

# Supplementary Material for “Credit and Inflation under Borrower’s Lack of Commitment” \*

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## Abstract

In this document we present three extensions of the basic model. First, we show that redistributive money transfers of money crowd out privately supplied credit and reduce risk sharing and welfare. Second, we introduce idiosyncratic uncertainty in labor productivity and show that existence of equilibrium requires the government moving away from the Friedman rule. Finally, we show that our basic result—deflation crowds out credit completely also holds when households value leisure.

**Keywords:** Monetary policy, existence of credit, Friedman rule, self-enforcing debt, risk sharing.

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# 1 Redistributive money injections

To simplify the exposition, we focus on the case in which the efficient allocation can be restored with monetary policy; that is, Assumption 2 in Díaz and Perera-Tallo (2010) holds. Money injections are redistributive:

**Assumption 3.**  $\tau_h < w_h/\bar{w}$ , and  $\tau_l > w_h/\bar{w}$ ,  $\tau_h/2 + \tau_l/2 = 1$ . We denote as  $\omega_i(\Psi)$  the household's income, after money injections, in units of the next period consumption good,

$$\omega_i(\Psi) = \frac{w_i + \theta \tau_i \mathbf{m}}{1 + \theta}. \quad (1.1)$$

The next proposition establishes the range of transfers for which there is credit in equilibrium

**Proposition 6.** *If  $\tau_h \in [\frac{\chi}{\bar{w}}, \frac{w_h}{\bar{w}})$  there is credit when the inflation rate is positive. If  $\tau_h \in [1, \frac{\chi}{\bar{w}})$  there is  $\hat{\theta}(\tau_h) > 0$  such that there is credit for moderate levels of inflation,  $\theta \in [0, \tilde{\theta}(\tau_h))$ , where  $\tilde{\theta}(\tau_h)$  is an increasing function of  $\tau_h$ .*

*Proof.* Let us write the household's problem:

$$\begin{aligned} V(w_i, b, d, \Psi) = & \max \{ \beta u(c') + \beta V(w_{-i}, b', d', \Psi) \} \\ \text{s. t.} & (1 + \theta)(c' + d') + q(\Psi) b' \leq (1 + \theta)\omega_i(\Psi) + b + d, \\ & c' \geq 0, \\ & d' \geq 0, \\ & V(w_{-i}, b', d', \Psi) \geq V_D(w_{-i}, 0, \Psi). \end{aligned} \quad (1.2)$$

The default problem is

$$\begin{aligned} V_D(w_i, a, \Psi) = & \max \{ \beta u(z') + \beta V_D(w_{-i}, a', \Psi) \} \\ \text{s. t.} & (1 + \theta)(z' + a') \leq (1 + \theta)\omega_i(\Psi) + a, \\ & z' \geq 0, \\ & a' \geq 0. \end{aligned} \quad (1.3)$$

It is straight forward to show that when  $\mathbf{m} = \bar{w}$ ,

$$V(w_i, b, d, \Psi) = V\left(\omega_i(\Psi), b, d, \left(\theta, \frac{\omega_i(\Psi)}{\bar{w}}\right)\right), \quad (1.4)$$

$$V_D(w_i, 0, \Psi) = V_D\left(\omega_i(\Psi), 0, \left(\theta, \frac{\omega_i(\Psi)}{\bar{w}}\right)\right). \quad (1.5)$$

We already have shown in Proposition 2 in Díaz and Perera-Tallo (2010) that there is an incentive compatible steady state with a positive amount of credit when  $w_h > \chi$ , transfers are neutral,

$\tau_h = \frac{w_h}{\bar{w}}$ , and  $\theta \geq 0$ . Thus, there must be an incentive compatible steady state when transfers are redistributive and  $\omega_i(\Psi) > \chi$ . Moreover, the consumption allocation and the price of the bond must be the same that their respective counterparts in the incentive compatible steady state with neutral transfers and  $w_h > \chi$ . Following the same argument, by Corollary 2 in Díaz and Perera-Tallo (2010), if  $\omega_i(\Psi) \leq \chi$ , there is no credit. Since, by Assumption 2 in Díaz and Perera-Tallo (2010), we know that  $w_h > \chi$ , it follows that:

If  $\tau_h \geq \frac{\chi}{\bar{w}}$  and  $\theta \geq 0$ , then  $\omega_i(\Psi) > \chi$ . Thus there is an incentive compatible steady state with a positive amount of credit.

If  $\tau_h \in [1, \frac{\chi}{\bar{w}})$ , there exists  $\tilde{\theta}(\tau_h) > 0$  such that  $\omega_h(\tilde{\theta}(\tau_h), \tau_h) = \chi$ . For  $\theta \in [0, \tilde{\theta}(\tau_h))$ ,  $\omega_i(\Psi) > \chi$  and there is credit in equilibrium. For any  $\theta \geq \tilde{\theta}(\tau_h)$ , the  $\omega_i(\Psi) \leq \chi$  and credit cannot be sustained in equilibrium.  $\square$

**Proposition Supp. 1.** Consider any transfer scheme such that  $\tau_h \in [\frac{\chi}{\bar{w}}, \frac{w_h}{\bar{w}})$ . For any  $\theta \geq 0$ , the utility of a high productivity defector,  $V_D(h, 0, \Psi)$  decreases with  $\tau_h$ . There exists a strictly positive rate of inflation  $\theta^D(\tau_h) > 0$  such that for any  $\theta \in [0, \theta^D(\tau_h))$  the utility of defecting decreases with the level of inflation and increases afterwards. There exists  $\hat{\theta}(\tau_h) > \theta^D(\tau_h)$  such that for any  $\theta \geq \hat{\theta}(\tau_h)$ ,  $z_l = \omega_h(\Psi)$ . The utility of defecting when  $\theta$  is arbitrarily large converges to  $\frac{\beta u(\tau_h \bar{w}) + \beta^2 u(\tau_l \bar{w})}{1 - \beta^2}$ .

*Proof.* Using Proposition App. 1 in Díaz and Perera-Tallo (2010) we know that for any  $\theta > 0$  aggregate money balances are  $\mathbf{m} = \bar{w}$ . Using Proposition App. 2 in Díaz and Perera-Tallo (2010) we can write the problem solved by a high productivity defector as

$$\begin{aligned} V_D(h, 0, \Psi) = & \max_{z_l, z_h, a_l} \beta \frac{u(z_l)}{1 - \beta^2} + \beta^2 \frac{u(z_h)}{1 - \beta^2} \\ \text{s. t.} & (1 + \theta)z_h + z_l \leq \omega_h(\Psi) + (1 + \theta)\omega_l(\Psi), \\ & z_h \geq \omega_l(\Psi), \\ & \omega_i(\Psi) = \frac{w_i + \theta \tau_i \bar{w}}{1 + \theta}. \end{aligned} \tag{1.6}$$

The first order condition of the problem shown above for an interior solution,  $a_l(\theta, \tau_h) > 0$ , is:

$$(1 + \theta) u'(\omega_h(\Psi) - a_l(\theta, \tau_h)) = \beta u' \left( \omega_l(\Psi) + \frac{a_l(\theta, \tau_h)}{1 + \theta} \right) \tag{1.7}$$

In section 4.4 in Díaz and Perera-Tallo (2010) we show that  $V_D(h, 0, (0, \tau_h)) > \beta u(\bar{w}) / (1 - \beta)$  at  $\theta = 0$ . Strict concavity of the utility function ensures that there exists  $\hat{\theta}(\tau_h)$  such that for all  $\theta \geq \hat{\theta}(\tau_h)$ , defectors are liquidity constrained and they are in autarky,  $z_i(\theta, \tau_h) = \omega_{-i}(\theta, \tau_h)$ , where  $\hat{\theta}(\tau_h)$  satisfies

$$\frac{u'(\omega_h(\hat{\theta}(\tau_h), \tau_h))}{\beta u'(\omega_l(\hat{\theta}(\tau_h), \tau_h))} = \frac{1}{1 + \hat{\theta}(\tau_h)}, \tag{1.8}$$

where the left hand side part of the above equality (the marginal rate of substitution) is an increasing function of  $\theta$  due to the fact that  $\omega_h(\theta, \tau_h)$  decreases with  $\theta$ ,  $\omega_l(\theta, \tau_h)$  increases with  $\theta$ , and utility is strictly concave. Since  $\tau_h \geq \varepsilon/\bar{w}$  and, by Assumption 2 in Díaz and Perera-Tallo (2010),  $w_h > \varepsilon$ , then, for all  $\theta \geq \hat{\theta}(\tau_h)$ ,  $\omega_h(\theta, \tau_h) > \varepsilon$ . Since the allocation  $(\varepsilon, 2\bar{w} - \varepsilon)$  yields the same utility than the full commitment allocation, strict concavity of the utility function ensures that, for all  $\theta \geq \hat{\theta}(\tau_h)$ ,  $V_D(h, 0, (\theta, \tau_h)) < \beta u(\bar{w})/(1 - \beta)$ . For all  $\theta \in (0, \hat{\theta}(\tau_h))$ , defectors are not liquidity constrained,  $a_l(\theta, \tau_h) = z_h - \omega_l(\Psi) > 0$ , hence, the derivative of  $V_D(h, 0, (\theta, \tau_h))$  with respect to the inflation rate is (using the Envelope Theorem) is

$$\frac{\partial V_D(h, 0, (\theta, \tau_h))}{\partial \theta} = \frac{\beta u'(z_l)}{(1 - \beta^2)} \left[ \theta \frac{\partial \omega_l(\Psi)}{\partial \theta} - \frac{a_l(\theta, \tau_h)}{1 + \theta} \right]. \quad (1.9)$$

