule

Taking the first order condition of (52) yields

$$s_{Gold} = \alpha. \tag{53}$$

Intuition: The more important is capital in the production function the more we should save.

The economics behind the golden rule

In steady state, an alternative to write the problem is:

$$\tilde{c}^* = \tilde{y}^* - (n + g + \delta)\tilde{k}^*. \tag{54}$$

Now take the derivative with respect to the steady state capital stock per efficient worker:

$$\alpha(\tilde{k}^*)^{\alpha-1} = n + g + \delta \tag{55}$$

$$MPK = n + g + \delta. \tag{56}$$

The marginal gain of savings need to equal its marginal cost (the effective depreciation rate).

Are we saving enough?

To assess whether Spain has a savings rate consistent with the golden rule, consider the following data facts

1. The capital output ratio is 2.75: $k = 2.75y$.
2. Capital depreciation is 10 percent of yearly output: $\delta k = 0.1y$.
3. The capital share of income is 30%
4. Output growth is 3%.

Are we saving enough? II

Combining 1 and 2 tells us that the depreciation rate is 3.6%.

According to our model, 3 implies $MPK^*k = 0.3y$. Combining with 1 we have $MPK = 0.11$.

According to our model, 4 implies $n + g = 0.03$.

Hence, $MPK > n + g + \delta$, i.e., we save to little.

Can we rationalize a high MPK

We can draw two possible conclusions from a high MPK:

1. Either we need to change savings incentives. Reforming the pension system is one aspect economists have advocated.
2. Or optimizing long-run consumption per worker is not optimal. If we discount future consumption relative to today's consumption, the golden rule is not optimal. A yearly discount rate of over 4% would be needed to explain the high capital returns.

Introducing human capital

Why human capital

So far, we assume all workers are equally productive across time and countries. However,

- the number of average years of schooling varies substantially within a country over time and across countries.
- the quality of schooling varies substantially within a country over time and across countries.
- within a country at a point in time, income differences across education groups are large suggesting that education matters for productivity.

Production

To introduce human capital, we make a small change to the production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \quad (57)$$

$$H(t) = \exp(\psi\, u)L(t), \quad (58)$$

where $L(t)$ is the amount of labor, and $H(t)$ is the amount of total human capital. Total human capital not only depends on the amount of labor but also in the time invested in learning, u.

Note that

$$\frac{\partial H(t)}{\partial u} = \psi \exp(\psi u)L(t) = \psi H(t) \tag{59}$$

$$\frac{\partial \ln H(t)}{\partial u} = \psi. \tag{60}$$

That is, a change in u translates into ψ percent more human capital. One way to interpret ψ is to think about the quality of the education system for a fixed time spend in it.

Steady state

We need to define again a variable that has a steady state. We will define again:

$$\tilde{k}(t) = \frac{K(t)}{A(t)L(t)} \tag{61}$$

$$\tilde{y}(t) = \frac{Y(t)}{A(t)L(t)} = \tilde{k}(t)^{\alpha} \left(\exp(\psi u)\right)^{1-\alpha} \tag{62}$$

Rewriting the capital accumulation equation

$$\dot{K}(t) = sK(t)^{\alpha}\left(A(t)H(t)\right)^{1-\alpha} - \delta K(t) \tag{63}$$

$$\frac{\dot{K}(t)}{K(t)} = s\tilde{k}(t)^{\alpha-1}\left(\exp(\psi u)\right)^{1-\alpha} - \delta \tag{64}$$

We need to find again an expression for $\frac{\dot{K}(t)}{K(t)}$.

Rewriting the capital accumulation equation II

Note that

$$\tilde{k} = \frac{K(t)}{A(t)L(t)} \tag{65}$$

$$\ln \tilde{k} = \ln K(t) - \ln A(t) - \ln L(t) \tag{66}$$

Now take the derivative with respect to time:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{67}$$

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - g - n \tag{68}$$

Rewriting the capital accumulation equation III

Combining the equations yields:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} + n + g = s\tilde{k}(t)^{\alpha-1} \left(\exp(\psi u)\right)^{1-\alpha} - \delta \tag{69}$$

$$\dot{\tilde{k}}(t) = s\tilde{k}(t)^{\alpha} \left(\exp(\psi u)\right)^{1-\alpha} - (n + g + \delta)\tilde{k}(t). \tag{70}$$

Note, this is almost the same dynamic system as in the model without human capital. The only difference is the additional education term.

