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INFORMATION IN TULLOCK CONTESTS*

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Abstract

We study the effect of changes of players' information on the equilibrium efforts and payoffs of Tullock contests in which the common value of the prize is uncertain. When the diseconomies of scale in exerting effort increase at a large (small) rate, in contests with symmetric information expected effort decreases (increases) as players become better informed, while in two-player contests with asymmetric information a player with information advantage exerts less (more) effort, in expectation, than his opponent. In classic Tullock contests with symmetric information the equilibrium expected effort and payoff are invariant to the information available to the players. And when information is asymmetric, a player's information advantage is rewarded. Moreover, in two-player contests, while both players exert the same expected effort regardless of their information, expected effort is smaller when one player has information advantage than when both players have the same information. Interestingly, the player with information advantage wins the prize less frequently than his opponent.

Keywords: Information Advantage; Asymmetric Information; Common-Value; Tullock Contests.

JEL Classification: C72, D44, D82

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1 Introduction

Tullock contests (see Tullock 1980) are perhaps the most widely studied models in the literature on imperfectly discriminating contests. In a Tullock contest each player's probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all players. We study Tullock contests in which the players' common value for the prize is uncertain. Our aim is to understand how changes in the information available to the players' affects their equilibrium behavior and payoffs.

We provide a simple framework in which players' uncertainty is described by a probability space. The common value is a random variable on this space, and the common cost of effort is described by a differentiable, increasing and convex function, $c(x)$. (Our results also apply when the uncertain cost of effort is $c(x)$ multiplied by a random variable.) Players have a common prior belief, but upon the realization of the state of nature, and before taking action, each player obtains some information pertaining to the realized state. The interim information endowment of each player at the moment of taking action is described by a σ-field of subsets (*events*) of the state space: a player knows which events in his information field have occurred, and which have not.

This representation of players' uncertainty and information is natural, and encompasses the most general structures. It includes as a particular case situations in which each player observes some event containing the realized state of nature from a finite or countably infinite partition of the space of states of nature, but it also includes common situations in which a player information cannot be described by a partition of the state space. This is the case when, for example, each player observes a noisy signal of the realized value, and the value is a continuous random variable such that the smallest sigma field for which it is measurable is not generated by a partition of its support.

In this setting, we characterize the equilibria of a contest by a system of equations involving conditional expectations. Using this characterization, the law of iterated expectations, and Jensen's and Cauchy-Schwartz's inequalities, we derive interesting comparative static properties of the equilibria of Tullock contests. It turns our that the impact of changes in players' information on equilibrium efforts depends on a certain measure of the curvature of the cost effort. Specifically, on whether the function $\varphi(x) = xc'(x)$ is convex or concave. Intuitively, the convexity/concavity of φ relates to the rate at which the diseconomies of scale in exerting effort increase with effort. Discussing the implications of these results for a certain class of contests $\mathcal{T}$, in which the cost of effort is a function of the form $c(x) = x^\alpha$ with $\alpha \in [1, \infty)$, which are clear cut, help us providing intuition. Classic Tullock contests (for which $c(x) = x$) form an important subclass of $\mathcal{T}$ that has been extensively studied in the literature.

In contests with symmetric information, the unique, symmetric and interior equilibrium is identified by a simple equation. This equation reveals that a player's effort decreases with the number of players. Using this equation we study the impact on equilibrium effort of changes in the information available to the players. We show that when the diseconomies of scale in exerting effort increase at a large (small) rate, i.e., when the function φ is convex (concave), in expectation players exert less (more) effort the better is their information. For contests in $\mathcal{T}$, we calculate explicitly players' equilibrium strategy, and show that the players' expected effort decreases the better informed they are; however, the players' expected cost of effort is invariant to changes in their information, and since all players win the prize with the same probability (because equilibrium is symmetric) the expected payoff of a player is also invariant to changes in the players' information.

Next, we consider contests with asymmetric information. For two-player contests, we show that when the diseconomies of scale in exerting effort increase at a large (small) rate, in expectation a player with information advantage exert less (more) effort than his opponent. An implication of this result for contests in the class $\mathcal{T}$ is that even though the players' expected cost of effort is the same, the player with information advantage exerts, in expectation, less effort.

Finally, we study classic Tullock contests. Using our results for contests with symmetric information, we obtain explicitly the players' equilibrium strategy in this scenario, and derive as simple corollaries the main properties of equilibrium: expected effort and payoff are invariant to changes in the information available to the players, and decrease with the number of players. Likewise, our results for two-player contests with asymmetric information readily imply that in any two-player classic Tullock contest the expected effort of both players is the same. We show by example that these results do not hold in contest with more than two players. Then we calculate explicitly the players' equilibrium strategies in two-player classic Tullock contests in which one player has information advantage, and derive the main properties of equilibrium. Interestingly, players' expected effort is smaller than when they have symmetric information. Moreover, the player with information advantage wins the prize less frequently than his opponent. However, the payoff of the player with information advantage is greater or equal to that of his opponent (i.e., information advantage is rewarded).

Lastly, we show that (with some qualification) information advantages are rewarded in classic Tullock contests with any number of players, that is, if a player i has better information than some other player j, then the expected payoff of player i is greater than or equal to that of player j. This result holds for any two players with rankable information fields, regardless of the information endowments of the other players in the contest. The arguments behind our result rely on the proof of the theorem of Einy *et al.* (2002), which shows that in any Bayesian Cournot equilibrium of an oligopolistic industry with linear costs a firm's information advantage is rewarded.

There is an extensive literature on Tullock contests that we shall not attempt to review. For the complete information case, Baye and Hoppe (2003) have identified a variety of economic settings (rent-seeking, innovation tournaments, patent races), which are strategically equivalent to a Tullock contest. Skaperdas (1996) and Clark and Riis (1998) provide axiomatic characterizations of Tullock contests. Perez-Castrillo and Verdier (1992), Baye, Kovenock and de Vries (1994), Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Yamazaki (2008) and Chowdhury and Sheremeta (2009) study existence and uniqueness of equilibrium. Skaperdas and Gan (1995), Glazer and Konrad (1999), Konrad (2002), Cohen and Sela (2005) and Franke *et al.* (2011) study the effects of changes in the payoff structure on the behavior of players, and Schweinzer and Segev (2012) and Fu and Lu (2013) study optimal prize structures.

The literature on Tullock contests with incomplete information is more recent and sparse. Fey 2008 and Wasser 2011 study rent-seeking games under asymmetric information. Einy *et al.* (2015) show that under standard assumptions Tullock contests with asymmetric information have pure strategy Bayesian Nash equilibria, although they neither characterize equilibrium strategies nor do they study their properties.

The present paper builds on the insights and results of Warneryd (2003) and Einy *et al.* (2017). Warneryd (2003) studies two-player generalized Tullock contests in which the players' cost of effort is linear, and the value is a continuous random variable. In this setting, Warneryd (2003) considers the equilibria of contests with the information structures arising when each player either observes the value, or has only the information provided by the common prior. Our results for two-player contests extend Warneryd (2003)'s results to contests with general information structures, including those in which the players' information endowments are not rankable. In addition, we either obtain extensions of the results to contests with more than two players, or identify examples showing that they do not extend. Further, we show that in contests with any number of players, a player's information advantage over another player – not necessarily the extreme one considered in Warneryd (2003) – is rewarded, regardless of the information of other players.

Einy *et al.* (2017) study the impact of changes of the players' information on the equilibria of Tullock contests with symmetric information. Using direct, simple methods, which do not rely on high order derivatives of the cost function, we obtain extensions of its results. Moreover, in some scenarios we are able to calculate the equilibrium explicitly, uncovering interesting additional features.

The paper is organized as follows: Section 2 describes our setting and provides a characterization of the equilibrium of a Tullock contest. Section 3 studies contests with symmetric information, while Section 4 studies contests with asymmetric information. Section 5 studies classic Tullock contests. Section 6 concludes. An Appendix contains the technical proofs.