For  $\theta = 0$  the derivative shown in (1.9) is negative. For  $\theta = \hat{\theta}(\tau_h)$  the derivative is positive. Hence, continuity ensures that there exists an odd number of points at which  $\frac{\partial V_D(h, 0, (\theta, \tau_h))}{\partial \theta} = 0$ . Using the Envelope Theorem, the second derivative at those points is

$$\frac{\partial^2 V_D(h, 0, (\theta, \tau_h))}{\partial \theta^2} = \frac{\beta u'(z_l)}{(1 - \beta^2)} \left[ \theta \frac{\partial^2 \omega_l(\Psi)}{\partial \theta^2} + \frac{1 + 2\theta}{1 + \theta} \frac{\partial \omega_l(\Psi)}{\partial \theta} - \frac{1}{1 + \theta} \frac{\partial a_l(\theta, \tau_h)}{\partial \theta} \right]. \quad (1.10)$$

Differentiating (1.7) with respect to  $\theta$  and assuming that  $\frac{\partial V_D(h, 0, (\theta, \tau_h))}{\partial \theta} = 0$  we find that

$$\frac{\partial a_l(\theta, \tau_h)}{\partial \theta} = \frac{u'(z_l)}{(1 + \theta)u''(z_l) + \frac{\beta}{1 + \theta}u''(z_h)} - \frac{\partial \omega_l(\Psi)}{\partial \theta} < 0, \quad (1.11)$$

ensuring that  $\frac{\partial^2 V_D(h, 0, (\theta, \tau_h))}{\partial \theta^2} > 0$  at all the points for which the first derivative shown in (1.9) is zero. This implies that there exists a unique  $\theta^D(\tau_h) \in (0, \hat{\theta}(\tau_h))$  for which the derivative shown in (1.9) is zero. Hence,  $V_D(h, 0, (\theta, \tau_h))$  is greater than the utility yielded by the full commitment allocation for  $\theta = 0$ , it decreases for any  $\theta \in (0, \theta^D(\tau_h))$  and it increases afterwards. For any  $\theta \geq \hat{\theta}(\tau_h)$  the default allocation is  $(\omega_h(\Psi), \omega_l(\Psi))$ , which is feasible,  $\omega_h(\Psi) + \omega_l(\Psi) = 2\bar{w}$ , and it converges to  $(\tau_h \bar{w}, \tau_l \bar{w})$  as  $\theta$  becomes arbitrarily large. Hence, for all  $\theta > 0$ ,  $V_D(h, 0, (\theta, \tau_h)) < V_D(h, 0, (0, \tau_h))$ .

Finally, it is easy to check that total resources  $\omega_h(\Psi) + (1 + \theta)\omega_l(\Psi)$  decrease with  $\tau_h$  (recall that  $\tau_l = 2 - \tau_h$ ) when  $\theta > 0$ . Thus, given  $\theta$ , the utility of defecting decreases with  $\tau_h$ .  $\square$

**Proposition 7.** *Consider any transfer scheme such that  $\tau_h \in [\frac{\varepsilon}{\bar{w}}, \frac{w_h}{\bar{w}})$ . Then, there exists  $\theta^*(\tau_h) > 0$  such that  $V_D(h, 0, (\theta^*(\tau_h), \tau_h)) = \frac{\beta u(\bar{w})}{1 - \beta}$ , and for all  $\theta > \theta^*(\tau_h)$ ,  $V_D(h, 0, (\theta, \tau_h)) < \frac{\beta u(\bar{w})}{1 - \beta}$ . The optimal inflation rate  $\theta^*(\tau_h)$  decreases with  $\tau_h$ .*

*Proof.* In section 4.4 in Díaz and Perera-Tallo (2010) we have shown that at  $\theta = 0$  the defector's value function satisfies  $V_D(h, 0, \Psi) > \beta u(\bar{w})/(1 - \beta)$ . By Proposition Supp. 1 we know that  $V_D(h, 0, \Psi)$  has a minimum for a given  $\theta^D(\tau_h) > 0$ . Moreover,  $\lim_{\theta \rightarrow \infty} V_D(h, 0, \Psi) = \frac{\beta u(\tau_h \bar{w})}{1 - \beta^2} +$

$\frac{\beta^2 u(\tau_l \bar{w})}{1-\beta^2}$ . Since  $\tau_h \geq \varepsilon/\bar{w}$ , by strict concavity of the utility function,  $\frac{\beta u(\tau_h \bar{w})}{1-\beta^2} + \frac{\beta^2 u(\tau_l \bar{w})}{1-\beta^2} \leq \frac{\beta u(\bar{w})}{1-\beta}$ . Hence, for any  $\theta \geq \theta^D(\tau_h)$ ,  $V_D(h, 0, \Psi) \leq \frac{\beta u(\bar{w})}{(1-\beta)}$ . Thus, there exists  $0 < \underline{\theta}^*(\tau_h) < \theta^D(\tau_h)$  for which  $V_D(h, 0, (\underline{\theta}^*(\tau_h), \tau_h)) = \frac{\beta u(\bar{w})}{(1-\beta)}$ . Since  $V_D(h, 0, (\theta, \tau_h))$  decreases with  $\tau_h$  for any  $\theta > 0$ , then  $\underline{\theta}^*(\tau_h)$  decreases with  $\tau_h$ .  $\square$

**Proposition 8.** *Consider any transfer scheme such that  $\tau_h \in [\frac{\chi}{\bar{w}}, \frac{\varepsilon}{\bar{w}}]$ . Then, there exists  $0 < \underline{\theta}^*(\tau_h) < \bar{\theta}^*(\tau_h)$  such that for all  $\theta \in [\underline{\theta}^*(\tau_h), \bar{\theta}^*(\tau_h)]$  the incentive compatible steady state equilibrium is such that  $c_l = c_h = \bar{w}$ , where  $\underline{\theta}^*(\tau_h)$  decreases with  $\tau_h$ , and  $\bar{\theta}^*(\tau_h)$  increases with  $\tau_h$ .*

*Proof.* In section 4.4 in Díaz and Perera-Tallo (2010) we have shown that at  $\theta = 0$  the defector's value function satisfies  $V_D(h, 0, \Psi) > \beta u(\bar{w})/(1-\beta)$ . By Proposition Supp. 1 we know that  $V_D(h, 0, \Psi)$  has a minimum for a given  $\theta^D(\tau_h) > 0$ . Moreover,  $\lim_{\theta \rightarrow \infty} V_D(h, 0, \Psi) = \frac{\beta^2 u(\tau_h \bar{w})}{1-\beta^2} + \frac{\beta^2 u(\tau_l \bar{w})}{1-\beta^2}$ . Since  $\tau_h \in [\chi/\bar{w}, \varepsilon/\bar{w}]$ , by strict concavity of the utility function,

$$\frac{\beta u(\bar{w})}{1-\beta} < \frac{\beta u(\tau_h \bar{w})}{1-\beta^2} + \frac{\beta^2 u(\tau_l \bar{w})}{1-\beta^2} \leq \frac{\beta u(\chi)}{1-\beta^2} + \frac{\beta^2 u(2\bar{w} - \chi)}{1-\beta^2}. \quad (1.12)$$