Solving for the steady state

Solving for the steady state:

$$\tilde{k}^* = \left(\frac{s}{n + g + \delta} \right)^{\frac{1}{1-\alpha}} \exp(\psi u) \quad (71)$$

$$\tilde{y}^* = \tilde{k}(t)^{\alpha} \left(\exp(\psi u) \right)^{1-\alpha} = \left(\frac{s}{n + g + \delta} \right)^{\frac{\alpha}{1-\alpha}} \exp(\psi u) \quad (72)$$

$$\tilde{c}^* = (1 - s)\tilde{y}^* = (1 - s) \left(\frac{s}{n + g + \delta} \right)^{\frac{\alpha}{1-\alpha}} \exp(\psi u). \quad (73)$$

Output per capita in steady state

Hence, output per worker in steady state is:

$$\left(\frac{Y(t)}{L(t)}\right)^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi\, u) \tag{74}$$

Output per capita is increasing in the amount of education. A more educated workforce is more productive and, thereby, allows each worker to produce more.

Growth in steady state

We can ask again about the growth rate in steady state. As education is assumed to be constant, nothing really changes:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{75}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^{*} = n + g \tag{76}$$

$$\left(\frac{\dot{k}(t)}{k(t)}\right)^{*} = g. \tag{77}$$

That is, capital per capita (and output/consumption per capita) grows at the rate of technological progress.

Transition dynamics

We have already seen that changes in population growth rates and saving rates can have rich transition dynamics for output per worker. We can now also analyze changes in education. In general, output per worker is:

$$y(t) = \frac{Y(t)}{L(t)} = \tilde{k}(t)^{\alpha} A(t) \left(\exp(\psi\, u)\right)^{1-\alpha} \tag{78}$$

Increasing the time spend in education, u, or the quality of education, ψ, has a an initial level impact on output per worker of $\left(\exp(\psi\, u)\right)^{1-\alpha}$.

Transition dynamics II

Moreover, it leads to transition dynamics through capital accumulation. Remember that, in steady state,

$$\tilde{k}^* = \left(\frac{s}{n+g+\delta} \right)^{\frac{1}{1-\alpha}} \exp(\psi u). \tag{79}$$

By increasing human capital, we increase the marginal product of capital leading to additional capital accumulation:

$$\dot{\tilde{k}}(t) = s\tilde{k}(t)^{\alpha} \left(\exp(\psi u)\right)^{1-\alpha} - (n+g+\delta)\tilde{k}(t) > 0. \tag{80}$$

The additional capital further increases output until we have reached the new, higher capital per efficient worker steady state:

$$\frac{\dot{y}(t)}{y(t)} = \alpha \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} + g. \tag{81}$$

By the same logic as before, output (capital) per worker grows particularly fast initially.

Mankiw et al. (1992) consider also the augmented linear regression model where they measure human capital by the share of people with secondary schooling:

$$\left(\frac{Y(t)}{L(t)}\right)^{*} = A(t)\exp(\psi\, u)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{82}$$

$$\ln y(t) = \ln A(0) + gt + \frac{\alpha}{1-\alpha}s - \frac{\alpha}{1-\alpha}(n+g+\delta) + \psi\, u \tag{83}$$

$$\ln y(t) = \beta_0 + \beta_1 s + \beta_2(n+0.05) + \beta_3 u + \epsilon(t). \tag{84}$$

Empirics of the Solow model revisited II

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	6.89	7.81	8.63
	(1.17)	(1.19)	(2.19)
ln(I/GDP)	0.69	0.70	0.28
	(0.13)	(0.15)	(0.39)
$\ln(n + g + \delta)$	−1.73	−1.50	−1.07
	(0.41)	(0.40)	(0.75)
ln(SCHOOL)	0.66	0.73	0.76
	(0.07)	(0.10)	(0.29)
$\overline{R}^2$	0.78	0.77	0.24

- The three variables together explain almost 80% in cross-country variation in GDP per capita.
- As expected, more schooling increases output per capita.
- The implied α is more reasonable.

Drawbacks of the linear regression model

The linear regression model approach has several drawbacks:

- Endogeneity of variables is a serious issue. Unobservables, such as management capacity, are likely correlated with education (and savings rates).
- We have to assume a steady state.
- Our measures may be quite imprecise. In particular, the regression does not control for the quality of education.
- The R^2 does not tell us what part of the distribution the regression model fits well.

Development accounting

Development accounting is an alternative approach to ask how good the Solow model is in explaining cross country income differences. It relates observable inputs to GDP per capita through the production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{85}$$

$$Y(t)^{1-\alpha} = \left(\frac{K(t)}{Y(t)}\right)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{86}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)H(t) \tag{87}$$

$$\frac{Y(t)}{L(t)} = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi u). \tag{88}$$

Note, any cross country differences in s or n, the heart of the Solow model, should be reflected in the capital output ratio.