2 Common-Value Tullock Contests

A group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize by exerting effort. Players' uncertainty is described by a probability space $(\Omega, \mathcal{F}, p)$, where Ω is the set of states of nature, $\mathcal{F}$ is a σ-field of subsets of Ω, and p is a probability measure on $(\Omega, \mathcal{F})$ representing the players' common prior belief. Players' common value for the prize is an $\mathcal{F}$-measurable and bounded random variable $V : \Omega \to \mathbb{R}_{++}$. Players' common cost of effort is a differentiable, strictly increasing and convex function $c : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $c(0) = 0$. The private information of player $i \in N$ is described by a σ-subfield of $\mathcal{F}$, which we denote by $\mathcal{F}_i$. This means that for *any* event $A \in \mathcal{F}_i$ player i knows whether the realized state of nature is contained in A; in particular, if $\mathcal{F}_i$ is generated by a finite or countably infinite partition of Ω, then i knows the element of the partition containing the realized state of nature.

A *common-value Tullock contest* (to which we will henceforth refer to simply as a *Tullock contest*) starts by a move of nature that selects a state ω from Ω, about which every player i receives the information described by $\mathcal{F}_i$. Then the players simultaneously choose their effort levels, $x = (x_1, ..., x_n) \in \mathbb{R}^n_+$. The prize is awarded to the players in a probabilistic fashion, using a contest success function $\rho : \mathbb{R}^n_+ \to \triangle^n$, where $\triangle^n$ is the n-simplex. Specifically, if $x \in \mathbb{R}^n_+ \backslash \{0\}$, then the probability that player $i \in N$ wins the prize is

$$\rho_i(x) = \frac{x_i}{\sum_{k=1}^n x_k}, \tag{1}$$

whereas if $x = 0$, i.e., if no player exerts effort, then the prize is allocated according to some fixed probability vector $\rho(0) \in \triangle^n$. (When clear from the context, henceforth we use 0 to denote either the zero vector in $\mathbb{R}^n$ or the real number.) Hence, the payoff of player $i \in N$ is

$$u_i(\omega, x) = \rho_i(x)V(\omega) - c(x_i). \tag{2}$$

For any $\mathcal{F}$-measurable random variable f, we denote by $E[f \mid \mathcal{F}_i]$ a random variable which is (a version of) the conditional expectation with respect to the σ-field $\mathcal{F}_i$ – see, e.g., Borkar (1995) for a formal definition. Also, for any two random variables f and g, we write $f = g$, $f > g$, or $f \geq g$ when each of these relations hold almost everywhere on Ω.

A Tullock contest defines a Bayesian game in which a pure strategy for player $i \in N$ is an $\mathcal{F}_i$-measurable and integrable function $X_i : \Omega \to \mathbb{R}_+$, which describes i's choice of effort in each state of nature. (The measurability restriction implies that player i can condition its effort only on his private information.) We denote by S_i the set of strategies of player i, and by $S = \times_{i=1}^n S_i$ the set of strategy profiles. Given a strategy profile $X = (X_1, ..., X_n) \in S$ we denote by X_{-i} the profile obtained from X by suppressing the strategy of player i. Throughout the paper we restrict attention to pure strategies.

An *equilibrium* of a Tullock contest is a Bayesian Nash equilibrium of the Bayesian game defined by the contest; that is, it is a strategy profile $X = (X_1, ..., X_n)$ such that for every $i \in N$ and every $X_i' \in S_i$,

$$E[u_i(\cdot, X(\cdot)] \geq E[u_i(\cdot, X_{-i}(\cdot), X_i'(\cdot))], \tag{3}$$

or equivalently,

$$E[u_i(\cdot, (X(\cdot)) \mid \mathcal{F}_i] \geq E[u_i(\cdot, (X_{-i}(\cdot), X_i'(\cdot)) \mid \mathcal{F}_i] \tag{4}$$

almost everywhere on Ω. Einy *et al.* (2015) provide conditions that imply the existence of equilibrium in Tullock contests in which the players' information subfields are generated by finite or countably infinite partitions of Ω.

Our first remark shows that in any equilibrium total effort is positive almost everywhere on Ω. (If players exert no effort at some positive probability event, any player can secure the prize in that event by exerting a negligible effort, which would be a profitable deviation for some player.) Hence the vector $\rho(0) \in \triangle^n$ used to allocate the prize when no player exerts effort does not affect the set of equilibria. Thus, we describe a Tullock contest by a collection $T = (N, (\Omega, \mathcal{F}, p), \{\mathcal{F}_i\}_{i \in N}, V, c)$, omitting any reference to the vector $\rho(0)$.

Remark 1. *If X is an equilibrium of a Tullock contest, then $\sum_{i \in N} X_i > 0$.*

Proof. Assume by way of contradiction that there is an equilibrium X and a positive-measure set $B \in \mathcal{F}$ such that $X_1 = ... = X_n = 0$ on B. Let i be a player for whom $\rho_i(0) \leq 1/2$. Since X_i is $\mathcal{F}_i$-measurable there is $A_i \in \mathcal{F}_i$ such that $X_i = 0$ on A_i and $B \subset A_i$. Consider a strategy $X_i' = \varepsilon \cdot 1_{A_i} + X_i \cdot 1_{\Omega \setminus A_i} \in S_i$ for some $\varepsilon > 0$. Then $\rho_i(X) \leq \rho_i(X_{-i}, X_i')$ on A_i, and $\rho_i(X_{-i}, X_i') = 1$ on B. Thus, by switching from X_i to X_i', player i's payoff increases by at least $E[V \mid B] \cdot p(B)/2 - c(\varepsilon) \cdot p(A_i)$. As $c(0) = 0$ and c is continuous at 0, this expression is positive for ε sufficiently small. Therefore, in expectation X_i' is profitable deviation, which contradicts that X is an equilibrium. ■

The results we derive below extend to Tullock contests in which players' cost of effort is uncertain, i.e., state dependent, so long as it has a multiplicative structure. Our next remark, which makes this claim precise, follows immediately from the equivalence in terms of expected payoffs of a Tullock contest with this cost structure and a Tullock contest as described above.

Remark 2. *The set of equilibria of a Tullock contests in which the cost of effort is given for* $(\omega, x) \in \Omega \times \mathbb{R}_+$ *by* $W(\omega)c(x)$, *where* W *is an* $\mathcal{F}$*-measurable random variable* $W : \Omega \to \mathbb{R}_+$ *such that* $\inf W > 0$, *coincides with that of the Tullock contest in which the value is* $V' = V/W$, *the cost of effort is* c, *and the players' prior belief is the probability measure* p' *on* $(\Omega, \mathcal{F})$ *given for* $\omega \in \Omega$ *by* $dp'(\omega) = (W(\omega)/E[W])dp(\omega)$.

The literature has studied *generalized* Tullock contests, in which the contest success function ρ is given for $i \in N$ and $x \in \mathbb{R}^n_+ \setminus \{0\}$ by

$$\rho_i(x) = \frac{g(x_i)}{\sum_{k=1}^{n} g(x_k)},$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and satisfies $g(0) = 0$. In these contests, the payoff of player i is

$$u_i(\omega, x) = \frac{g(x_i)}{\sum_{j=1}^{n} g(x_j)} V(\omega) - c(x_i) = \frac{y_i}{\sum_{j=1}^{n} y_j} V(\omega) - \hat{c}(y_i),$$

where $y_j = g(x_j)$ for $j \in N$, and $\hat{c}(y_i) = c(g^{-1}(y_i))$. Hence there is a bijection between the equilibrium sets of a generalized Tullock contest in which the *score* function is g and the cost of effort is c, and a Tullock contest in which the cost of effort is $\hat{c}$. Thus, the results we derive below apply to generalized Tullock contests, as established in the following remark.

Remark 3. *The profile* $(X_1, ..., X_n)$ *is an equilibrium of a generalized Tullock contests* $(N, (\Omega, \mathcal{F}, p), \{\mathcal{F}_i\}_{i \in N}, V, c)$, *in which the score function is* g, *if and only if* $(g(X_1), ..., g(X_n))$ *is an equilibrium of the Tullock contest* $(N, (\Omega, \mathcal{F}, p), \{\mathcal{F}_i\}_{i \in N}, V, \hat{c})$, *where* $\hat{c} = g^{-1} \circ c$.