Thus, there exist two inflation rates  $\underline{\theta}^*(\tau_h)$  and  $\bar{\theta}^*(\tau_h)$ , that satisfy  $0 < \underline{\theta}^*(\tau_h) < \bar{\theta}(\tau_h) < \bar{\theta}^*(\tau_h)$ , for which  $V_D(h, 0, \Psi) = \frac{\beta u(\bar{w})}{(1-\beta)}$ . Since  $V_D(h, 0, (\theta, \tau_h))$  decreases with  $\tau_h$  for any  $\theta > 0$ , and decreases with  $\theta$  for any  $\theta < \bar{\theta}(\tau_h)$ , then  $\underline{\theta}^*(\tau_h)$  decreases with  $\tau_h$ . Since  $V_D(h, 0, (\theta, \tau_h))$  decreases with  $\tau_h$  for any  $\theta > 0$ , and increases with  $\theta$  for any  $\theta > \bar{\theta}(\tau_h)$ ,  $\bar{\theta}^*(\tau_h)$  increases with  $\tau_h$ .  $\square$

**Corollary 3.** *Consider any transfer scheme such that  $\tau_h \in [1, \frac{w_h}{\bar{w}}]$ . Then, given  $\theta > 0$ , welfare is an increasing function of  $\tau_h$ .*

*Proof.* The consumption allocation,  $(c_l, 2\bar{w} - c_l)$ , at the incentive compatible symmetric steady state satisfies

$$u(c_l) + \beta u(2\bar{w} - c_l) = \max \{(1 + \beta)u(\bar{w}), u(z_l) + \beta u(z_h)\}, \quad (1.13)$$

where  $(z_l, z_h)$  is the allocation chosen by the high productivity defector. By Proposition Supp. 1 it is easy to show that the utility of defecting increases with higher redistribution, i.e., lower  $\tau_h$ , given  $\theta > 0$ . Hence  $u(z_l) + \beta u(z_h)$  increases and  $u(c_l) + \beta u(2\bar{w} - c_l)$  must increase, which implies that  $c_l$  increases, too. Since the highest welfare is attained for  $c_l = c_h$ , strict concavity of the utility function and the welfare function imply that welfare must fall.  $\square$

## 2 Idiosyncratic uncertainty

Here we modify our benchmark model economy so that agents' productivity can be represented as an idiosyncratic shock. We assume that productivity at period  $t$  is drawn from the set  $\mathbf{W} = \{w_1, \dots, w_n\}$ , where  $w_i < w_{i+1}$ , for any  $i = 1, \dots, n$ , and  $w_1 > 0$ . The shock is Markov with transition matrix  $\Pi = (\pi_{ij})_{n \times n}$ . The associated stationary distribution of productivities is the vector  $(\varsigma_1, \dots, \varsigma_n)$ . We assume that there are complete markets except for the fact that households cannot commit to repaying their debts. That is, households trade Arrow securities in bonds. Notice, however, that money balances are not contingent. The following assumption ensures that trade is always welfare improving,

**Assumption Supp. 1.**

$$(u(\bar{w}), \dots, u(\bar{w})) H > (u(w_1), \dots, u(w_n)) H, \quad (2.1)$$

where  $H$  is the matrix  $(I - \beta \Pi)^{-1}$  and  $\bar{w} = \langle \varsigma, (w_1, \dots, w_n) \rangle$ .

This assumption says that autarky gives lower utility than does full risk sharing. It is the natural extension of Assumption 2 in Díaz and Perera-Tallo (2010). Moreover, we assume that transfers are neutral,  $\tau_i = w_i/\bar{w}$ . The following assumption will simplify our analysis,

**Assumption Supp. 2.** *The elements of the transition matrix  $\Pi$  satisfy  $\pi_{ij} \in (0, 1)$  for all  $i, j$ .*

This assumption implies that the transition matrix has a unique ergodic set and it is the matrix itself. As in our benchmark economy, default entails being banned from the credit market. Defaulters can only save using money balances.

### 2.1 The household's problem at the financial markets stage

To simplify the exposition, we will write only the inflation rate as the aggregate state variable  $\Psi$ , since we assume throughout this section that the transfer policy,  $\tau_h$ , does not change. The problem solved by a household of productivity level  $i$  is

$$\begin{aligned} V(i, b, d, \theta) &= \max_{c'_j, b'_s, d'_j} \sum_{j=1}^n \pi_{ij} \left[ \beta u(c'_j) + \beta V(j, b'_j, d'_j, \theta) \right] \\ \text{s. t.} \quad &(1 + \theta) c'_j + \sum_{s=1}^n q(i, s) b'_s + (1 + \theta) d'_j \leq w_i \left(1 + \theta \frac{m}{\bar{w}}\right) + b + d, \text{ for all } j, \\ &V(j, b'_j, d'_j, \theta) \geq V_D(j, 0, \theta), \text{ for all } j, \\ &d'_j \geq 0, \text{ for all } j. \end{aligned} \quad (2.2)$$

where  $V_D(i, a, \theta)$  satisfies

$$\begin{aligned}
V_D(i, a, \theta) &= \max_{\substack{z'_j \geq 0 \\ a'_j \geq 0}} \sum_{j=1}^n \pi_{ij} \left[ \beta u(z'_j) + \beta V_D(j, a'_j, \theta) \right] \\
\text{s. t.} \quad & (1 + \theta) z'_j + (1 + \theta) a'_j \leq w_i \left( 1 + \theta \frac{m}{w} \right) + a, \text{ for all } j, \\
& a'_j \geq 0, \text{ for all } j.
\end{aligned} \tag{2.3}$$

## 2.2 Equilibrium

We denote as  $X = \{\mathbf{W} \times \mathbf{R} \times [0, \bar{d}]\}$  the set where the individual state variable  $x$  can take values. The set where the defector's state variable takes place is  $X^D = \{\mathbf{W} \times [0, \bar{d}]\}$ . We assume that  $\bar{d}$  is sufficiently large. We construct a Markov process for the individual state variables from the Markov process on the shocks and the decision rules of the agents (see Huggett 1993 or Hopenhayn and Prescott 1992 for details). Let  $\mathcal{B}$  be the  $\sigma$ -algebra generated in  $X$  by, say, the open intervals. A probability measure  $\mu$  over  $\mathcal{B}$  exhaustively describes the economy by stating how many households there are of each type. Let  $P(x, B)$  denote the probability that a type  $\{x\}$  has of becoming of a type in  $B \subset \mathcal{B}$ . The function  $P$  naturally describes how the economy moves over time by generating a probability measure for tomorrow,  $\mu'$ , given a probability measure,  $\mu$ , today. The exact way in which this occurs is

$$\mu'(B) = \int_X P(x, B) d\mu. \tag{2.4}$$

To find a steady state, we look for the measure of households  $\mu$  such that given the prices implied by that measure, households actions reproduce the same measure  $\mu$  in the following period.

**Definition Supp. 1.** *An incentive compatible steady state equilibrium for this economy is a monetary policy,  $\theta$ , an aggregate amount of money balances  $\mathbf{m}$ , a set of bond prices,  $q(j, s)$ , for all  $j, s = 1, \dots, n$ , a set of functions  $\left\{ V(x, \theta), \left( g^{c_j}(x, \theta), g^{b_j}(x, \theta), g^{d_j}(x, \theta) \right)_{j=1}^n \right\}$ ,  $\left\{ V_D(x^D, \theta), \left( g^{z_j}(x^D, \theta), g^{a_j}(x^D, \theta) \right)_{j=1}^n \right\}$  and a measure of households  $\mu$  such that:*

1. *given  $\mu, \theta, q(j, s), \mathbf{m}$ , and the functions  $\left\{ V(x, \theta), \left( g^{c_j}(x, \theta), g^{b_j}(x, \theta), g^{d_j}(x, \theta) \right)_{j=1}^n \right\}$  solve the household's problem, shown in (2.2),*
2.  *$\left\{ V_D(x^D, \theta), \left( g^{z_j}(x^D, \theta), g^{a_j}(x^D, \theta) \right)_{j=1}^n \right\}$  solve the default problem, shown in (2.3),*
3. *markets clear,*

$$(a) \int_X \left( \sum_{j=1}^n g^{c_j}(i, b, d) \pi_{ij} \right) d\mu = \int_X w d\mu,$$

$$(b) \int_X \left( \sum_{j=1}^n g^{bj}(i, b, d) \pi_{ij} \right) d\mu = 0,$$

$$(c) \mathbf{m} = \int_X \left( \sum_{j=1}^n (g^{cj}(i, b, d) + g^{dj}(i, b, d)) \pi_{ij} \right) d\mu,$$

4. and the measure of household is stationary,  $\mu(B) = \int_X P(x, B) d\mu$ , for all  $B \subset \mathcal{B}$ .