Development accounting II

$$y(t) = \frac{Y(t)}{L(t)} = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t) \exp(\psi u). \tag{89}$$

- We are going to assume $\alpha = 0.3$.
- We use micro estimates of the return to schooling for ψ.
- Note, different from the linear regression framework, we now fix values for α and ψ instead of letting the regression choose those that best fit the data. Moreover, we do not impose that all economies are in steady state, i.e., the production function holds in and out of steady state.

Development accounting III

Taking the U.S. as reference, we can ask what factor explains differences in output per capita relative to the U.S.:

$$\frac{y(t)}{y^{US}(t)} = \frac{\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{K^{US}(t)}{Y^{US}(t)}\right)^{\frac{\alpha}{1-\alpha}}} \frac{A(t)}{A^{US}(t)} \exp(\psi(u - u^{US})). \tag{90}$$

For example, the U.S. has around 11 years of schooling while the poorest countries have only 3. With a 10% return on schooling, we have:

$$\exp(0.1(3 - 11)) = 0.45, \tag{91}$$

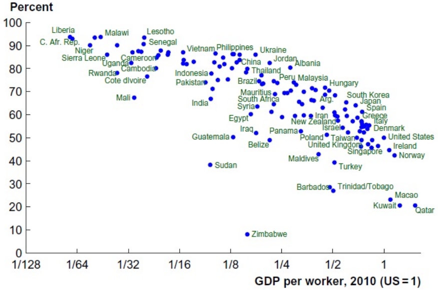
i.e., differences in education can explain a 55% lower output per capita in the poorest countries.

Development accounting IV

	GDP per worker, y	Capital/GDP $(K/Y)^{\alpha/(1-\alpha)}$	Human capital, h	TFP	Share due to TFP
United States	1.000	1.000	1.000	1.000	–
Hong Kong	0.854	1.086	0.833	0.944	48.9%
Singapore	0.845	1.105	0.764	1.001	45.8%
France	0.790	1.184	0.840	0.795	55.6%
Germany	0.740	1.078	0.918	0.748	57.0%
United Kingdom	0.733	1.015	0.780	0.925	46.1%
Japan	0.683	1.218	0.903	0.620	63.9%
South Korea	0.598	1.146	0.925	0.564	65.3%
Argentina	0.376	1.109	0.779	0.435	66.5%
Mexico	0.338	0.931	0.760	0.477	59.7%
Botswana	0.236	1.034	0.786	0.291	73.7%
South Africa	0.225	0.877	0.731	0.351	64.6%
Brazil	0.183	1.084	0.676	0.250	74.5%
Thailand	0.154	1.125	0.667	0.206	78.5%
China	0.136	1.137	0.713	0.168	82.9%
Indonesia	0.096	1.014	0.575	0.165	77.9%
India	0.096	0.827	0.533	0.217	67.0%
Kenya	0.037	0.819	0.618	0.073	87.3%
Malawi	0.021	1.107	0.507	0.038	93.6%
Average	0.212	0.979	0.705	0.307	63.8%
1/Average	4.720	1.021	1.418	3.260	69.2%

- The vast majority of income differences due to TFP differences.
- Capital to output ratios are relatively similar across countries.

Development accounting V



- TFP differences are important for all countries.
- TFP differences explain almost all the income differences for the poorest countries.

Development accounting VI

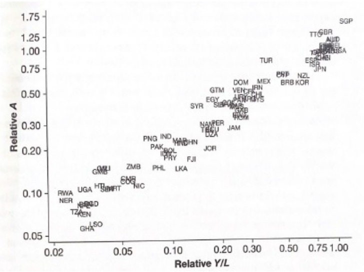
A different way to see the same point is to rewrite:

$$y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t) \exp(\psi u) \tag{92}$$

$$A(t) = \left(\frac{Y(t)}{K(t)}\right)^{\frac{\alpha}{1-\alpha}} \frac{Y(t)}{\exp(\psi u)}, \tag{93}$$

i.e., ask what technology level do we require to explain the observed output per capita. We will express this again relative to the U.S.

Development accounting VII



As expected, there is a strong correlation between output per capita and the inferred TFP, i.e., other factors explain relatively little.

Back to our three big questions

1. Why are we so rich and they so poor?
   - Different saving rates, population growth rates, education levels, and technology levels.
2. Why are there growth miracles?
   - Rapid accumulation of physical capital or increases in human capital.
3. What are the engines of long-run economic growth?
   - Technological progress.

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