Henceforth we denote by $\varphi : \mathbb{R}_+ \to \mathbb{R}$ the function given for $x \in \mathbb{R}_+$ by $\varphi(x) = xc'(x)$. Since c is convex, φ is increasing, and therefore φ^{-1} is well defined. Moreover, $\varphi(x) \geq c(x)$, i.e., φ overstates the cost of effort. (Since c is convex, marginal cost is above average cost, i.e., $c'(x) \geq c(x)/x$.) The function φ will be useful in characterizing the equilibria of a Tullock contest. Its curvature, which is a proxy for the rate of growth of the diseconomies on scale in exerting effort, plays an important role in our analysis.

Proposition 1 provides a system of equations that characterizes the set equilibria of a Tullock contest. This characterization will allow us to derive interesting properties of equilibria in a variety of settings.

Proposition 1. *If* $(X_1, ..., X_n)$ *is an equilibrium of a Tullock contest, then for all* $i \in N$,

$$\varphi(X_i) = E\left[\frac{X_i \bar{X}_{-i}}{\left(X_i + \bar{X}_{-i}\right)^2} V \mid \mathcal{F}_i\right], \tag{5}$$

where $\bar{X}_{-i} = \sum_{j \in N \setminus \{i\}} X_j$.

The system of equations (5) provides a full characterization of the interior and corner equilibria of a contest. The left-hand side of equation (5) is well-defined almost everywhere by Remark 1. Using $\varphi(X_i) = X_i c'(X_i)$, and noting that X_i is $\mathcal{F}_i$-measurable, and hence it may be factorized out of the expectation on the right-hand side of equation (5), we may write this equation as

$$X_i c'(X_i) = X_i E\left[\frac{\bar{X}_{-i}}{\left(X_i + \bar{X}_{-i}\right)^2} V \mid \mathcal{F}_i\right].$$

Thus, when $X_i > 0$, equation (5) simplifies to

$$c'(X_i) = E\left[\frac{\bar{X}_{-i}}{\left(X_i + \bar{X}_{-i}\right)^2} V \mid \mathcal{F}_i\right],$$

which has a simply interpretation: it merely requires that, conditional on player i's information, the marginal cost of effort equals its expected marginal benefit. Moreover, when player i exerts no effort in some event, $\varphi(0) = xc'(0)$ provides a precise approximation of the cost of exerting a small amount of effort, which must be larger than the benefit if exerting no effort is optimal, and hence equation (5) holds as well. Even though this intuition is simple, the proof of Proposition 1 requires dealing with some measure theoretic issues, and is therefore relegated to the Appendix.

In order to provide clear illustration of our findings, and facilitate understanding the role of the curvature of φ in our results, we will consider the class of Tullock contests $\mathcal{T}$ in which players' cost of effort is a function of the form $c(x) = x^\alpha$, for some $\alpha \in [1, \infty)$. For contests in this class we are able to calculate equilibrium explicitly and derive many interesting properties. We denote by $T(\alpha)$ the subclass of $\mathcal{T}$ identified by the parameter

$\alpha \in [1, \infty)$; e.g., $T(1)$ is the subclass of *classic* Tullock contests. Note that for a contest in $T(\alpha)$, the function φ, given for $x \in \mathbb{R}_+$ by $\varphi(x) = \alpha x^\alpha = \alpha c(x)$, is convex, and its inverse is $\varphi^{-1}(y) = (y/\alpha)^{1/\alpha}$.

3 Symmetric Information

In this section, we study the equilibria of Tullock contests with symmetric information, and derive some comparative static properties of the impact of changes in the information available to the players, and in the number of players, on equilibrium efforts and payoffs. Existence, uniqueness, symmetry and interiority of equilibrium in such contests has been established by Einy *et al.* (2017) using a different approach. Proposition 2 provides a simplified version of Proposition 1 for contests with symmetric information: in this scenario equilibrium is characterized by a simple equation. This simple characterization of equilibrium reveals some interesting properties.

Proposition 2. *In the unique, symmetric and interior equilibrium of a Tullock contests with symmetric information players' strategy X is the solution to the equation*

$$\varphi(X) = \frac{n-1}{n^2} E\left[V \mid \mathcal{G}\right], \tag{6}$$

where $\mathcal{G}$ is the σ-subfield of $\mathcal{F}$ describing the players' information. Hence $E\left[\varphi(X)\right] = (n-1) E\left[V\right]/n^2$ is independent of the players' information. Moreover, a player's effort decreases with the number of players in the contest.

Proof. Substituting $X_i = X$ and $\bar{X}_{-i} = (n-1)X$ in equation (5) of Proposition 1 we get

$$\varphi(X) = E\left[\frac{(n-1) X^2}{(X + (n-1)X)^2} V \mid \mathcal{G}\right] = \frac{n-1}{n^2} E\left[V \mid \mathcal{G}\right].$$

Since φ is increasing and $((n-1)/n^2)$ decreases with n, X decreases with n. ■

In the equilibrium of a Tullock contest with symmetric information, which is symmetric, each player wins the prize with equal probability. Thus, the expected payoff of a player is $E\left[V\right]/n$ minus his expected cost of effort. For a contest $T \in T(\alpha) \subset \mathcal{T}$, equation (6) becomes

$$\alpha X^{\alpha} = \frac{n-1}{n^2} E\left[V \mid \mathcal{G}\right]. \tag{7}$$

Taking expectation in this equation we readily calculate a player's expected cost of effort as

$$E\left[c(X)\right] = E\left[X^{\alpha}\right] = \frac{n-1}{\alpha n^2} E\left[V\right].$$

Thus, in equilibrium player's cost of effort is independent of their information. Also, players' total effort is

$$nX = n\left(\frac{n-1}{\alpha n^2}\right)^{\frac{1}{\alpha}} \left(E\left[V \mid \mathcal{G}\right]\right)^{\frac{1}{\alpha}}.$$

Since $n[(n-1)/(\alpha n^2)]^{1/\alpha}$ increases with n, total effort increases with n. We state these results in the following corollary.

Corollary 1. *The equilibrium of a Tullock contests with symmetric information $T \in T(\alpha) \subset \mathcal{T}$, is given by $X = [(n-1)\, E\left[V \mid \mathcal{G}\right])/(\alpha n^2)]^{1/\alpha}$, where $\mathcal{G}$ is the σ-subfield of $\mathcal{F}$ describing players' information. Hence total effort increases with the number of players. Moreover, a player's expected cost of effort is $E[c(X)] = (n-1)E\left[V\right]/(\alpha n^2)$, and his expected payoff is $E\left[V\right]/n - (n-1)\, E\left[V\right]/(\alpha n^2)$, independently of the players' information.*

Our next result shows that the curvature of the function φ, which is a proxy for size of the diseconomies of scale in exerting effort, determines whether players' expected effort increases or decreases as they become better informed: when φ is convex (concave) player's equilibrium effort is larger (smaller) the better informed they are.

Let $\mathcal{G}$ and $\mathcal{G}'$ be any two σ-subfields of $\mathcal{F}$, and assume that $\mathcal{G}'$ is finer than $\mathcal{G}$ (i.e., $\mathcal{G} \subset \mathcal{G}'$). If the realized state of nature is $\omega \in \Omega$, then for each $A \in \mathcal{G}$ such that $\omega \in A$ there exists $B \in \mathcal{G}'$ such that $B \subset A$ and $\omega \in B$; that is, players have more precise information about ω when their information is that given by $\mathcal{G}'$ than when it is that given by $\mathcal{G}$. Thus, players are better informed the finer is the σ-subfield describing their information.

Proposition 3. *Let $X_{\mathcal{G}}$ and $X_{\mathcal{G}'}$ be the equilibria of two identical Tullock contests with symmetric information, except that players' information is given by the σ-subfields $\mathcal{G}$ and $\mathcal{G}'$, respectively, where $\mathcal{G} \subset \mathcal{G}'$. If φ is convex, then $E[X_{\mathcal{G}}] \geq E[X_{\mathcal{G}'}]$, whereas if φ is concave, then $E[X_{\mathcal{G}}] \leq E[X_{\mathcal{G}'}]$.*

The following lemma will be useful in proving propositions 3 and 5.