## 2.3 Properties of symmetric steady states

**Proposition 9.** *In a symmetric steady state the consumption allocation is the efficient allocation,  $c_j = \bar{w}$ , for all  $j$ . The amount of precautionary savings does not depend on the productivity state,  $d_i = d \geq 0$ , for all  $i = 1, \dots, n$ . If  $1 + \theta > \beta$ , there are no precautionary money balances,  $d = 0$ .*

*Proof.* A symmetric steady state is characterized by the following set of equations:

$$q(j, s) \sum_{\kappa=1}^n \pi_{j\kappa} u'(c_\kappa) \geq \beta \pi_{js} \sum_{\kappa=1}^n \pi_{s\kappa} u'(c_\kappa), \text{ for all } j, s = 1, \dots, n, \quad (2.5)$$

$$(1 + \theta) u'(c_s) \geq \beta \sum_{\kappa=1}^n \pi_{s\kappa} u'(c_\kappa), \text{ for all } s = 1, \dots, n, \quad (2.6)$$

$$d_s \geq 0, \text{ for all } s = 1, \dots, n, \quad (2.7)$$

$$\sum_{j=1}^n \varsigma_j b_j = 0, \quad (2.8)$$

$$\mathbf{m} = \bar{w} + \sum_{j=1}^n \varsigma_j d_j, \quad (2.9)$$

$$(1 + \theta)(c_i + d_i) + \sum_{s=1}^n q(j, s) b_s = w_j \left( 1 + \theta \frac{\mathbf{m}}{\bar{w}} \right) + b_j + d_j, \text{ for all } j, i = 1, \dots, n. \quad (2.10)$$

Suppose that there exists a symmetric steady state in which consumption varies across states. Without loss of generality assume that there exist two states,  $j_1$  and  $j_2$ , for which  $c_{j_2} < c_{j_1}$ . Lets us denote as  $d_{j_1}$  and  $d_{j_2}$  the amount of precautionary savings associated to each state. Using the budget constraint shown in (2.10) we know that  $c_{j_1} + d_{j_1} = c_{j_2} + d_{j_2}$ . Therefore,  $d_{j_2} > d_{j_1} \geq 0$ . Since  $d_{j_2} > 0$  the first order condition shown in (2.6) should hold with strict equality. Since  $\beta \leq 1 + \theta$ , it follows that

$$u'(c_{j_2}) \leq \sum_{\kappa=1}^n \pi_{j_1\kappa} u'(c_\kappa). \quad (2.11)$$

Strict concavity of the instantaneous utility function, implies that  $u'(c_{j_1}) < u'(c_{j_2})$ . Therefore, there must exist  $j_3$  such that  $u'(c_{j_2}) < u'(c_{j_3})$ , which implies that  $d_{j_3} > d_{j_2}$ . Following the

argument and using Assumption Supp. 2, since the set of possible productivity estates is countable we should find  $d_{j_1} > d_{j_2}$  and we arrive to a contradiction. Therefore,  $c_j = c$  for any productivity state  $j$ . Applying market clearing conditions it should be the case that  $c_j = \bar{w}$  for all  $j = 1, \dots, n$ . This implies that the amount of precautionary money balances are constant across states,  $d_j = d$ , for all  $j$ . The first order condition (2.6) implies that if the inflation rate is higher than the rate of time preference,  $1 + \theta > \beta$ , the amount of precautionary money balances at the symmetric steady state is zero,  $d_j = 0$ , for all  $j = 1, \dots, n$ .  $\square$

As in the case of certainty, the efficient allocation is the one that maximizes the social planner's problem and implies full risk sharing,  $c_t = \bar{w}$ , for all households. Let us now turn to the decentralization of the full risk sharing allocation with full commitment. The household's problem is the one shown in (2.2) except for the incentive compatibility constraint, which has to be substituted by a non Ponzi scheme constraint. As in the deterministic case, we will focus our attention on symmetric steady states.

**Corollary 4.** *There is a unique symmetric steady state with full commitment in which there is full risk sharing,  $c_i = \bar{w}$  and the price of a contingent bond is  $q(i, j) = \beta \pi_{ij}$ , for all  $i, j = 1, \dots, n$ .*

*Proof.* It follows from Proposition 9, the first order condition shown in (2.5) with strict equality and equations (2.6)–(2.10).  $\square$

**Corollary 5.** *If there exists a symmetric incentive compatible steady state it is the full commitment symmetric steady state.*

*Proof.* Proposition 9 tells us that in a symmetric steady state the consumption allocation is the full risk sharing allocation, regardless of the existence of either full or limited commitment. Thus, if the full risk sharing allocation is not incentive compatible there is no other symmetric incentive compatible steady state.  $\square$

## 2.4 Sustainability of credit and efficiency

Next, we show some properties of the utility of defaulting for a high productivity defector.

**Lemma Supp. 1.** *Assume that either  $1 + \theta > \beta$  or  $\mathbf{m} = \bar{w}$ .  $V_D(i, a, \theta)$  is decreasing in  $\theta$ , for all  $\theta \geq \beta - 1$ .*

*Proof.* Let  $\theta_2 < \theta_1$ , and define

$$\begin{aligned}
W_1(i, a, (\theta_2, \theta_1)) &= \max_{\substack{z'_j \geq 0 \\ a'_j \geq 0}} \left\{ \sum_{j=1}^n \pi_{ij} \left[ \beta u(z'_j) + \beta V_D(j, a'_j, \theta_1) \right] \right\} \\
\text{s. t.} \quad & z'_j + a'_j \leq w_i + \frac{1}{1+\theta_2} a, \quad j = 1, \dots, n, \\
& a'_j \geq 0, \quad j = 1, \dots, n.
\end{aligned} \tag{2.12}$$

Clearly,  $W_1(i, a, (\theta_2, \theta_1)) > V_D(i, a, \theta_1)$ . Now define the sequence of functions  $\{W_\ell\}$  such that

$$\begin{aligned}
W_{\ell+1}(i, a, (\theta_2, \theta_1)) &= \max_{\substack{z'_j \geq 0 \\ a'_j \geq 0}} \left\{ \sum_{j=1}^n \pi_{ij} \left[ \beta u(z'_j) + \beta W_\ell(j, a'_j, (\theta_2, \theta_1)) \right] \right\} \\
\text{s. t.} \quad & z'_j + a'_j \leq w_i + \frac{1}{1+\theta_2} a, \quad j = 1, \dots, n, \\
& a'_j \geq 0, \quad j = 1, \dots, n.
\end{aligned} \tag{2.13}$$

Notice that  $V_D(i, a, \theta_2) > W_{\ell+1}(i, a, (\theta_2, \theta_1))$  and that  $W_{\ell+1}(i, a, (\theta_2, \theta_1)) > W_\ell(i, a, (\theta_2, \theta_1))$ , for all  $\ell$ . Hence, it follows that  $V_D(i, a, \theta_2) > V_D(i, a, \theta_1)$ .  $\square$

**Lemma Supp. 2.** *Assume that  $\mathbf{m} = \bar{w}$ . If  $\theta < -\frac{\bar{w}-w_1}{w_n-w_1}$ , then  $V_D(n, a, \theta) > \frac{\beta u(\bar{w})}{1-\beta}$ , for all  $\beta > 0$ ,  $a \geq 0$ .*

*Proof.* Let us assume that the household with high productivity has zero money balances at the time of default. If  $\theta < -\frac{\bar{w}-w_1}{w_n-w_1}$  it is possible to accumulate enough real money balances to ensure the full commitment level of consumption,  $\bar{w}$ , even if the productivity state is the lowest possible,  $w_1$ , thereafter. That is, the budget constraint at time  $t$  is

$$a_{t+1} = w_t + \frac{1}{1+\theta} a_t - z_t. \tag{2.14}$$

Let us assume that at  $t = 1$ , the period when the household defaults, the productivity is  $w_n$  and for any subsequent period  $t > 1$  is the lowest,  $w_1$ , and assume that  $z_t = \bar{w}$ . For this plan to be affordable it has to be the case that  $a_{t+1}$  in (2.14) is non negative. Assuming that at the period of default the household consumes  $\bar{w}$  we get that  $a_1 = w_n - \bar{w}$ . If  $\theta \leq -\frac{\bar{w}-w_1}{w_n-w_1}$  we can easily check that  $a_{t+1} \geq 0$ , for any  $t > 1$ . Therefore,  $V_D(n, 0, \theta) > \frac{\beta u(\bar{w})}{1-\beta}$ , or all  $\beta > 0$ . Since  $V_D(i, a, \theta)$  is increasing in the amount of real money balances, the result follows.  $\square$

**Corollary Supp. 1.** *Assume that  $\mathbf{m} = \bar{w}$ . If  $\theta < -\frac{\bar{w}-w_1}{w_n-w_1}$ , the full commitment allocation does not satisfy the incentive compatibility constraint for  $\beta > 0$ .*

The following Proposition ensures that there exists an inflation rate below which the full commitment allocation is not incentive compatible.

