Equilibrium Existence in Tullock Contests with Incomplete Information*

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Abstract

We show that under general assumptions a Tullock contest in which the information endowment of each contender is described by a countable partition has a pure strategy Bayesian Nash equilibrium.

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1 Introduction

The simplest form of *Tullock contest* is a *lottery* in which each player's probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all players. Tullock (1980) also considers a more general class of contests, in which the probability of success is taken to be the ratio between the individual and total "productivities" of efforts, where the productivity of effort is linked to the effort by a power function with a positive exponent that determines the returns to scale. Baye and Hoppe (2003) have identified a variety of economic settings (rent-seeking, innovation tournaments, patent races), which are strategically equivalent to Tullock contests. In addition, Tullock contests arise by design, e.g., in sport competition, internal labor markets – axiomatic justifications for this class of contests are offered by, e.g., Skaperdas (1996) and Clark and Riis (1998).

Existence of pure strategy Nash equilibria in Tullock contests with complete information has long been known – see, e.g., Perez-Castrillo and Verdier (1992) for symmetric contests, and Cornes and Hartly (2005) for asymmetric contests. Szidarovszky and Okuguchi (1997) established existence and uniqueness of equilibrium for a general class of Tullock contests (henceforth referred to as *Tullock SO-contests*), for which the "production function for lotteries" of each player is twice continuously differentiable, strictly increasing, concave, and vanishes at zero.

More recently, there has been a growing interest in studying the conditions under which a Tullock contests with incomplete information has a pure strategy Bayesian Nash equilibrium. In the private values two-player setting, Hurley and Shogren (1998) consider Tullock lotteries with one-sided asymmetric information, while Malueg and Yates (2004) and Fey (2008) study Tullock lotteries with ex ante symmetric players and two-sided private information represented by a binary type set. Fey (2008) also establishes existence of symmetric equilibrium when players' constant marginal costs of effort are independently and uniformly distributed; Ryvkin (2010) extends Fey (2008)'s results to more general multi-player symmetric contests. Warneryd (2003) studies two-player SO-Tullock contests in which players have a common value drawn from a continuous distribution and a common constant marginal cost of effort, and in which each player either observes the value or only knows the distribution from which the value is drawn; Warneryd (2012) extends this study to a multi-player setting. Wasser (2013a) considers modified, continuous versions of Tullock lotteries (see also Wasser (2013b), who establishes existence of equilibrium in private-value imperfectly discriminating contests with everywhere continuous success function).

We prove the existence of pure strategy Bayesian Nash equilibria in a class of incomplete information Tullock contests with several general features. In our setting, the contest success function as well as each player's value for the prize and cost of effort may depend on the state of nature. Moreover, players' information about the realized state of nature is described by a (countable) partition of the set of states of nature, which allows for a broad class of asymmetric information structures.

The class of *(generalized) Tullock contests* that we consider is characterized by the following three properties of the success function. At each state of nature, each player's probability of winning the prize is: (i) continuous with respect to the efforts of all players whenever the total effort is positive, (ii) non-decreasing and concave in his own effort, and (iii) equal to 1 if he is the only player who exerts positive effort. Tullock lotteries, and more generally Tullock SO-contests, satisfy these properties. But our class of contests is broader, and admits success functions that may be neither differentiable, nor additively separable in aggregating players' productivities, as well as success functions that are state-dependent. (However, when the set of states of nature is uncountable we introduce an assumption that limits the variability of the success function with the state of nature.) As for the players' cost functions, we assume that they are continuous, strictly increasing and convex, and vanish at zero.

Considering state-dependent success and cost functions, as well as non-linear cost functions, enhances the scope of applications of our result. Situations in which the marginal cost of effort is state-dependent and/or increasing are common; e.g., the opportunity cost of investing in an R&D project is likely to be state-dependent and increasing in the size of the investment if the available funds are limited. Moreover, as we allow a broad class of information structures, we in particular admit those in which some players are completely informed and others completely uninformed, as well as others, in which players may have partial and/or complementary information. With the value for the prize being a general function of the state of nature, our result would apply in the appropriate pure private value and common value settings.