Lemma 1. *Let $\mathcal{G}$ be a σ-subfield of $\mathcal{F}$, and let X and Y be random variables such that $\varphi(X)$ and $\varphi(Y)$ are integrable and satisfy $\varphi(X) = E\left[\varphi(Y) \mid \mathcal{G}\right]$. If φ is convex, then $E[X] \geq E[Y]$. whereas if φ is concave, then $E[X] \leq E[Y]$.*

Proof. Assume that φ is convex. By the law of iterated expectations (see, e.g., Theorem 34.4 of Billingsley (1995)) and the *conditional* Jensen's inequality (see, e.g., Corollary 3.1.1 (ii) of Borkar (1995)),

$$\begin{aligned} E[X] &= E[\varphi^{-1}(\varphi(X))] \\ &= E[\varphi^{-1}(E[\varphi(Y) \mid \mathcal{G}])] \\ &\geq E[\varphi^{-1}(\varphi\left(E[Y \mid \mathcal{G}]\right))] \\ &= E[E[Y \mid \mathcal{G}]] \\ &= E[Y]. \blacksquare \end{aligned}$$

With this lemma in hand we can easily prove Proposition 3.

Proof of Proposition 3. Since $\mathcal{G} \subset \mathcal{G}'$, equation (6) and the law of iterated expectations imply

$$\varphi(X_{\mathcal{G}}) = \frac{n-1}{n^2} E\left[V \mid \mathcal{G}\right] = E\left[\frac{n-1}{n^2} E\left[V \mid \mathcal{G}'\right] \mid \mathcal{G}\right] = E\left[\varphi(X_{\mathcal{G}'}) \mid \mathcal{G}\right].$$

Hence the conclusions of Proposition 3 follow from Lemma 1. $\blacksquare$

The concavity or convexity of the function φ depends on the curvature of the cost function. One can show that if the Arrow-Pratt curvature of c, $R_c(x) := xc''(x)/c'(x)$ is non-decreasing, then the function the φ is convex. For contests in the class $\mathcal{T}$, $\varphi''(x) = \alpha^2 \left(\alpha - 1\right) x^{\alpha-2} > 0$, if $\alpha > 1$, and $\varphi''(x) = 0$ if $\alpha = 1$; i.e., φ is convex. (Note that the $R_c(x) = \alpha\left(\alpha - 1\right) \geq 0$.) Thus, the following corollary is a direct implication of Proposition 3. When φ is strictly convex, i.e., $\alpha > 1$, and the value is a non-degenerate random variable, expected effort strictly decreases the finer is the subfield describing the players' information.

Corollary 2. *In the equilibrium of a Tullock contests with symmetric information $T \in \mathcal{T}$ players' expected effort decreases as they become better informed.*

4 Asymmetric Information

In this section we study the equilibria of contests with asymmetric information. Proposition 4 establishes an auxiliary result for two-player contests that has important implications. Note that Proposition 4 does not involve any assumption about the players' information.

Proposition 4. *In any equilibrium (X_1, X_2) of a two-player Tullock contest, $E[\varphi(X_1)] = E[\varphi(X_2)]$.*

Proof. Let (X_1, X_2) be an equilibrium of a two-player Tullock contest. Proposition 1 and the law of iterated expectations imply

$$\begin{aligned} E[\varphi(X_1)] &= E\left[E\left[\frac{X_1X_2}{(X_1+X_2)^2}V \mid \mathcal{F}_1\right]\right] \\ &= E\left[\frac{X_1X_2}{(X_1+X_2)^2}V\right] \\ &= E\left[E\left[\frac{X_2X_1}{(X_2+X_1)^2}V \mid \mathcal{F}_2\right]\right] \\ &= E\left[\varphi(X_2)\right]. \blacksquare \end{aligned}$$

For a contest $T \in \mathcal{T}$, $\varphi(x) = \alpha c(x)$ for some $\alpha \in [1, \infty)$. Thus, Proposition 4 implies

$$E[\alpha c(X_1)] = E[\varphi(X_1)] = E[\varphi(X_2)] = E[\alpha c(X_2)].$$

Hence the following corollary.

Corollary 2. *In any equilibrium (X_1, X_2) of a two-player Tullock contest in $\mathcal{T}$, $E[c(X_1)] = E[c(X_2)]$.*

Our next proposition establishes that in two-player contests a player with *information advantage* exerts less effort, in expectation, than his opponent. Formally, player $i \in N$ is said to have an information advantage over player $j \in N$ if $\mathcal{F}_j \subset \mathcal{F}_i$. As noted above, the finer is the information subfield of a player, the more precise is the player's information about the realized state of nature.

Proposition 5. *Let (X_1, X_2) be an equilibrium of a two-player Tullock contest in which player 2 has an information advantage over player 1. If φ is convex, then $E[X_1] \geq E[X_2]$, whereas if φ is concave, then $E[X_1] \leq E[X_2]$.*

Proof. Let (X_1, X_2) be an equilibrium. Since $\mathcal{F}_1 \subset \mathcal{F}_2$, Proposition 1 and the law of iterated expectation imply

$$\begin{aligned}
\varphi(X_1) &= E\left[\frac{X_1 X_2 V}{(X_1+X_2)^2} \mid \mathcal{F}_1\right] \\
&= E\left[E\left[\frac{X_2 X_1 V}{(X_1+X_2)^2} \mid \mathcal{F}_2\right] \mid \mathcal{F}_1\right] \\
&= E\left[\varphi(X_2) \mid \mathcal{F}_1\right].
\end{aligned}$$

Hence the conclusions of Proposition 5 follow from Lemma 1. ■

The following example identifies a three-player classic Tullock contest with a unique equilibrium. In this equilibrium players' expected cost of effort differ, and the expected effort of players 2 and 3, who have information advantage over player 1, is greater than that of player 1. Hence the results of this section do not extend to contests with more than two players.

Example 1 *Consider a three-player classic Tullock contest in which* $\Omega = \{\omega_1, \omega_2\}$, $p(\omega_1) = 1/8$, $V(\omega_1) = 1$, *and* $V(\omega_2) = 8$. *Players* 2 *and* 3 *observe the value; player 1 has only the prior information. The unique equilibrium is* $(X_1(\omega_1), X_1(\omega_2)) = (168/121, 168/121)$ *and* $(X_2(\omega_1), X_2(\omega_2)) = (X_3(\omega_1), X_3(\omega_2)) = (0, 224/121)$. *Hence for* $i \in \{2, 3\}$

$$E[c(X_1)] = E[X_1] = \frac{168}{121} < \left(\frac{7}{8}\right)\frac{224}{121} = E[X_i] = E[c(X_i)].$$

5 Classic Tullock Contests

In this section we study the properties of the equilibria of classic Tullock contests. We begin by stating corollaries describing the implications of our result in sections 3 and 4 for classic Tullock contests. Our first result follows immediately from Corollary 1. (Simply, set $\alpha = 1$ in the formulae.) The result that in classic Tullock contests with symmetric information changes in the information available to the players have no impact on their expected effort and payoff is derived by Einy *et al.* (2017) using an indirect approach. Instead, Corollary 3 provides explicitly the players' equilibrium strategy as well as their expected effort and payoff, which allows deriving additional properties of equilibrium.

Corollary 3. *In the equilibrium of a classic Tullock contests with symmetric information a player's strategy is* $X = (n-1)\, E\,[V \mid \mathcal{G}]\,/n^2$*, where* $\mathcal{G}$ *is the* σ*-subfield of* $\mathcal{F}$ *describing players' information. Hence the expected effort of a player is* $E[X] = (n-1)\, E\,[V]\,/n^2$*, and his expected payoff is* $E\,[V]\,/n^2$*, independently of the players' information.*

Next we study the properties of the equilibria of Tullock contests with asymmetric information. A direct implication of Corollary 2 is that in any equilibrium of a two-player *classic* Tullock contests both players exert the same expected effort regardless of their information. This result, which we state in Corollary 4, has been established by Warneryd (2003) in a setting in which $\mathcal{F}_1 = \{\varnothing, \Omega\}$ and $\mathcal{F}_2$ is the minimal σ-field with respect to which a continuously distributed V is measurable. Corollary 4, however, involves no assumptions on the distribution of the value or on the information of the players. Example 1 shows that this result does not extend to contests with more than two players.