**Proposition Supp. 2.** *Assume that  $\mathbf{m} = \bar{w}$ . Given  $\beta > 0$ , there is  $\bar{\theta}(\beta)$  such that if  $\theta < \bar{\theta}(\beta)$  then the full commitment allocation does not satisfy the incentive compatibility constraint.*

*Proof.* Assumption Supp. 1, Lemma Supp. 1 and Lemma Supp. 2 imply that there exist  $\bar{\theta}(\beta)$  such that  $V_D(n, 0, \bar{\theta}(\beta)) = \beta u(\bar{w})/(1 - \beta)$ . Hence, using Lemma Supp. 1,  $V_D(n, 0, \theta) > \beta u(\bar{w})/(1 - \beta)$  for any  $\theta < \bar{\theta}(\beta)$ .  $\square$

**Corollary Supp. 2.** *Assume that  $\mathbf{m} = \bar{w}$ . Given  $\beta > 0$ , for any  $\theta < \bar{\theta}(\beta)$ , the full commitment allocation is not the incentive compatible steady state allocation.*

**Lemma Supp. 3.** *Let us, for ease of exposition, write the economy-wide state variable as  $(\theta, d)$ , where  $d \geq 0$  denotes aggregate money balances. Assume that  $\mathbf{m} = \bar{w} + d$ .  $V_D(i, a, (\theta, d))$  is decreasing in  $d$ , for  $\beta - 1 \geq \theta > 0$ .*

*Proof.* Let  $d_2 < d_1$ , and define

$$\begin{aligned}
 W_1(i, a, (\theta, d_2, d_1)) &= \max_{\substack{z'_j \geq 0 \\ a'_j \geq 0}} \left\{ \sum_{j=1}^n \pi_{ij} \left[ \beta u(z'_j) + \beta V_D(j, a'_j, (\theta, d_1)) \right] \right\} \\
 \text{s. t.} \quad & z'_j + a'_j \leq w_i \frac{1 + \theta \left(1 + \frac{d_2}{\bar{w}}\right)}{1 + \theta} + \frac{1}{1 + \theta} a, \quad j = 1, \dots, n, \\
 & a'_j \geq 0, \quad j = 1, \dots, n.
 \end{aligned} \tag{2.15}$$

Clearly,  $W_1(i, a, (\theta, d_2, d_1)) > V_D(i, a, (\theta, d_1))$ . Now define the sequence of functions  $\{W_\ell\}$  such that

$$\begin{aligned}
 W_{\ell+1}(i, a, (\theta, d_2, d_1)) &= \max_{\substack{z'_j \geq 0 \\ a'_j \geq 0}} \left\{ \sum_{j=1}^n \pi_{ij} \left[ \beta u(z'_j) + \beta W_\ell(j, a'_j, (\theta, d_2, d_1)) \right] \right\} \\
 \text{s. t.} \quad & z'_j + a'_j \leq w_i \frac{1 + \theta \left(1 + \frac{d_2}{\bar{w}}\right)}{1 + \theta} + \frac{1}{1 + \theta} a, \quad j = 1, \dots, n, \\
 & a'_j \geq 0, \quad j = 1, \dots, n.
 \end{aligned} \tag{2.16}$$

Notice that  $V_D(i, a, (\theta, d_2)) = \lim_{\ell \rightarrow \infty} W_\ell(i, a, (\theta, d_2, d_1))$ , and that  $W_{\ell+1}(i, a, (\theta, d_2, d_1)) > W_\ell(i, a, (\theta, d_2, d_1))$ , for all  $\ell$ . Hence, it follows that  $V_D(i, a, (\theta, d_2)) > V_D(i, a, (\theta, d_1))$ .  $\square$

Now we can show the main result of this section.

**Proposition 10.** *There exists a threshold level for the discount factor,  $\beta^* > 0$ , such that the full risk sharing allocation is not incentive compatible if  $\beta < \beta^*$  and one of the two following conditions is satisfied:*

1. *the inflation rate is bounded away from the Friedman rule and not too high,  $\theta \in (\beta - 1, \bar{\theta}(\beta))$ ,*
2. *the government follows the Friedman rule,  $\theta = \beta - 1$ , and the volume of precautionary savings is sufficiently low,  $0 \leq d < \bar{d}(\beta)$ .*

*Proof.* This proof has two parts. **(1)** Assume that  $\theta > \beta - 1$ , which implies that  $\mathbf{m} = \bar{w}$ . In this case we can apply Corollary Supp. 1, which implies that for any  $\beta \leq 1 - \frac{\bar{w}-w_1}{w_n-w_1}$  that  $V_D(n, 0, (\beta - 1, 0)) > \beta u(\bar{w})/(1 - \beta)$ . However, it could be the case that, for all  $\beta < 1$ , the inequality  $V_D(n, 0, (\beta - 1, 0)) > \beta u(\bar{w})/(1 - \beta)$  holds, case in which  $\beta^* = 1$ . If this were not the case, continuity of  $V_D(n, 0, (\beta - 1, 0))$  with respect to the discount factor implies that there is  $\beta^* > 0$  such that for all  $\beta < \beta^*$   $V_D(n, 0, (\beta - 1, 0)) > \beta u(\bar{w})/(1 - \beta)$ . Thus, if the efficient allocation does not satisfy the ICC, it cannot be the incentive compatible steady state allocation. By Proposition Supp. 2, for any  $\beta > 0$  there exists  $\bar{\theta}(\beta)$  for which  $V_D(n, 0, (\bar{\theta}(\beta), 0)) = \beta u(\bar{w})/(1 - \beta)$ , hence, it must be the case that  $\bar{\theta}(\beta) > \beta - 1$ , and that  $V_D(n, 0, (\theta, 0)) > \beta u(\bar{w})/(1 - \beta)$  for any  $\beta - 1 < \theta < \bar{\theta}(\beta)$ .

**(2)** Assume that  $\theta = \beta - 1$ . Since, for all  $\beta < \beta^*$ ,  $V_D(n, 0, (\beta - 1, 0)) > \beta u(\bar{w})/(1 - \beta)$ , continuity of  $V_D$  and Lemma Supp. 3 ensure that there exists  $\bar{d}(\beta) \geq 0$  such that for all  $d \in [0, \bar{d}(\beta)]$ ,  $V_D(n, 0, (\beta - 1, d)) > \beta u(\bar{w})/(1 - \beta)$ . Notice that if  $\beta^* = 1$ , then  $d(\beta^*) > 0$ .  $\square$

This Proposition, coupled with Corollary 5, implies that the government cannot restore efficiency by setting the inflation rate equal to the preference discount rate, unless the amount of precautionary money balances is large. That is, unless the price level is sufficiently low. It could be thought that if the amount of precautionary money balances is arbitrarily large, efficiency can be restored at the Friedman rule. This is not always the case. This argument is summarized in the following Proposition, which mirrors Proposition 10.

**Proposition Supp. 3.** *If  $\beta^* = 1$ , then for all  $d > 0$  there exist a lower bound for the discount factor,  $0 < \underline{\beta}(d) < \beta^*$ , such that if  $\beta \in (\underline{\beta}(d), \beta^*)$ , the full risk sharing allocation is not incentive compatible at the Friedman rule. If  $\beta^* < 1$ , there exists a threshold for the amount of precautionary money balances,  $\bar{d} > 0$ , and an upper bound  $\bar{\beta}(d) \in (\underline{\beta}(d), \beta^*)$  such that, for all  $d \in [0, \bar{d}]$  and  $\beta \in (\underline{\beta}(d), \bar{\beta}(d))$ , the full risk sharing allocation is not incentive compatible at the Friedman rule.*

*Proof.* Let us denote as  $\omega_i(\theta, d) = w_i \left(1 + \frac{\theta}{1+\theta} \frac{d}{\bar{w}}\right)$  the after tax real labor income. Let us denote as  $\hat{\theta}(d)$  the inflation rate for which  $\omega_n(\hat{\theta}(d), d) = \bar{w}$ . Notice that  $\hat{\theta}(d) = -\frac{(w_n-\bar{w})\bar{w}}{(w_n-\bar{w})\bar{w}+dw_n} \in (-1, 0)$ . Let us further define  $\hat{\beta}(d) = 1 + \hat{\theta}(d)$ . It follows that

$$V_D(n, 0, (\beta - 1, d)) < \frac{\beta u(\bar{w})}{1 - \beta}, \quad \beta \leq \hat{\beta}(d). \quad (2.17)$$

Let us analyze two different cases. **(1)**  $\beta^* = 1$ . In this case  $\omega_i(\theta, 0) = w_i$ . Thus, it follows from (2.17), Proposition 10 and continuity that there exists  $\underline{\beta}(d) \in (0, 1)$  such that for all  $\beta \in (0, \underline{\beta}(d))$  the full risk sharing allocation is not incentive compatible for any  $d \geq 0$ .