Our proof of existence of pure strategy Bayesian Nash equilibrium in Tullock contests builds on Reny (1999)'s equilibrium existence result for games with discontinuous payoff functions. Payoffs in contests satisfying the properties (i) – (iii) described above are discontinuous at strategies prescribing zero effort at some states of nature, and hence Reny (1999)'s theorem provides a continuity-bypassing tool. (The hint of the usefulness of such an approach is present already in Baye *et al.* (1993), who include an application of their equilibrium existence result for games with discontinuities to proportional Tullock contests with complete information.) The main step of our proof shows that a contest in this class is a *better-reply-secure* game, which is one of the main premises of Reny (1999)'s theorem. Better reply security of the expected payoff functions is a weakening of the usual continuity requirement.

It is worth noting that Nash's existence theorem remains a viable alternative to establishing existence of equilibrium in Tullock contests. Following this approach Einy *et al.* (2013), the discussion paper upon which the current work is based, provides a proof that considers "truncated" contests in which players choose efforts from a compact interval with a positive lower bound, in which the expected payoff functions are continuous and therefore existence of equilibrium is assured by Nash's theorem. (The idea to bound the effort sets away from zero, and then let the bounds drop, is already present in Fey (2008) – see the proof of his Theorem 1 and footnote 12.) The crux of the proof in Einy *et al.* (2013) is to show that a limit point of the sequence of equilibria of truncated contests with a lower bound on players' efforts approaching zero is an equilibrium in the original contest. Ewerhart and Quartieri (2013) also follow this approach to establish existence of equilibrium in a setting in which the state space is finite, and players have differentiable cost functions and face budget constraints. (In addition, they show that equilibrium is unique, and derive results on rent dissipation in their setting. See also Ewerhart (2014) for results on existence and uniqueness of equilibrium when there is a continuum of types that are independently drawn.)

2 Tullock Contests with Incomplete Information

A group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize by choosing a level of *effort* in $\mathbb{R}_+$. Players' uncertainty about the state of nature is described by a probability measure p (representing the players' common prior belief) over a measurable space $(\Omega, \mathcal{F})$ of states of nature. The private information about the state of nature of player $i \in N$ is described by an $\mathcal{F}$-measurable and countable partition Π_i of Ω, specifying for each $\omega \in \Omega$ the event $\pi_i(\omega)$ containing ω that agent i observes. (Henceforth finite or countably infinite sets will be referred to as *countable*.) W.l.o.g.

we assume that $p(\pi_i) > 0$ for each $\pi_i \in \Pi_i$.

The value for the prize of each player $i \in N$ is given by an $\mathcal{F}$-measurable and integrable random variable $V_i : \Omega \to \mathbb{R}_{++}$, i.e., if $\omega \in \Omega$ is realized then player i's value for the prize is $V_i(\omega)$. The cost of effort of each player $i \in N$ is given by function $c_i : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ that is jointly measurable,[1] and is such that

(i) for every $x \in \mathbb{R}_+$ the random variable $c_i(\cdot, x)$ is integrable, and

(ii) for any $\omega \in \Omega$ the function $c_i(\omega, \cdot)$ is strictly increasing, continuous, convex, and vanishes at 0.

Upon observing the event containing the realized state of nature $\omega \in \Omega$ players simultaneously choose their effort levels $x = (x_1, ..., x_n) \in \mathbb{R}^n_+$. The prize is awarded to players in a probabilistic fashion, according to a state-dependent *success function* that specifies a probability distribution in the $(n-1)$-dimensional simplex Δ^{n-1}, $\rho : \Omega \times \mathbb{R}^n_+ \to \Delta^{n-1}$.