Corollary 4. *In any equilibrium of a two-player classic Tullock contest both players exert the same expected effort.*

In what follows, we study other properties of the equilibria of classic Tullock contests in which a player has information advantage. We begin by studying two-player contests. Our next proposition derives explicitly the unique interior equilibrium of such contests. We shall use these formulae to derive interesting properties of these contests.

Proposition 6. *In an interior equilibrium of a two-player classic Tullock contest in which player 2 has information advantage over player 1, i.e.,* $\mathcal{F}_1 \subset \mathcal{F}_2$*, players' strategies are* $(X_1, X_2) = (Z^2, Z\sqrt{E[V \mid \mathcal{F}_2]} - Z^2)$*, where* $Z = E[\sqrt{E\,[V \mid \mathcal{F}_2]} \mid \mathcal{F}_1]/2$*.*

Proof. Let (X_1, X_2) be an interior equilibrium. Since $\varphi(x) = x$, Proposition 1 implies

$$X_2 = E\left[\frac{X_1 X_2 V}{(X_1 + X_2)^2} \mid \mathcal{F}_2\right].$$

Since both X_1 and X_2 are $\mathcal{F}_2$-measurable (because $\mathcal{F}_1 \subset \mathcal{F}_2$) and $X_2 > 0$, this equation may be written as

$$1 = E\left[\frac{X_1 V}{(X_1 + X_2)^2} \mid \mathcal{F}_2\right] = \frac{X_1 E\,[V \mid \mathcal{F}_2]}{(X_1 + X_2)^2}.$$

Hence

$$X_2 = \sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]} - X_1. \tag{8}$$

Also by Proposition 1,

$$X_1 = E\left[\frac{X_1 X_2 V}{(X_1+X_2)^2} \mid \mathcal{F}_1\right],$$

and since $X_1 > 0$ is $\mathcal{F}_1$-measurable, we may write this equation as

$$1 = E\left[\frac{X_2 V}{(X_1+X_2)^2} \mid \mathcal{F}_1\right].$$

By the law of iterated expectations

$$E\left[\frac{X_2 V}{(X_1+X_2)^2} \mid \mathcal{F}_1\right] = E\left[E\left[\frac{X_2 V}{(X_1+X_2)^2} \mid \mathcal{F}_2\right] \mid \mathcal{F}_1\right].$$

Substituting X_2 from equation (8) and recalling that X_1 is $\mathcal{F}_2$-measurable, we get

$$\begin{aligned}
1 &= E\left[E\left[\frac{\left(\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]} - X_1\right)V}{\left(X_1 + \left(\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]} - X_1\right)\right)^2} \mid \mathcal{F}_2\right] \mid \mathcal{F}_1\right] \\
&= E\left[E\left[\frac{V}{\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]}} - \frac{V}{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_2\right] \mid \mathcal{F}_1\right] \\
&= \frac{1}{\sqrt{X_1}} E\left[\frac{E[V \mid \mathcal{F}_2]}{\sqrt{E[V \mid \mathcal{F}_2]}} \mid \mathcal{F}_1\right] - E\left[\frac{E[V \mid \mathcal{F}_2]}{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_1\right] \\
&= \frac{E\left[\sqrt{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_1\right]}{\sqrt{X_1}} - 1.
\end{aligned}$$

Hence

$$\sqrt{X_1} = \frac{E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]}{2}. \tag{9}$$

The formulae given in Proposition 6 follows from equations (8) and (9). ■

The following corollary is an interesting direct implication of Proposition 6 for classic Tullock contests in which one player observes the value before taking action and the other player has only the prior information.

Corollary 5. *Consider a two-player classic Tullock contest in which player 2 observes the state of nature and player 1 has only the prior information, i.e., $\mathcal{F}_2 = \mathcal{F}$ and $\mathcal{F}_1 = \{\Omega, \varnothing\}$.*

If $\inf V \geq (E[\sqrt{V}])^2/4$*, i.e., the value distribution is not too dispersed, then the unique equilibrium of the contests is* $(X_1, X_2) = (E[\sqrt{V}])^2/4, E[\sqrt{V}](2\sqrt{V} - E[\sqrt{V}])/4)$.

In two-player classic Tullock contest, when players have symmetric information each player's expected effort is $E[V]/4$ (Corollary 2), while when one observes the value and the other player has only the prior information their expected effort is $E[\sqrt{V}])^2/4$ (Corollary 5). Since $(E[\sqrt{V}])^2 \leq E[V]$ by Jensen's inequality, this implies that in expectation players exert less effort, and hence capture a larger share of the surplus, in the latter scenario. Warneryd (2003) obtains this result (see his Proposition 5) when V is a continuous random variable. Our next corollary shows that this inequality holds in any classic Tullock contest in which a player has information advantage, even if it is not as extreme as that of Corollary 5.

Corollary 6. *In the interior equilibrium of a two-player classic Tullock contest in which one player has information advantage, players' expected effort is less than or equal to that under symmetric information.*

Proof. Let (X_1, X_2) be an interior equilibrium of two-player classic Tullock contest in which player 2 has an information advantage over player 1, i.e., $\mathcal{F}_1 \subset \mathcal{F}_2$. By Corollary 4, $E[X_2] = E[X_1]$. Also, Proposition 6, and Jensen's inequality imply

$$E[X_1] = \frac{E\left[\left(E\left[\sqrt{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_1\right]\right)^2\right]}{4} \leq \frac{E[V]}{4},$$

which establishes the corollary, since by Corollary 1, $E[V]/4$ is the equilibrium expected effort when players have symmetric information. ■

Our next result establishes that in a two-player classic Tullock contest a player with information advantage tends to win the prize less frequently than his opponent. Warneryd (2003) arrives at the same result (see his Proposition 2) is the restrictive setting he considers.

Proposition 7. *In the interior equilibrium of a two-player classic Tullock contest a player with information advantage wins the prize less frequently than his opponent.*

Proof. Assume that player 2 has information advantage over player 1, and let (X_1, X_2) be an interior equilibrium. By Proposition 6

$$\begin{aligned} E\left[\frac{X_1}{X_1+X_2}\right] &= E\left[\frac{X_1}{X_1+\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]}-X_1}\right] \\ &= E\left[\frac{\sqrt{X_1}}{\sqrt{E[V \mid \mathcal{F}_2]}}\right] \\ &= \frac{1}{2}E\left[\frac{E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]}{\sqrt{E[V \mid \mathcal{F}_2]}}\right] \\ \text{(by Jensen's Inequality)} &\geq \frac{1}{2}. \blacksquare \end{aligned}$$

Example 2 describes an eight-player classic Tullock contest in which a player who has an information advantage over the other players wins the prize more frequently than every other player, which shows that Proposition 7 does not extend to contests with more than two players.