**(2)**  $\beta^* < 1$ . Let us define  $d(\beta^*)$  as the amount for which  $\omega_n(\hat{\theta}(d(\beta^*)), d(\beta^*)) = \bar{w}$ . Notice that  $\hat{\theta}(d(\beta^*)) = \beta^* - 1 < 0$ . Let  $f : [0, d(\beta^*)] \rightarrow \mathbb{R}$  be such that  $f(d) = \max_{\beta \in [0, \beta^*]} \left( V_D(n, 0, (\beta - 1, d)) \right)$

$-\frac{\beta u(\bar{w})}{1-\beta}$ ). It follows from Weierstrass Theorem that  $f(d)$  is continuous. Furthermore, Lemma Supp. 2 implies that  $f(0) > 0$  and it follows from (2.17) that  $f(d(\beta^*)) < 0$ . It follows from Lemma Supp. 3 that  $f(d)$  is strictly decreasing function of  $d$ . Therefore, there is a unique  $\bar{d} > 0$  such that  $f(\bar{d}) = 0$ . This implies that for all  $d \in [0, \bar{d}]$ , there exists  $\tilde{\beta} \in (\hat{\beta}(d), \beta^*)$  such that

$$V_D(n, 0, (\tilde{\beta} - 1, d)) > \frac{\tilde{\beta} u(\bar{w})}{1 - \tilde{\beta}}. \quad (2.18)$$

Moreover, since  $V_D(n, 0, (\beta - 1, d))$  strictly decreases with  $d$ , then for all  $d > 0$ ,

$$V_D(n, 0, (\beta^* - 1, d)) > \frac{\beta^* u(\bar{w})}{1 - \beta^*}. \quad (2.19)$$

It follows from continuity and (2.17), (2.18), and (2.19) that for all  $d \in [0, \bar{d}]$  there are  $\underline{\beta}(d)$ , and  $\bar{\beta}(d)$ , that satisfy  $\hat{\beta}(d) < \underline{\beta}(d) < \bar{\beta}(d) < \beta^*$  such that  $V_D(n, 0, (\underline{\beta}(d) - 1, d)) = \frac{\underline{\beta}(d) u(\bar{w})}{1 - \underline{\beta}(d)}$  and  $V_D(n, 0, (\bar{\beta}(d) - 1, d)) = \frac{\bar{\beta}(d) u(\bar{w})}{1 - \bar{\beta}(d)}$ , and, for all  $\beta \in (\underline{\beta}(d), \bar{\beta}(d))$ , the full risk sharing allocation is not incentive compatible,  $V_D(n, 0, (\beta - 1, d)) > \frac{\beta u(\bar{w})}{1 - \beta}$ .  $\square$

The following Proposition states that, given any inflation rate, there is no full risk sharing whenever households are sufficiently impatient.

**Proposition Supp. 4.** *Assume that  $\mathbf{m} = \bar{w}$ . For any inflation rate  $\tilde{\theta} \in \mathbf{R}$  there is an associated discount factor,  $\beta(\tilde{\theta})$ , such that if  $\beta < \beta(\tilde{\theta})$  and  $\theta \leq \tilde{\theta}$ , the full commitment allocation does not satisfy the incentive compatibility constraint.*

*Proof.* Let us take the default function for the household with highest productivity and compare  $V_D(n, 0, \bar{\theta}) - \frac{\beta u(\bar{w})}{1-\beta} > \beta u(w_n) + \frac{\beta^2}{1-\beta} u(w_1) - \beta u(\bar{w}) - \frac{\beta^2}{1-\beta} u(\bar{w}) = \beta [u(w_n) - u(\bar{w})] + \frac{\beta^2}{1-\beta} (u(w_1) - u(\bar{w}))$ . Taking limits with respect to the discount factor,  $\lim_{\beta \rightarrow 0} \left( \frac{V_D(n, 0, \bar{\theta})}{\beta} - \frac{\beta u(\bar{w})}{1-\beta} \right) \geq \lim_{\beta \rightarrow 0} \left( \beta u(w_n) - \beta u(\bar{w}) + \frac{\beta^2}{1-\beta} (u(w_1) - u(\bar{w})) \right) = u(w_n) - u(\bar{w}) > 0$ . It follows from continuity of  $V_D(i, a, \theta)$  that there exists  $\bar{\theta}$  such that its associated discount factor satisfies  $V_D(n, 0, \bar{\theta}) = \frac{\beta(\bar{\theta}) u(\bar{w})}{1 - \beta(\bar{\theta})}$  and that for all  $\beta < \beta(\bar{\theta})$ ,  $V_D(n, 0, \bar{\theta}) > \frac{\beta u(\bar{w})}{1 - \beta}$ . Furthermore, it follows from Lemma Supp. 1 that if  $\theta \leq \bar{\theta}$  and  $\beta \leq \beta(\bar{\theta})$ ,  $V_D(n, 0, \theta) \geq V_D(n, 0, \bar{\theta}) > \frac{\beta u(\bar{w})}{1 - \beta}$ .  $\square$

**Corollary Supp. 3.** *Assume that  $\mathbf{m} = \bar{w}$ . There is  $\beta(0)$  such that if  $\beta \leq \beta(0)$  and  $\theta \leq 0$ , the full commitment allocation is not incentive compatible.*

### 2.4.1 No existence of equilibrium at the Friedman rule

**Proposition 11.** *If the full risk sharing allocation is not incentive compatible at the Friedman rule, then there is no incentive compatible equilibrium at the Friedman rule.*

*Proof.* We prove it by contradiction. Let us suppose that there exists an equilibrium. Corollary 5 implies that the only symmetric steady state is the full commitment steady state. Thus, the incentive compatible steady state cannot be symmetric. Hence, consumption depends on the individual state variable  $x_t = \{j_t, j_{t-1}, d_{t-1}, b_{t-1}\}$ . Using the first order condition with respect to precautionary money holdings, the convexity of the marginal utility,  $u'''(\cdot) > 0$ , and the fact that consumption varies across states we find that  $u'(c(x_t)) \geq E[u'(c(x_{t+1}))/x_t] > u'(E[c(x_{t+1})/x_t])$ , which implies that consumption is growing in expected terms,  $E[c(x_{t+1})/x_t] > c(x_t)$ . Adding across households,  $\int_{X_{t+1}} [c_{t+1}(x_{t+1})] d\mu_{t+1} = \int_{X_t} E[c(x_{t+1})/x_t] d\mu_t > \int_{X_t} c_t(x_t) d\mu_t = \int_{X_t} w(x_t) d\mu_t = \int_{X_{t+1}} w(x_{t+1}) d\mu_{t+1}$ , we find that the equilibrium allocation is not feasible. Thus, we incur in a contradiction and, hence, equilibrium cannot exist.  $\square$

Proposition 11 says that if the return to money is equal to the preference discount rate, there is no equilibrium. The reason is the following: although bonds are contingent, the credit restriction induced by limited commitment implies a market incompleteness. Households cannot borrow as much as they want. If the government deflates at the preference time rate the price of the bond must be equal to the discount factor. As Bewley (1983) shows, at the preference discount rate, consumption and money holdings must grow without bound. Hence, equilibrium cannot exist.

## 2.5 Inflation and credit

**Proposition 12.** *For any  $\beta < \beta^*$ , there exists  $\theta^*(\beta) > \beta - 1$  such that if  $\theta \geq \theta^*(\beta)$  the full risk sharing allocation satisfies the incentive compatibility constraint and, therefore, the equilibrium is efficient.*

*Proof.* Lemma Supp. 1 and Lemma Supp. 2, Assumption Supp. 1, and the continuity of  $V_D(i, a, \theta)$  with respect to  $\theta$  ensure that there exists  $\theta^* > \beta - 1$  such that  $V_D(n, 0, \theta) = \frac{\beta u(\bar{w})}{1-\beta}$ . It follows from Proposition 10 that  $\theta^* > \beta - 1$ .  $\square$

Thus, our main result holds: monetary policy can restore efficiency, provided that households are sufficiently impatient. Nevertheless, we want to stress that the key effect of idiosyncratic uncertainty is that the Friedman rule is no longer efficient. Finally, we want to point out a difference between this economy and our benchmark economy. In the economy without uncertainty, households do not hold precautionary money balances when there is credit. In this economy with idiosyncratic uncertainty, households do not hold precautionary money balances at the efficient inflation rate,

$\theta \geq \theta^*$ . If the amount of credit is constrained below its efficient level, households may hold precautionary money balances in equilibrium.