Denote by $\mathbf{0} \in \mathbb{R}^n_+$ the zero vector. The class of *(generalized) Tullock contests* that we consider is characterized, in addition to (i) and (ii) above, by some simple properties of the success function. Specifically, for each $(\omega, x) \in \Omega \times \mathbb{R}^n_+$ and $i \in N$:

(iii) $\rho(\omega, \cdot)$ is continuous on $\mathbb{R}^n_+ \backslash \{\mathbf{0}\}$;

(iv) $\rho_i(\omega, x_{-i}, \overline{x}_i)$ is non-decreasing and concave in the effort $\overline{x}_i$ of player i;

(v) $\rho_i(\omega, x) = 1$ whenever $x_i > 0$ and $x_j = 0$ for all $j \in N \backslash \{i\}$; and

(vi) $\rho_i(\cdot, x)$ is measurable with respect to Π_i.

Condition (vi) is very strong – when $n = 2$ it implies that $\rho_i(\cdot, x)$ is measurable with respect to the information partitions of both players, and hence (essentially) state-independent. However, if the set of states of nature Ω is countable, then condition (vi) will not be needed for our result. Henceforth (vi) will be imposed only if Ω is uncountable.

A *Tullock lottery* is a particular Tullock contest in which the state-independent success function ρ^T is given for each $x \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$ and $i \in N$ by

$$\rho^T_i(x) = \frac{x_i}{\sum_{j=1}^n x_j}. \tag{1}$$

It is easy to see that ρ^T satisfies conditions (iii) – (vi). More generally, conditions (iii) – (vi) are satisfied by any success function ρ that is given for any $\omega \in \Omega$, $x \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$

[1] Joint measurability of c_i, i.e., its measurability w.r.t. the tensor-product of $\mathcal{F}$ and the Borel σ-algebra on $\mathbb{R}_+$, assures that players' expected payoffs in the contest are well defined – see (3) below.

and $i \in N$ by

$$\rho_i(\omega, x) = \frac{g_i(\omega, x_i)}{\sum_{j=1}^{n} g_j(\omega, x_j)}, \tag{2}$$

where, for every $j \in N$, the state-dependent *production function for lotteries* $g_j(\omega, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, continuous, concave, and vanishes at 0 for every $\omega \in \Omega$ (and $g_j(\cdot, x_j)$ is Π_j-measurable for every $x_j \in \mathbb{R}_+$ if Ω is uncountable). Thus, our class includes the incomplete information version of Tullock SO-contests (see Szidarovszky and Okuguchi (1997)) where the functions $g_1, ..., g_n$ are in addition state independent and twice continuously differentiable. In particular, the commonly assumed contest success function given by $g_i(\omega, x_i) = x_i^r$ is a member of our class when the "impact parameter" r is in $(0, 1]$.

A Tullock contest with incomplete information is formally represented by a collection $(N, (\Omega, \mathcal{F}, p), \{\Pi_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho)$. This representation can accommodate the familiar Harsanyi's types model (with countable type sets). The present framework, however, is more amenable to study the impact of changes in the information structure, and has been used in the corresponding literature (see, e.g., Einy *et al.* (2001) and Malueg and Orzach (2012) for studies of common-value first- and second-price auctions).

In a Tullock contest, a pure strategy of player $i \in N$ is a Π_i-measurable function $X_i : \Omega \to \mathbb{R}_+$ (i.e., X_i is constant on every element of Π_i), that represents i's choice of effort in each state of nature following the observation of his private information. We denote by S_i the set of strategies of player i, and by $S = \times_{i=1}^{n} S_i$ the set of strategy profiles. For any strategy $X_i \in S_i$ and $\pi_i \in \Pi_i$, $X_i(\pi_i)$ stands for the constant value of X_i on π_i. Also, given a strategy profile $X = (X_1, ..., X_n) \in S$, we denote by X_{-i} the profile obtained from X by suppressing the strategy of player $i \in N$. Throughout the paper we restrict attention to pure strategies.