Example 2 *Consider an eight-player classic Tullock contest in which* $\Omega = \{\omega_1, \omega_2\}$, $p(\omega_1) = 1/2$, $V(\omega_1) = 1$, *and* $V(\omega_2) = 2$. *Player 8 observes the value; all the other players have only the prior information. The unique equilibrium* X *of this contest is* $X_1 = ... = X_7 = (x, x)$ *and* $X_8 = (0, y)$, *where*

$$x = \frac{7\sqrt{229}+139}{1575}, \; y = \frac{56\sqrt{229}-238}{1575}.$$

Thus, the ex-ante probability that player $i \in \{1, 2, ..., 7\}$ *wins the prize is*

$$\frac{1}{2}(\frac{1}{7}+\frac{x}{7x+y}) = \frac{\sqrt{229}+37}{420},$$

whereas the ex-ante probability that player 8 *win the prize is*

$$1-7\left(\frac{\sqrt{229}+37}{420}\right) = \frac{161-7\sqrt{229}}{420} > \frac{\sqrt{229}+37}{420}.$$

Next we examine the impact of information advantages on payoffs. More information allows an individual to better tune his actions to the realized state of nature. However, in a strategic setting the lack of information serves as a potentially useful commitment instrument, and hence having more information is a mixed blessing, which impact on payoffs is generally dubious. Proposition 8 establishes that in a two-player classic Tullock contest the expected payoff of a player with information advantage is greater or equal to that of his opponent; that is, information advantages are rewarded. Warneryd (2003) derives an analogous result for contests in which one player has full information and the other player has only the prior information, and in which the value is a continuous random variable. Proposition 7, however, involves no assumption about the value, and applies to any information advantage, not necessarily the extreme information advantage assumed in Warneryd (2003).

Proposition 8. *In the interior equilibrium of a two-player classic Tullock contest the payoff of a player with information advantage is greater than or equal to that of his opponent.*

Proof. Assume that player 2 has information advantage over other player 1, i.e., $\mathcal{F}_1 \subset \mathcal{F}_2$, and let X be the interior equilibrium. Since $E[X_1] = E[X_2]$ by Corollary 4, we show that

$$\begin{aligned} E[u_2(\cdot, X(\cdot)] - E[u_1(\cdot, X(\cdot)] &= E\left[\frac{X_2 V}{X_1 + X_2} - X_2\right] - E\left[\frac{X_1 V}{X_1 + X_2} - X_1\right] \\ &= E\left[\frac{X_2 - X_1}{X_1 + X_2} V\right] \geq 0. \end{aligned}$$

Using the formulae of Proposition 6 we get

$$\begin{aligned} E\left[\frac{X_2 - X_1}{X_1 + X_2} V\right] &= E\left[\frac{\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]} - X_1 - X_1}{X_1 + \sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]} - X_1} V\right] \\ &= E\left[\frac{\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]} - 2X_1}{\sqrt{X_1}\sqrt{E[V \mid \mathcal{F}_2]}} V\right] \\ &= E[V] - 2E\left[\frac{\sqrt{X_1}}{\sqrt{E[V \mid \mathcal{F}_2]}} V\right] \\ &= E[V] - E\left[\frac{E\left[\sqrt{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_1\right]}{\sqrt{E[V \mid \mathcal{F}_2]}} V\right]. \end{aligned}$$

We show that

$$\left(E\left[\frac{E\left[\sqrt{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_1\right]}{\sqrt{E[V \mid \mathcal{F}_2]}} V\right]\right)^2 \leq (E[V])^2,$$

which establishes the proposition. By Jensen's Inequality

$$E\left[\left(E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]\right)^2\right] \leq E[V].$$

Also

$$E\left[\frac{V}{E[V \mid \mathcal{F}_2]}\right] = E\left[E\left[\frac{V}{E[V \mid \mathcal{F}_2]} \mid \mathcal{F}_2\right]\right] = E\left[\frac{E[V \mid \mathcal{F}_2]}{E[V \mid \mathcal{F}_2]}\right] = 1.$$

Hence

$$\begin{aligned}
\left(E\left[\frac{E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]}{\sqrt{E[V \mid \mathcal{F}_2]}}V\right]\right)^2 &= \left(E\left[\left(E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]\sqrt{V}\right)\sqrt{\frac{V}{E[V \mid \mathcal{F}_2]}}\right]\right)^2 \\
\text{(by Cauchy-Schwartz's Inequality)} &\leq E\left[\left(E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]\sqrt{V}\right)^2\right] E\left[\frac{V}{E[V \mid \mathcal{F}_2]}\right] \\
&= E\left[\left(E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]\sqrt{V}\right)^2\right] \\
\text{(by Cauchy-Schwartz's Inequality)} &\leq E\left[\left(E\left[\sqrt{E\left[V \mid \mathcal{F}_2\right]} \mid \mathcal{F}_1\right]\right)^2\right] E[V] \\
\text{(by Jensen's Inequality)} &\leq (E[V])^2 . \blacksquare
\end{aligned}$$

Proposition 9, our last result, shows that (with some qualification) information advantages are rewarded in classic Tullock contests with any number of players. Proposition 9 is proved by observing a formal equivalence between a Tullock contest and a Cournot oligopoly with asymmetric information, and by appealing to (the proof of) a result of Einy *et al.* (2002) which shows that the equilibria of such industries have the desired property. It applies to both interior and corner equilibria, but it does not imply Proposition 8, as it assumes that in equilibrium total effort is bounded above zero.

Proposition 9. *Let X be an equilibrium of a classic Tullock contest such that* $\inf \sum_{j=1}^{n} X_j >$ 0. *If player i has information advantage over other player j, i.e., $\mathcal{F}_j \subset \mathcal{F}_i$, then $E[u_i(\cdot, X(\cdot)] \geq$ $E[u_j(\cdot, X(\cdot)]$.*

Proof. For $X = (X_1, ..., X_n) \in S$ and $\omega \in \Omega$, the payoff of each player $i \in N$ may be written as

$$\begin{aligned}
u_i(\omega, X(\omega)) &= \frac{X_i(\omega)}{\sum_{j=1}^{n} X_j(\omega)}V(\omega) - c(X_i(\omega)) \\
&= P(\omega, \sum\nolimits_{j=1}^{n} X_j(\omega))X_i(\omega)) - C(\omega, X_i(\omega)),
\end{aligned}$$

where the functions $P : \Omega \times \mathbb{R}_{++} \to \mathbb{R}_+$ and $C : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ are defined as

$$P(\omega, x) = \frac{V(\omega)}{x}, \text{ and } C(\omega, x) = c(x). \tag{10}$$

Thus, if X is an equilibrium of the contests, then X is an equilibrium of the oligopolistic industry $(N, (\Omega, \mathcal{F}, p), (\mathcal{F}_i)_{i \in N}, P, C)$, where P is the inverse market demand and C is the firms' cost function.

Einy *et al.* (2002) show that an information advantage is rewarded in any equilibrium of an oligopolistic industry under certain conditions on the inverse demand and cost functions. Some of the conditions are not satisfied, however, by the function P in (10). Fortunately, the proof of Einy *et al.* (2002) applies to the present setting provided that

$$E\left[1_{X_i > 0} \times \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i\right] = 0 \tag{11}$$

holds for every $i \in N$. Equation (11) immediately yields equation (2.6) of Einy *et al.* (2002), page 157, from which point on their proof applies without change. We establish that equation (11) holds in the Appendix. ■

It is worth noticing that Proposition 9 that does not involve any assumption about the information of the players whose information fields are *not* being compared: a player's information advantage over another player (again, not necessarily an extreme advantage) is rewarded regardless of the information endowments of the other players; that is, its conclusion holds whenever two players have rankable information.

We conclude with a remark showing that the qualification in Proposition 9 on the sum of equilibrium efforts being bounded above zero holds under some general conditions. The proof of this remark is given in the Appendix.

Remark 4. *Let X be an equilibrium of a classic Tullock contest in which either (i) $\mathcal{F}_1, ..., \mathcal{F}_n$ are finite, or (ii) $n = 2$ and $\inf V > 0$. If player i has information advantage over player j, then $E[u_i(\cdot, X(\cdot)] \geq E[u_j(\cdot, X(\cdot)]$.*

6 Conclusions

We provide a general framework well suited for studying the outcomes generated by Tullock contest under incomplete information. We characterize the equilibria of a contest as the solutions to a system of equations involving conditional expectations. Simple calculations, the use of the law of iterated expectation and well know inequalities allows us to derive interesting properties equilibria. For a simple class of contests, which includes the widely studied classic Tullock contests, equilibrium can be calculated explicitly.

In contests with symmetric information, the players' expected effort increases (decreases) with their level of information when exerting effort is subject to diseconomies of scale that grow at a large rate. In two-player contests with asymmetric information, a player with information advantage exert less (more) effort than his opponent when the diseconomies of scale in exerting effort increase at a large (small) rate – this result does not extend to contests with more than two players.