### 3 Endogenous labor supply

In the model economies that we have studied above, inflation has no real effects besides the effect on credit. Here we explore the robustness of our results when households value leisure. We assume that households are endowed with  $\bar{n}$  units of time each period that they can devote to work,  $n$ , or for leisure purposes,  $\ell = \bar{n} - n$ . For simplicity of exposition, we assume that the period utility function is additively separable in consumption and leisure,  $u(c) + v(\ell)$ , and is twice continuously differentiable with the first partial derivatives strictly positive,  $u'(c) > 0$ , and  $v'(\ell) > 0$ . Moreover, they satisfy the INADA conditions  $u'(c) \rightarrow \infty$  when  $c \rightarrow 0$ ,  $v'(\ell) \rightarrow \infty$  when  $\ell \rightarrow 0$ , and  $v'(\ell) \rightarrow 0$  when  $\ell \rightarrow \bar{n}$ . The last two assumptions guarantee that households always work, which simplify our analysis without any loss of its generality. Finally, the Hessian is semi-definite negative. The timing is as follows: households work and consume at the first stage, they decide whether to default or not at the second stage. Finally, they receive earnings and money transfers and participate in the credit market at the third stage.

#### 3.1 The household's problem at the financial markets stage

At the financial market stage the individual state variable is  $(i, b, d, n)$ , where  $i$  denotes household's labor productivity,  $b$  amount of bonds,  $d$  is the amount of precautionary money balances and  $n$  is the amount of hours worked. The aggregate state variable is summarized by  $\Psi = (\theta, \tau_h)$ . The household's problem at this stage is

$$\begin{aligned}
V(i, b, d, n, \Psi) &= \max_{c', d', n'} \{ \beta u(c') + \beta v(\bar{n} - n') + \beta V(-i, b', d', n', \Psi) \} \\
\text{s. t.} \quad & (1 + \theta)(c' + d') + q(\Psi) b' \leq w_i n + b + d + \theta \tau_i \mathbf{m}, \\
& c' \geq 0, d' \geq 0, 0 \leq n' \leq \bar{n}, \\
& V(-i, b', d', n', \Psi) \geq V_D(-i, 0, n', \Psi).
\end{aligned} \tag{3.1}$$

where  $c$  denotes next period consumption. The default problem is

$$\begin{aligned}
V_D(i, a, \eta, \Psi) &= \max_{z', a', \eta'} \{ \beta u(z') + \beta v(\bar{n} - \eta') + \beta V_D(-i, a', \eta', \Psi) \} \\
\text{s. t.} \quad & (1 + \theta)(z' + a') \leq w_i \eta + a + \theta \tau_i \mathbf{m}, \\
& z' \geq 0, a' \geq 0, 0 \leq \eta' \leq \bar{n}.
\end{aligned} \tag{3.2}$$

The incentive compatibility constraint  $V(-i, b', d', n', \Psi) \geq V_D(-i, 0, n', \Psi)$  implicitly determines a solvency constraint, an endogenous debt limit,  $b' \geq \underline{b}(i, d, n, \Psi)$ , which is contingent on the individual state.

### 3.2 Efficiency and full commitment

We focus our attention on symmetric steady states. Efficiency entails full risk sharing in consumption,  $c_l = c_h = y$ , where  $y$  is aggregate output,  $y = w_h n_h/2 + w_l n_l/2$ , and that the marginal rate of substitution between consumption and leisure equals the real wage,

$$v'(\bar{n} - n_j) = w_j u'(y), \quad j = h, l. \quad (3.3)$$

This allocation can be decentralized with full commitment as long as the government follows the Friedman rule. If the government sets  $1 + \theta > \beta$ , households work fewer hours,

$$v'(\bar{n} - n_j) = w_j \frac{\beta}{1 + \theta} u'(y), \quad j = h, l, \quad (3.4)$$

and output is lower than in the efficient allocation. Regardless of the value of  $\theta$ , there is full risk sharing in consumption and the equilibrium price of the bond is equal to the discount factor,  $q = \beta$ . High productivity households work longer hours than low productivity households. At the financial markets stage the utility of a household with productivity  $i$  is

$$\beta \frac{u(y) + v(\bar{n} - n_{-i})}{1 - \beta} + \beta^2 \frac{[u(y) + v(\bar{n} - n_i)]}{1 - \beta^2}. \quad (3.5)$$

This implies that high productivity households have higher utility than low productivity households and, as in the benchmark economy, those are the households that have the incentives to default in the setting with limited commitment.

### 3.3 The incentive compatible symmetric steady state

As in our benchmark economy, the incentive compatible symmetric steady state is either the steady state with full commitment or the ICC is binding for the high productivity household and the low productivity household is credit constrained. Let us focus on the latter case.

Before characterizing further the incentive compatible steady state we need to characterize the allocation chosen by a defector. The default allocation, however, is not symmetric since the household has already worked when it decides whether to default or not, and the number of hours worked depends on having access to the credit market. Nevertheless, the utility of defecting must be greater than or equal to the utility associated to not having access to the credit market and working each period exactly the number of hours chosen by a household that participates in the credit market,

$$\begin{aligned} \tilde{V}_D(i, a_i, n_i, \Psi) &= \max_{z_{-i}, a_{-i}} \left\{ \beta u(z_{-i}) + v(\bar{n} - n_{-i}) + \beta \tilde{V}_D(-i, a_{-i}, n_{-i}, \Psi) \right\} \\ \text{s. t.} \quad &(1 + \theta) z_{-i} + (1 + \theta) a_{-i} = w_i n_i + a_i + \theta \tau_i \mathbf{m}, \quad i = h, l, \\ &a_{-i} \geq 0, \quad j = h, l. \end{aligned} \quad (3.6)$$

This problem is symmetric and its solution will help us to characterize the incentive compatible steady state.

### 3.3.1 No existence of credit with deflation

As in our benchmark economy, there is no credit with deflation. Notice that, taking as given the labor allocation chosen by a household that participates in the credit market, we can apply Proposition 1 in Díaz and Perera-Tallo (2010), which implies that, for any  $\theta < 0$ , the utility of participating in the credit market,  $V(i, b_i, d_i, n_i, \Psi)$ , is lower than  $\tilde{V}_D(h, 0, n_i, \Psi)$ . Since, by the Maximum Principle, the utility of default,  $V_D(h, 0, n_i, \Psi)$ , is greater than or equal than  $\tilde{V}_D(h, 0, n_i, \Psi)$ , the result is established. Thus, there cannot be credit in equilibrium. Nevertheless, high productivity households must be indifferent between saving in bonds or money,  $q = 1 + \theta$ . The equilibrium allocation is characterized by the price of bonds being equal to the inflation factor,  $q = 1 + \theta$ , limited insurance,  $c_l \geq c_h$ , and the following set of equations.

$$(1 + \theta) u'(c_l) = \beta u'(c_h), \quad j = h, l, \quad (3.7)$$

$$d_j \geq 0, \quad j = l, h, \quad (3.8)$$

$$v'(\bar{n} - n_j) = w_j \frac{\beta}{1 + \theta} u'(c_{-j}), \quad j = h, l, \quad (3.9)$$

$$(1 + \theta) c_j + (1 + \theta) d_j = w_{-j} n_{-j} + d_{-j} + \theta \tau_{-j} \mathbf{m}, \quad j = l, h, \quad (3.10)$$

$$c_l + c_h = w_l n_l + w_h n_h, \quad (3.11)$$

$$\mathbf{m} = \frac{1}{2} c_l + \frac{1}{2} c_h + \frac{1}{2} d_l + \frac{1}{2} d_h. \quad (3.12)$$

It is easy to check that if  $1 + \theta > \beta$  low productivity households are liquidity constrained,  $d_h = 0$ , whereas high productivity households may not,  $d_l > 0$ . In those cases, inflation does not distort the number of hours worked by low productivity households. Inflation, however, reduces the number of hours worked by high productivity households. Finally, the equilibrium allocation is efficient if the government follows the Friedman rule.