For each strategy profile $X = (X_1, ..., X_n) \in S$ we write

$$U_i(X) \equiv E[\rho_i(\cdot, X(\cdot)) V_i(\cdot) - c_i(\cdot, X_i(\cdot))]. \tag{3}$$

for the expected payoff of player i. Also, for $\pi_i \in \Pi_i$, we write

$$U_i(X \mid \pi_i) \equiv E[\rho_i(\cdot, X(\cdot)) V_i(\cdot) - c_i(\cdot, X_i(\cdot)) \mid \pi_i].$$

for the expected payoff of player i conditional on π_i.

An n-tuple of strategies $X^* = (X_1^*, ..., X_n^*)$ is a Bayesian Nash equilibrium if

$$U_i(X^*) \geq U_i(X_{-i}^*, X_i) \tag{4}$$

for every player $i \in N$, and every strategy $X_i \in S_i$; or equivalently, if

$$U_i(X^* \mid \pi_i) \geq U_i(X^*_{-i}, x_i \mid \pi_i) \tag{5}$$

for every $i \in N$, every $\pi_i \in \Pi_i$, and every effort $x_i \in \mathbb{R}_+$ (viewed here as a strategy in S_i with a constant value x_i on the set π_i).

3 Existence of Equilibrium

In this section we state and prove our result.

Theorem. *Every incomplete information Tullock contest satisfying (i) – (vi) has a pure strategy Bayesian Nash equilibrium.*[2]

Proof. Let $C = (N, (\Omega, \mathcal{F}, p), \{\Pi_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$ be a Tullock contest.

Step 1. We show that it entails no loss of generality to assume that the set of states of nature Ω is *countable*, and that each $\omega \in \Omega$ occurs with positive probability (and, in particular, $\{\omega\}$ is measurable, i.e., $\mathcal{F} = 2^{\Omega}$). These assumptions on C will be maintained henceforth.

If Ω is countable, let Ω' be the set of all atoms (minimal measurable sets with positive probability) in the probability space $(\Omega, \mathcal{F}, p)$. Define the probability distribution p' on Ω' by

$$p'(\{\omega'\}) = p(\omega') \text{ for every } \omega' \in \Omega', \tag{6}$$

and, for every $i \in N$, consider the partition Π'_i of Ω' that consists of the sets

$$\pi'_i = \{\omega' \in \Omega' \mid \omega' \subset \pi_i\} \text{ for every } \pi_i \in \Pi_i. \tag{7}$$

The players' values $\{V_i\}_{i\in N}$ and costs $\{c_i\}_{i\in N}$, as well as the contest success function ρ, all of which are $\mathcal{F}$-measurable, can be viewed as functions on Ω' in the natural way. Hence $C' = (N, (\Omega', 2^{\Omega'}, p'), \{\Pi'_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$ also constitutes a Tullock contest. Since every strategy X_i of player $i \in N$ takes the constant value $X_i(\pi_i)$ on each $\pi_i \in \Pi_i$, it is identifiable with his strategy X'_i in C' that takes the value $X_i(\pi_i)$ on each $\pi'_i \in \Pi'_i$ (where $\pi_i = \cup \pi'_i$ up to a zero-probability set). The map $X_i \to X'_i$ is a bijection for each i, and there an obvious equality of the expected payoffs under a

[2]Recall that (vi) is imposed only when the underlying set of states of nature Ω is uncountable.

strategy profile X in C and under the corresponding strategy profile X' in C'. Hence, with the above identification of strategies the contests C and C' are equivalent; note that C' satisfies the required properties.