Our results for two-player classic Tullock contests are clear cut: while both players exert the same expected effort regardless of their information, players exert more effort when they are symmetrically informed than when one of the players has information advantage. Moreover, the player with information advantage capture more surplus than his opponent, even though he wins the prize less frequently. Further, while the result on the equality of expected effort and the frequency with which a player with information advantage wins the prize do not extend to contests with more than two players, information advantages are rewarded in classic Tullock contests with any number of players.

There are several interesting open questions. It is unclear whether the result on the reward of information advantages extends to contests in which the cost of effort is not linear. Whereas Einy *et al.* (2002) provide an example showing that their result for Cournot oligopolies does not extend to industries in which firms produce the good with diseconomies of scale, finding an example with this feature in our setting has shown to be elusive. Also, the study of the equilibria of contests in which player's information is not rankable is particularly challenging. Our result for two-player Tullock contests, showing that when there are large diseconomies of scale in exerting effort, in expectation players exert more effort when their information is symmetric (and coarse) than when one player has information advantage over the other, may suggest that a principal who organizes a contest with the objective of maximizing expected total effort should maintain participants symmetrically informed, and minimize information disclosure. However, such conclusion is not warranted, since we do not know how much effort players exert when their information is non-rankable. Nevertheless, we believe our framework and methods are suitable for these tasks.

7 Appendix

Proof of Proposition 1. Let X be an equilibrium and let $i \in N$. For any $\varepsilon \in \mathbb{R}$ define $X'_{i,\varepsilon} := \max\{X_i + \varepsilon, 0\} \in S_i$. Inequality (4) implies

$$E[u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i] \geq E[u_i(\cdot, X_{-i}(\cdot), X'_{i,\varepsilon}(\cdot)) \mid \mathcal{F}_i].$$

It follows that, for any $\varepsilon > 0$,

$$E\left[\frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{\varepsilon} \mid \mathcal{F}_i\right] \leq 0, \tag{12}$$

and

$$E\left[\frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,-\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{-\varepsilon} \mid \mathcal{F}_i\right] \geq 0. \tag{13}$$

As X_i and $X'_{i,-\varepsilon}$ are $\mathcal{F}_i$-measurable and non-negative, multiplying by $X'_{i,-\varepsilon}$ both sides of inequalities (12) and (13) we obtain

$$E\left[X_i(\cdot) \times \frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{\varepsilon} \mid \mathcal{F}_i\right] \leq 0 \tag{14}$$

and

$$E\left[X'_{i,-\varepsilon}(\cdot) \times \frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,-\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{-\varepsilon} \mid \mathcal{F}_i\right] \geq 0. \tag{15}$$

For every $\omega \in \Omega$ the function $u_i(\omega, x)$ is concave in the variable x_i, and hence for any $\varepsilon > 0$

$$\left|X_i(\omega) \times \frac{u_i(\omega, X_{-i}(\omega), X'_{i,\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{\varepsilon}\right| \tag{16}$$

$$\leq \quad X_i(\omega) \times \max\left\{\left|\frac{d}{dx_i}u_i(\omega, X(\omega))\right|, \left|\frac{d}{dx_i}u_i(\omega, X_{-i}(\omega), X_{i,\varepsilon}(\omega))\right|\right\} \tag{17}$$

and

$$\left| X'_{i,-\varepsilon}(\omega) \times \frac{u_i(\omega, X_{-i}(\omega), X'_{i,-\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{-\varepsilon} \right| \tag{18}$$

$$\leq \quad X'_{i,-\varepsilon}(\omega) \times \max\{\left|\frac{d}{dx_i}u_i(\omega, X(\omega))\right|, \left|\frac{d}{dx_i}u_i(\omega, X_{-i}(\omega), X_{i,-\varepsilon}(\omega))\right|\} \tag{19}$$

(The partial derivative $\frac{d}{dx_i}u_i(\omega, x)$ may not be defined when $x_i + \overline{x}_{-i} = 0$. However, the bounds in (17) and (19) vanish in such a case, being multiples of $x_i = 0$, and are thus well-defined.)

Since the cost function c is convex and strictly increasing, there exists $b > 0$ such that $c(b) > \overline{v} := \sup V$. (Recall that V has bounded support.) It follows that X_i is bounded from above by b almost everywhere on Ω, as otherwise the expected equilibrium payoff of player i would be negative conditional on some positive-measure event $A_i \in \mathcal{F}_i$, making it profitable for player i to deviate to $Y_i = 1_{\Omega \setminus A_i} \times X_i$. Now, notice that

$$\frac{d}{dx_i}u_i(\omega, x) = \frac{\overline{x}_{-i}}{(x_i + \overline{x}_{-i})^2}V(\omega) - c'(x_i) \tag{20}$$

whenever $x_i + \overline{x}_{-i} > 0$. Since $X_i \leq b$ as argued above, it can be easily seen from (20) that for all $\varepsilon \in (0, 1]$ the random variables

$$\left|X_i(\cdot) \times \frac{d}{dx_i}u_i(\cdot, X(\cdot))\right|,$$
$$\left|X_i(\cdot) \times \frac{d}{dx_i}u_i(\cdot, X_{-i}(\cdot), X_{i,\varepsilon}(\cdot))\right|,$$
$$\left|X'_{i,-\varepsilon}(\cdot) \times \frac{d}{dx_i}u_i(\cdot, X(\cdot))\right|, \text{ and}$$
$$\left|X'_{i,-\varepsilon}(\cdot) \times \frac{d}{dx_i}u_i(\cdot, X_{-i}(\cdot), X_{i,-\varepsilon}(\cdot))\right|$$

and are bounded from above by $\frac{1}{2}\overline{v} + c'(b + 1)$. (While the second factor in each of these random variables may be undertermined, each such variable is 0 when the first factor is 0.) In particular, the terms in (17) and (19) are bounded from above by $\frac{1}{2}\overline{v} + c'(b+1)$ when $\varepsilon \in$

$(0,1]$. Additionally, for every $\omega \in \Omega$

$$\begin{aligned} & \lim_{\varepsilon \to 0+} X_i(\omega) \times \frac{u_i(\omega, X_{-i}(\omega), X'_{i,\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{\varepsilon} \\ = & \lim_{\varepsilon \to 0+} X'_{i,-\varepsilon}(\omega) \times \frac{u_i(\omega, X_{-i}(\omega), X'_{i,-\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{-\varepsilon} \\ = & X_i(\omega) \times \frac{d}{dx_i} u_i(\omega, X(\omega)). \end{aligned} \tag{21}$$

Given (16)-(17), (18)-(19), and the boundedness arguments above, (21) leads to the following equalities by the conditional dominated convergence theorem (see Corollary 3.1.1 (iv) in Borkar (1995)):

$$\begin{aligned} & \lim_{\varepsilon \to 0+} E\left[X_i(\cdot) \times \frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{\varepsilon} \mid \mathcal{F}_i\right] \\ = & \lim_{\varepsilon \to 0+} E\left[X'_{i,-\varepsilon}(\cdot) \times \frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,-\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{-\varepsilon} \mid \mathcal{F}_i\right] \\ = & E\left[X_i(\cdot) \times \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i\right]. \end{aligned} \tag{22}$$

From (14), (15) and (22) we now obtain

$$E\left[X_i(\cdot) \times \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i\right] = 0.$$

Using (20) this becomes

$$\begin{aligned} 0 &= E\left[\frac{X_i \overline{X}_{-i}}{\left(X_i + \overline{X}_{-i}\right)^2} V - X_i c'(X_i) \mid \mathcal{F}_i\right] \\ &= E\left[\frac{X_i \overline{X}_{-i}}{\left(X_i + \overline{X}_{-i}\right)^2} V \mid \mathcal{F}_i\right] - E\left[X_i c'(X_i) \mid \mathcal{F}_i\right] \\ &= E\left[\frac{X_i \overline{X}_{-i}}{\left(X_i + \overline{X}_{-i}\right)^2} V \mid \mathcal{F}_i\right] - X_i c'(X_i), \end{aligned}$$

i.e.,

$$\varphi(X_i) = E\left[\frac{X_i \overline{X}_{-i}}{\left(X_i + \overline{X}_{-i}\right)^2} V \mid \mathcal{F}_i\right]. \blacksquare$$