### 3.3.2 The case of zero inflation

To characterize the equilibrium allocation let us use the following assumptions:

**Definition Supp. 2.** *The elasticity of substitution is:  $\varepsilon_{1,2}^s(c_1, c_2) = \frac{\partial \ln(c_2/c_1)}{\partial \ln[u'(c_1)/\beta u'(c_2)]}$ .*

**Assumption Supp. 3.** *The Arrow-Pratt coefficient of the instantaneous utility function has a positive lower bound,  $\varepsilon_{1,2}^s(c_1, c_2) > \frac{1}{\sigma} > 0$ .*

**Assumption Supp. 4.** *Fluctuation of labor productivity is sufficiently large,  $\frac{w_h}{w_l} > \left(\frac{1}{\beta}\right)^{\max\{1, \frac{1}{\sigma}\}}$ .*

In this case, Assumption Supp. 4 ensures that high productivity defectors save,  $a_h > 0$ . Thus, as in our benchmark economy, households attain the same allocation by either participating in the credit market or defaulting on their debts. To see this, let us particularize equations (3.7)–(3.12) for  $\theta = 0$ , and set an amount of bonds  $b_h$  that satisfies

$$b_h = \frac{w_l n_l - c_h}{2}. \quad (3.13)$$

The amount of bonds held by low productivity households is  $b_l = -b_h$  and the price of the bond is  $q = 1$ . Hence, households choose the same consumption and labor allocation either with credit or without it.

### 3.3.3 Credit and inflation

In order to concentrate on the effects of inflation on leisure we are going to assume that money injections are neutral. Thus,

$$\tau_i = \frac{2 w_i n_i}{w_h n_h + w_l n_l}. \quad (3.14)$$

**Proposition Supp. 5.** *If  $\theta \geq 0$ , the equilibrium amount of credit is positive and the price of the bond satisfies  $q \in [\beta, 1]$ .*

To prove this Proposition we need some preliminary results. First, we start by constructing the equilibrium of an economy with an exogenous debt constraint.

**Definition Supp. 3.** *Given  $\theta \geq 0$  and for any  $q \in [\beta, 1]$ , let the allocation  $\left( \hat{c}_l(q, \Psi), \hat{c}_h(q, \Psi), \hat{n}_l(q, \Psi), \hat{n}_h(q, \Psi) \right)$  satisfy*

$$q u'(\hat{c}_l(q, \Psi)) = \beta u'(\hat{c}_h(q, \Psi)), \quad (3.15)$$

$$v'(\bar{n} - \hat{n}_i(q, \Psi)) = \beta \frac{w_i}{1 + \theta} u'(\hat{c}_{-i}(q, \Psi)), \quad i = h, l, \quad (3.16)$$

$$\hat{c}_l(q, \Psi) + \hat{c}_h(q, \Psi) = w_h \hat{n}_h(q, \Psi) + w_l \hat{n}_l(q, \Psi). \quad (3.17)$$

**Lemma Supp. 4.** *Let us define*

$$\underline{b}(q, \Psi) = \frac{1 + \theta}{1 + q} (w_h \hat{n}_h(q, \Psi) - \hat{c}_l(q, \Psi)). \quad (3.18)$$

*Given Assumption Supp. 3 and Assumption Supp. 4, then  $\underline{b}(q, \Psi) > 0$ .*

*Proof.* We know that  $v'(\bar{n} - \hat{n}_h(q, \Psi)) = \frac{w_h}{w_l} \frac{\beta}{q} v'(\bar{n} - \hat{n}_l(q, \Psi)) > v'(\bar{n} - \hat{n}_l(q, \Psi))$ , which implies that  $\hat{n}_h(q, \Psi) > \hat{n}_l(q, \Psi)$ , where in the latter inequality we have used Assumption Supp. 3. If

$c_i > c_{-i}$ , we can obtain  $\ln\left(\frac{u'(c_i)}{\beta u'(c_{-i})}\right) = \ln\left(\frac{u'\left(\frac{c_i+c_{-i}}{2}\right)}{\beta u'\left(\frac{c_i+c_{-i}}{2}\right)}\right) + \frac{\partial\left(\ln\left(\frac{u'\left(\lambda\frac{c_i+c_{-i}}{2}+(1-\lambda)c_i\right)}{\beta u'\left(\lambda\frac{c_i+c_{-i}}{2}+(1-\lambda)c_{-i}\right)}\right)\right)}{\partial\ln(c_{-i}/c_i)} \ln(c_{-i}/c_i) < \ln\left(\frac{1}{\beta}\right) + \bar{\sigma} \ln(c_{-i}/c_i)$ , which implies that  $\frac{u'(c_i)}{\beta u'(c_{-i})} < \frac{1}{\beta} \left(\frac{c_{-i}}{c_i}\right)^{\bar{\sigma}}$ . Therefore,  $\frac{u'(w_h(\hat{n}_h(q, \Psi)))}{\beta u'(w_l(\hat{n}_l(q, \Psi)))} < \frac{1}{\beta} \left(\frac{w_l(\hat{n}_l(q, \Psi))}{w_h(\hat{n}_h(q, \Psi))}\right)^{\bar{\sigma}} < \frac{1}{\beta} \left(\frac{w_l}{w_h}\right)^{\bar{\sigma}} < 1 \leq \frac{1}{q} = \frac{u'(\hat{c}_l(q, \Psi))}{\beta u'(\hat{c}_h(q, \Psi))}$ . The above inequality, together with feasibility implies that  $w_h \hat{n}_h(q, \Psi) > \hat{c}_l(q, \Psi)$ , and, thus,  $\underline{b}(q, \Psi) > 0$ .  $\square$

**Definition Supp. 4.** Let us define  $\mathbf{m} = w_h \hat{n}_h(q, \Psi) + w_l \hat{n}_l(q, \Psi)$ . Let us construct the following problem faced by the household with productivity  $i$ ,

$$\begin{aligned} \widehat{V}(i, b_i, n_i, q, \Psi) &= \max_{c_{-i}, n_{-i}} \left\{ \beta u(c_{-i}) + \beta v(\bar{n} - n_{-i}) + \beta \widehat{V}(-i, b_{-i}, n_{-i}, q, \Psi) \right\} \\ \text{s. t.} \quad &(1 + \theta) c_{-i} + q b_{-i} \leq w_i n_i + b_i + \theta \tau u_i \mathbf{m}, \\ &c_{-i} \geq 0, 0 \leq n_{-i} \leq \bar{n}, \\ &b_{-i} \geq -\underline{b}(q, \Psi), \\ &\text{if } i = h, -i = l, \text{ and viceversa.} \end{aligned} \tag{3.19}$$

The solution to this problem is the allocation  $(\hat{c}_l(q, \Psi), \hat{c}_h(q, \Psi), \hat{n}_l(q, \Psi), \hat{n}_h(q, \Psi))$  characterized by equations (3.15)–(3.17).

Now we can prove Proposition Supp. 5

*Proof.* if  $\theta = 0$  we have already seen that  $V_D(h, 0, \hat{n}_h(q, \Psi), (0, \tau_h)) = \widehat{V}(h, \underline{b}(1, \Psi), \hat{n}_h(1, \Psi), 1, \Psi)$ . For any  $\theta > 0$ , there are two possible cases. If  $V_D(h, 0, \hat{n}_h(\beta, \Psi), \Psi) \leq \widehat{V}(h, \underline{b}(\beta, \Psi), \hat{n}_h(\beta, \Psi), \beta, \Psi)$ , then  $(\hat{c}_l(\beta, \Psi), \hat{c}_h(\beta, \Psi), \hat{n}_l(\beta, \Psi), \hat{n}_h(\beta, \Psi))$  is the equilibrium allocation and the amount of credit is  $\underline{b}(\beta, \Psi)$ . If, on the contrary,  $V_D(h, 0, \hat{n}_h(\beta, \Psi), \Psi) > \widehat{V}(h, \underline{b}(\beta, \Psi), \hat{n}_h(\beta, \Psi), \beta, \Psi)$ , then it must be the case that  $V_D(h, 0, \hat{n}_h(1, \Psi), \Psi) < \widehat{V}(h, \underline{b}(1, \Psi), \hat{n}_h(1, \Psi), 1, \Psi)$ . This is so because  $\theta > 0$  and  $\underline{b}(1, \Psi) > 0$ . It follows from the maximum Theorem that there exists  $q^* \in (\beta, 1)$  such that  $V_D(h, 0, \hat{n}_h(q^*, \Psi), \Psi) = \widehat{V}(h, \underline{b}(q^*, \Psi), \hat{n}_h(q^*, \Psi), q^*, \Psi)$ . Thus, the equilibrium allocation is  $(\hat{c}_l(q^*, \Psi), \hat{c}_h(q^*, \Psi), \hat{n}_l(q^*, \Psi), \hat{n}_h(q^*, \Psi))$ , and the amount of credit is  $\underline{b}(q^*, \Psi)$ .  $\square$

Thus, for any positive inflation rate, there exists a price of the bond and a borrowing limit such that the high productivity household is indifferent between participating in the credit market or defaulting on its debts. Hence, there is an incentive compatible amount of credit. Whether inflation can increase welfare depends on the elasticity of substitution between consumption and leisure and, therefore, it is a quantitative question. Nevertheless, the only efficient inflation rate is the Friedman rule.

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