Assume next that Ω is uncountable. Let Ω' be the set of all positive probability elements of Π, where $\Pi = \vee_{i\in N}\Pi_i$ is the coarsest partition of Ω that refines each Π_i. Since each Π_i is countable, so is Ω'. Define the probability distribution p' on Ω' as in (6) above, and, for every $i \in N$, consider the partition Π'_i of Ω' that consists of the sets defined as in (7). Furthermore, for every $\omega' \in \Omega'$, $x \in \mathbb{R}^n_+$, and $i \in N$, define

$$V'_i(\omega') \equiv E[V_i \mid \omega'] \text{ and } c'_i(\omega', x_i) \equiv E[c_i(\cdot, x_i) \mid \omega'],$$

and note that

$$\rho'_i(\omega', x) \equiv \rho_i(\omega, x) \text{ if } \omega \in \omega'$$

is well-defined as ρ_i is Π_i-measurable by condition (vi). It is easy to see that the functions $\{V'_i\}_{i\in N}$ are integrable on Ω', and that $\{c'_i\}_{i\in N}$ and ρ' satisfy conditions (i)–(v) with Ω' as the new set of states of nature (specifically, the continuity of $c'_i(\omega', \cdot)$ in condition (ii) is an implication of the dominated convergence theorem and the assumptions on c_i). Thus, $C' = (N, (\Omega', 2^{\Omega'}, p'), \{\Pi'_i\}_{i\in N}, \{V'_i\}_{i\in N}, \{c'_i\}_{i\in N}, \rho')$ also constitutes a Tullock contest. Denote by U'_i the expected payoff function of player i in C'.

As in the case of countable Ω, every strategy X_i of player $i \in N$ that takes the constant value $X_i(\pi_i)$ on each $\pi_i \in \Pi_i$, is identifiable with his strategy X'_i in C' that takes the value $X_i(\pi_i)$ on each $\pi'_i \in \Pi'_i$ (where $\pi_i = \cup \pi'_i$ up to a zero-probability set); the map $X_i \to X'_i$ is a bijection. Given a strategy profile $X = (X_1, ..., X_n)$, observe that

$$\begin{aligned} U_i(X) &= \sum_{\omega'\in\Omega'} \left(\rho'_i(\omega', X'(\omega')) \cdot E[V_i \mid \omega'] - E[c_i(\cdot, X'_i(\omega')) \mid \omega']\right) \cdot p(\omega') \\ &= \sum_{\omega'\in\Omega'} \left(\rho'_i(\omega', X'(\omega')) \cdot V'_i(\omega') - c'_i(\omega', X'_i(\omega'))\right) \cdot p'(\{\omega'\}) \\ &= U'_i(X'). \end{aligned}$$

Hence, under the above identification of strategies the contests C and C' are equivalent.

Step 2. We construct a "bounded" variant $\overline{C}$ of the given contest C, in which the strategy sets are compact.

Since the cost function of each player is strictly increasing and convex in the player's effort, $\lim_{x_i \to \infty} c_i(\omega, x_i) = \infty$ for every $\omega \in \Omega$, and hence $\lim_{x_i \to \infty} E\left[c_i(\cdot, x_i) \mid \pi_i\right] = \infty$ by Fatou's lemma for every $i \in N$ and $\pi_i \in \Pi_i$. It follows that for every $i \in N$ and $\pi_i \in \Pi_i$ there exists $Q^i_{\pi_i} > 0$ such that $E[V_i \mid \pi_i] < E[c_i(\cdot, Q^i_{\pi_i}) \mid \pi_i]$. Since $E[c_i(\cdot, 0) \mid \pi_i] = 0$, and since $E[c_i(\cdot, x_i) \mid \pi_i]$ is continuous in x_i on the interval $\left[0, Q^i_{\pi_i}\right]$ by the dominated convergence theorem and the monotonicity of c_i in x_i, there exists $0 < \overline{Q}^i_{\pi_i} < Q^i_{\pi_i}$ such that

$$E[V_i \mid \pi_i] < E[c_i(\cdot, \overline{Q}^i_{\pi_i}) \mid \pi_i] < E[V_i \mid \pi_i] + 1. \tag{8}$$

(The first inequality ensures that choosing effort $\overline{Q}^i_{\pi_i}$ gives player i a negative expected payoff conditional on π_i. The second inequality ensures that, when player i's effort choices are bounded from above by $\overline{Q}^i_{\pi_i}$ on every π_i, his ex ante expected cost is finite, since V_i is integrable by assumption.)