Proof of equation (11) used in the proof of Proposition 9. Let X be an equilibrium of the contest. We rely on the proof of the first part of Proposition 1 and the notations therein. Note first that for every $\omega \in \Omega$

$$\begin{aligned}
&\lim_{\varepsilon \to 0+} 1_{X_i>0}(\omega) \cdot \frac{u_i(\omega, X_{-i}(\omega), X'_{i,\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{\varepsilon} \\
= &\lim_{\varepsilon \to 0+} 1_{X_i>0}(\omega) \cdot \frac{u_i(\omega, X_{-i}(\omega), X'_{i,-\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{-\varepsilon} \\
= &\ 1_{X_i>0}(\omega) \cdot \frac{d}{dx_i} u_i(\omega, X(\omega)).
\end{aligned} \tag{23}$$

Next, for every ω the function $u_i(\omega, x)$ is concave in the variable x_i, and hence for any $\varepsilon \in \left(0, \frac{a}{2}\right)$ and $\omega \in \Omega$

$$\begin{aligned}
&\left|\frac{u_i(\omega, X_{-i}(\omega), X'_{i,\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{\varepsilon}\right| \\
\leq &\ \max\{\left|\frac{d}{dx_i} u_i(\omega, X(\omega))\right|, \left|\frac{d}{dx_i} u_i(\omega, X_{-i}(\omega), X_{i,\varepsilon}(\omega))\right|\}
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&\left|\frac{u_i(\omega, X_{-i}(\omega), X'_{i,-\varepsilon}(\omega)) - u_i(\omega, X(\omega))}{-\varepsilon}\right| \\
\leq &\ \max\{\left|\frac{d}{dx_i} u_i(\omega, X(\omega))\right|, \left|\frac{d}{dx_i} u_i(\omega, X_{-i}(\omega), X_{i,-\varepsilon}(\omega))\right|\}.
\end{aligned} \tag{25}$$

As X_i is bounded by some b (as argued in the proof of Proposition 1), and $\inf \sum_{i \in N} X_i := a > 0$ by assumption (implying in particular that $\bar{X}_{-i} + X_{i,-\varepsilon} \geq \frac{a}{2}$ for $\varepsilon \in \left(0, \frac{a}{2}\right)$), it follows from (20) that the right-hand side functions in both (24) and (25) are bounded from above by $\frac{4b+2a}{a^2}\overline{v} + c'(b + \frac{a}{2})$ when $\varepsilon \in \left(0, \frac{a}{2}\right)$. Using this fact, (23), and the conditional dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{\varepsilon \to 0+} E\left[1_{X_i>0}(\cdot) \times \frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{\varepsilon} \mid \mathcal{F}_i\right] \\
= &\lim_{\varepsilon \to 0+} E\left[1_{X_i>0}(\cdot) \times \frac{u_i(\cdot, X_{-i}(\cdot), X'_{i,-\varepsilon}(\cdot)) - u_i(\cdot, X(\cdot))}{-\varepsilon} \mid \mathcal{F}_i\right] \\
= &\ E\left[1_{X_i>0}(\cdot) \times \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i\right].
\end{aligned} \tag{26}$$

As $1_{X_i>0}$ is $\mathcal{F}_i$-measurable and can be extracted from the expectation, by using (**??**) – with all three terms multiplied by $1_{X_i>0}$ – and (26), we obtain

$$E\left[1_{X_i>0}(\cdot) \times \frac{d}{dx_i} u_i(\cdot, X(\cdot)) \mid \mathcal{F}_i\right] = 0,$$

which is the desired (11). ■

Proof Remark 4. We show that if either conditions (i) or (ii) hold, then in any equilibrium of a classic Tullock contest X the inequality $\inf \sum_{j=1}^{n} X_j > 0$ holds, and hence the conclusion of Remark 4 follows from Proposition 9.

Case (i). As $\sum_{i \in N} X_i$ is measurable w.r.t. $\vee_{i \in N} \mathcal{F}_i$ (the smallest σ-field containing each $\mathcal{F}_i$), which is finite, the probabilities $p\left(\sum_{i \in N} X_i \geq a\right)$ can take only finitely many values in $[0,1]$. Let $\delta = \max_{a>0} p\left(\sum_{i \in N} X_i \geq a\right)$, and suppose that it is attained at $a_0 > 0$. By Remark 1, $\sum_{i \in N} X_i > 0$ in any equilibrium X, and hence $\lim_{a \searrow 0} p\left(\sum_{i \in N} X_i \geq a\right) = p\left(\sum_{i \in N} X_i > 0\right) = 1$. Therefore $\delta = 1$ and a_0 is the desired bound for the equilibrium sum of efforts.

Case (ii). Assume w.l.o.g. that player 2 has an information advantage over player 1. Write $\underline{v} := \inf V$ and $\bar{v} = \sup V$, and let $\varepsilon > 0$ be such that $c(3\varepsilon) < \frac{\underline{v}}{4}$. (Such value exists since $c(0) = 0$ and c is continuous at 0.) Also, let $a \in (0, \varepsilon)$ be such that

$$\frac{2a}{\varepsilon + 2a} < \frac{\left[c(\varepsilon) - c(\frac{\varepsilon}{2})\right]}{\overline{v}}.$$

(Such value exists because the left-hand side vanishes when $a \to 0+$, while the right-hand side is positive.) Now consider an equilibrium X in the contest. We will show that $X_1 \geq a$. Assume by the way of contradiction that this is false. Then there exists a positive-measure set $A_1 \in \mathcal{F}_1$ such that $X_1 < a$ on A_1. We show that $X_2 \leq \varepsilon$ a.e. on A_1.

Indeed, suppose to the contrary that $X_2 > \varepsilon$ on some positive-measure $A_2 \in \mathcal{F}_2$ which is a subset of A_1. Consider a strategy

$$X_2' = \frac{\varepsilon}{2} \cdot 1_{A_2} + X_i \cdot 1_{\Omega \setminus A_2}$$

in S_2. Then, by switching from X_2 to X_2', player 2 decreases his expected reward by at most $\frac{2a}{\varepsilon+2a} \overline{v} \cdot p(A_2)$, and simultaneously decreases his expected cost by at least $\left[c(\varepsilon) - c(\frac{\varepsilon}{2})\right] \cdot p(A_2)$. By the choice of a, the first expression is smaller than the second, and hence deviating to X_2' is, in expectation, profitable for player 2, in contradiction to X being an equilibrium.

It follows that $\max\{X_1, X_2\} \leq \varepsilon$ a.e. on A_1. Let i be a player for whom $E(\rho_i(X) \mid A_1) \leq \frac{1}{2}$, and consider a strategy

$$X_i'' = 3\varepsilon \cdot 1_{A_1} + X_i \cdot 1_{\Omega \setminus A_1}$$

Since $A_1 \in \mathcal{F}_1 \subset \mathcal{F}_2$, X_i'' is measurable w.r.t. both $\mathcal{F}_1$ and $\mathcal{F}_2$. Hence $X_i'' \in S_i$. Notice that $\rho_i(X) \leq \rho_i(X_{-i}, X_i'')$ a.e. on A_1, and that $E(\rho_i(X_{-i}, X_i'') \mid A_1) \geq \frac{3}{4}$ (this is due to the fact that, a.e. on A_1, $\rho_i(X_{-i}, X_i'') \geq \frac{3\varepsilon}{3\varepsilon+\varepsilon} = \frac{3}{4}$). Thus, by switching from X_i to X_i'' player i improves his expected reward by at least $\frac{1}{4}\underline{v} \cdot p(A_1)$, while incurring an expected cost increase of at most $c(3\varepsilon) \cdot p(A_1)$. By the choice of ε, such a deviation leads to a net gain in the expected utility, in contradiction to X being an equilibrium. We conclude that, indeed, $X_1 \geq a$. ■

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