Consider a variant $\overline{C}$ of the given contest C, in which the effort set of each player i is restricted to be the bounded interval $\left[0, \overline{Q}^i_{\pi_i}\right]$ given his information set π_i. In $\overline{C}$, the set of strategies of player i, $\overline{S}_i$, is identifiable with the *compact* and metrizable product set $\times_{\pi_i \in \Pi_i} \left[0, \overline{Q}^i_{\pi_i}\right]$ via the the bijection $X_i \longleftrightarrow \left(X_i\left(\pi_i\right)\right)_{\pi_i \in \Pi_i}$, and player i's expected payoff function U_i is *concave* in i's own strategy (as $c_i(\cdot, x_i)$ is convex by (ii) and $\rho_i(\cdot, x)$ is concave in x_i by (iv)).

For each $i \in N$, the expected payoff function U_i is *not* continuous on $\overline{S} = \times_{i=1}^n \overline{S}_i$, but we will show that it is continuous on $\overline{S}_+$, where $\overline{S}_+ \subset \overline{S}$ is the set that consists of strategy-profiles X such that $X(\omega) \neq \mathbf{0}$ for every $\omega \in \Omega$. Indeed, consider a sequence $\left\{X^k\right\}_{k=1}^{\infty} \subset \overline{S}$ of strategy profiles that converge (pointwise) to a profile $X \in \overline{S}_+$. Then

$$\lim_{k \to \infty} E[\rho_i\left(\cdot, X^k\left(\cdot\right)\right) V_i(\cdot)] = E[\rho_i\left(\cdot, X\left(\cdot\right)\right) V_i(\cdot)]$$

by the dominated convergence theorem (note that $\lim_{k \to \infty} \rho_i\left(\omega, X^k\left(\omega\right)\right) = \rho_i\left(\omega, X\left(\omega\right)\right)$ for every $\omega \in \Omega$ as ρ is continuous on $\mathbb{R}^n_+ \backslash \{\mathbf{0}\}$ by condition (iii)), and

$$\lim_{k \to \infty} E[c_i\left(\cdot, X^k_i\left(\cdot\right)\right)] = E[c_i\left(\cdot, X_i\left(\cdot\right)\right)],$$

also by the dominated convergence theorem[3], as the continuity of the cost function in effort is ensured by condition (ii). It follows from (3) that

$$\lim_{k \to \infty} U_i\left(X^k\right) = U_i\left(X\right).$$

[3]The cost of i is bounded from above by the function that is equal to $c_i(\cdot, \overline{Q}^i_{\pi_i})$ on each π_i, which is integrable by the second inequality in (8).

Each function U_i is also *lower semi-continuous* in the variable $X_i \in \overline{S}_i$; i.e., for a fixed $X_{-i} \in \overline{S}_{-i} \equiv \times_{j \neq i} \overline{S}_j$ and every sequence $\{X_i^k\}_{k=1}^{\infty} \subset \overline{S}_i$ that converges (pointwise) to X_i, $\liminf_{k\to\infty} U_i\left(X_{-i}, X_i^k\right) \geq U_i\left(X_{-i}, X_i\right)$. Indeed, for every $\omega \in \Omega$

$$\begin{aligned} & \liminf_{k\to\infty} \left[\rho_i\left(\omega, X_{-i}(\omega), X_i^k(\omega)\right) V_i(\omega) - c_i\left(\omega, X_i^k(\omega)\right)\right] \\ \geq \; & \rho_i\left(\omega, X_{-i}(\omega), X_i(\omega)\right) V_i(\omega) - c_i\left(\omega, X_i(\omega)\right), \end{aligned}$$

since ρ_i is lower semi-continuous in $x_i \in \mathbb{R}_+$ (as follows from conditions (iii) and (v)), and c_i is continuous in $x_i \in \mathbb{R}_+$. This inequality implies, by (3) and Fatou's lemma, that

$$\liminf_{k\to\infty} U_i\left(X_{-i}, X_i^k\right) \geq U_i\left(X_{-i}, X_i\right).$$

Given the compactness of $\overline{S}_i$ and the concavity of U_i in the variable $X_i \in \overline{S}_i$, for each $i \in N$, existence of equilibrium in $\overline{C}$ is guaranteed by Theorem 3.1 of Reny (1999), provided $\overline{C}$ is in addition *better-reply-secure*: if (a) $\left\{X^k\right\}_{k=1}^{\infty} \subset \overline{S}$ is a sequence such that the (pointwise) limit $X \equiv \lim_{k\to\infty} X^k$ exists and X is not a Bayesian Nash equilibrium in $\overline{C}$; and (b) $w_i \equiv \lim_{k\to\infty} U_i(X^k)$ exists for every $i \in N$, then there must be some player i that can *secure* a payoff greater than w_i at X, i.e., there exist $Y_i \in \overline{S}_i$, $z_i > w_i$, and an open neighborhood $W \subset \overline{S}_{-i}$ of X_{-i} such that $U_i(X'_{-i}, Y_i) \geq z_i$ for every $X'_{-i} \in W$.

Step 3. We show that $\overline{C}$ is better-reply-secure.

Let $\left\{X^k\right\}_{k=1}^{\infty}$, X, and $(w_i)_{i\in N}$ be as above. If $X \in \overline{S}_+$, then the functions $(U_i)_{i\in N}$ are continuous at X and hence $w_i = U_i(X)$ for every $i \in N$. Since X is not an equilibrium by assumption, there exist $i \in N$ and $Y_i \in \overline{S}_i$ such that

$$U_i(X_{-i}, Y_i) > w_i + \varepsilon \tag{9}$$

for some $\varepsilon > 0$. It can be assumed w.l.o.g. that Y_i is strictly positive in all states of nature, as U_i is lower semi-continuous in the i^{th} variable. By the continuity of U_i at $(X_{-i}, Y_i) \in \overline{S}_+$, $U_i(X'_{-i}, Y_i) \geq z_i \equiv w_i + \frac{\varepsilon}{2}$ for every X'_{-i} in some open neighborhood W of X_{-i}, and thus i can secure at X a payoff greater than w_i.

Assume now that $X \in \overline{S} \backslash \overline{S}_+$; thus, $X(\omega^*) = \mathbf{0}$ for some $\omega^* \in \Omega$. Since Ω is countable, the set $\left(\Delta^{n-1}\right)^{\Omega}$ is metrizable and hence sequentially compact in the product topology. We can therefore consider an accumulation point $\left(\widetilde{p}\left(\omega\right)\right)_{\omega\in\Omega}$ of the sequence $\left\{\left(\rho\left(\omega, X^k\left(\omega\right)\right)\right)_{\omega\in\Omega}\right\}_{k=1}^{\infty}$. Assume w.l.o.g. (passing to a subsequence if necessary) that

$\lim_{k\to\infty}\left(\rho\left(\omega, X^k(\omega)\right)\right)_{\omega\in\Omega} = \left(\widetilde{p}(\omega)\right)_{\omega\in\Omega}$. Define, for every $\omega \in \Omega$ and $i \in N$,

$$\widetilde{w}_i(\omega) \equiv \widetilde{p}_i(\omega)V_i(\omega) - c_i(\omega, X_i(\omega)).$$

By the continuity of the cost function and the dominated convergence theorem, $w_i = E(\widetilde{w}_i(\cdot))$.

Since $\widetilde{p}(\omega^*)$ is a probability vector and $n \geq 2$, there exists $i \in N$ for whom

$$\widetilde{p}_i(\omega^*) < 1. \tag{10}$$

For any $0 < \varepsilon < 1$, consider a strategy $Y_i^\varepsilon \in \overline{S}_i$ given by $Y_i^\varepsilon(\omega) \equiv \max\{X_i(\omega), \varepsilon\overline{Q}^i_{\pi_i(\omega)}\}$ for every $\omega \in \Omega$. (In particular, $Y_i^\varepsilon(\pi_i(\omega^*)) = \varepsilon\overline{Q}^i_{\pi_i(\omega^*)}$.) For any $\omega \in \Omega$ with $X(\omega) \neq \mathbf{0}$, since ρ_i is continuous at $X(\omega) \neq \mathbf{0}$,

$$\lim_{\varepsilon\to 0+} \rho_i(\omega, X_{-i}(\omega), Y_i^\varepsilon(\omega)) = \lim_{k\to\infty} \rho_i\left(\omega, X^k(\omega)\right) = \widetilde{p}_i(\omega),$$

and therefore

$$\lim_{\varepsilon\to 0+} \left[\rho_i(\omega, X_{-i}(\omega), Y_i^\varepsilon(\omega)) V_i(\omega) - c_i(\omega, Y_i^\varepsilon(\omega))\right] = \widetilde{w}_i(\omega). \tag{11}$$

For any $\omega \in \Omega$ with $X(\omega) = \mathbf{0}$,

$$\lim_{\varepsilon\to 0+} \left[\rho_i\left(\omega, \mathbf{0}_{-i}, \varepsilon\overline{Q}^i_{\pi_i(\omega)}\right) V_i(\omega) - c_i\left(\omega, \varepsilon\overline{Q}^i_{\pi_i(\omega)}\right)\right] = V_i(\omega) \geq \widetilde{w}_i(\omega) \tag{12}$$

by property (v) of ρ, with a *strict inequality* for $\omega = \omega^*$ as follows from (10) and the assumption that every $V_i(\omega)$ is strictly positive. It is then implied by (3), (11) and (12) and the dominated convergence theorem that[4]

$$\lim_{\varepsilon\to 0+} U_i(X_{-i}, Y_i^\varepsilon) > E(\widetilde{w}_i) = w_i. \tag{13}$$

Now fix some $\varepsilon > 0$ for which $U_i(X_{-i}, Y_i^\varepsilon) > w_i + \varepsilon$, which exists by (13), and write $Y_i \equiv Y_i^\varepsilon$. By definition, (X_{-i}, Y_i) satisfies (9); hence the arguments following (9) show that i can secure a payoff greater than w_i. Thus $\overline{C}$ is better-reply-secure.

We conclude that $\overline{C}$ possesses some Bayesian Nash equilibrium X^*. In particular, X^* satisfies (5) for every $i \in N$, $\pi_i \in \Pi_i$, and $x_i \in \left[0, \overline{Q}^i_{\pi_i}\right]$. But note that every $x_i > \overline{Q}^i_{\pi_i}$ leads to a negative expected payoff to player i conditional on $\pi_i \in \Pi_i$ (this follows from the first inequality in (8)), which can be improved upon by lowering

[4] Recall that the cost of i is bounded from above by an integrable function that is equal to $c_i(\cdot, \overline{Q}^i_{\pi_i})$ on each π_i, and that w.l.o.g. (following step 1 of the proof) $p(\{\omega^*\}) > 0$.

the effort on π_i to zero. Thus, in contemplating a unilateral deviation from $X_i^*(\pi_i)$ conditional on π_i, player i is never worse off by limiting himself to efforts $0 \leq x_i \leq \overline{Q}^i_{\pi_i}$. But this means that X^* satisfies (5) for every $x_i \in \mathbb{R}_+$. Since this is the case for every $i \in N$ and $\pi_i \in \Pi_i$, X^* is a Bayesian Nash equilibrium of the original contest C. ■

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