Building socio-economic Networks: How many conferences should you attend?*

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Abstract

Profiting from spillovers is one of the key reasons behind technological collaboration agreements, agglommeration and even social stratification. Previous research has tended to ignore the costs of activities to internalize the externalities. We propose and solve a simple model for the tradeoff between costly socialization and direct productive effort. Our model can explain phenomena like the recent explosion in R&D collaborations or the decline in social capital, while giving a workable framework within which to analyze public policies towards network formation.

Key words: Research and development, social capital, knowledge spillovers, technological collaboration, network formation.

JEL Classification: L22, L51, O31, O38

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1 Introduction

External effects (spillovers) pervade economies and societies in general. Both inter and intra industry cross fertilization between firms have been the object of study, since at least the work of Marshall (1890) for the former, and Jacobs (1969) for the latter. Social interactions also have a crucial importance in the determination of individuals' well-being, as pointed early by Becker (1974) and recently emphasized by the literature on "social capital" (Putnam 2000.)[1]

Given the importance and pervasiveness of these external effects, it is natural that individuals and firms may want to control and manipulate the size and scope of those external benefits to their advantage. For example, regional economists have shown convincingly that economic agents agglomerate in few locations in the economic landscape, precisely in order to reap these localization externalities (Ciccone and Hall 1996). In a similar vein, d'Aspremont and Jacquemin (1988)[2] demonstrate that it is difficult to understand technological collaboration agreement between firms ("joint ventures" and other similar contracts) without thinking that these are done to control external effects. The persistent stratification of social groups among many dimensions (income, race, education) is *prima facie* evidence of the desire of social groups to arrange themselves so as to internalize spillovers (Tiébout 1956, Bénabou 1993).

The previous literature, however, has mostly overlooked that the process of socialization is expensive. For instance, the studies of strategic alliances between firms do not recognize that the formation of collaborative agreements detracts resources that could be used e.g. in the R&D effort. The work on localization does take into account that the choice (or change) of location is costly. But it typically assumes that once location is chosen, the external effect flow freely to all agents in the given location. This disregards the fact that access to many externalities arising from location requires some sort of effort. For example, Akerlof (1997) discusses the ethnographic observation that individuals moving to different locations in order to benefit from social public goods, often encounter difficulties in reaping the benefit from the move.

Our aim is to understand how economic agents decide to allocate resources when both "socialization" and their own productive effort are expensive. This will allow us, in addition, to propose a framework for the analysis of optimal public policies.

For this purpose, we analyze a two-stage model. In the first stage, agents make a costly

[1]Sobel (2002) offers a critical survey of this literature.

[2]See also Suzumura (1992).

socialization effort. The collection of such efforts determines how they benefit from the productive effort of others. In other words, the marginal value of the external effect depends on the collective socialization effort. In the second stage, agents undertake a costly productive effort, which delivers two types of benefits, an external and an internal one. The internal benefit is partly random and heterogeneous. The realization of this random component is known at the time when productive effort is taken, but unknown when deciding socialization effort. This allows to have idiosyncratic heterogeneity in productive decisions, net of external effects.

An innovation of our study is that the socialization effort is generic. That is to say, socialization efforts are not ear-marked for each candidate partner, and fine-tuned to each specific relationship. Socializing is not amenable here to elaborating a nominal list of intended relationships, as in the literature on network formation surveyed by Jackson (2005). Rather, we assume that agents devote a (joint) amount of resources to socializing with others, whomever these others are. Socialization is thus captured by a scalar, rather than a vector of decisions telling how much to socialize with every other agent. This way to model the socialization process seems realistic to us.[3] It also improves greatly the tractability of the model, and opens the door to a full-fledged welfare analysis and to the proposal of public policies.

It is important to note that modelling socialization as a compound effort does not prevent the emergence of a rich pattern of social relationships. In fact, even though the socialization effort is somehow anonymous, our model allows for ties involving different pairs of partners to be assigned different strengths, depending on the identity of this tie's partners. More precisely, we assume that the socialization effort of a given agent determines the aggregate strength of ties in which this agent is involved. Then, the collection of socialization efforts by the remaining agents determines how this overall strength of ties is distributed (disaggregated) into different ties' intensities. We take each tie's strength to be proportional to the socialization effort undertaken by the partners in the tie.

We first characterize the equilibria of the model. This equilibrium turns out to be unique, when a sufficiently large number of firms is implicated. The uniqueness result is somewhat surprising given the complication of the model, but more importantly, it is very useful to obtain welfare conclusions. Furthermore, this equilibrium is symmetric (all the companies

[3]The researchers go to fairs, or congresses to listen, to be listened to, and to meet other investigators in general. More generally, face-to-face meetings among agents that share a common location often result from random encounters among these agents, as the early literature on segregation indexes already points out (Bell 1955).

invest the same). A second important question is how a change in the return to the innovation affect the relative effort of research and socialization. The answer is that the effect is relatively stronger on socialization. This would explain, for example, the explosion in agreements of collaboration in R&D in the recent past (Caloghirou, Ioannides, Vonortas 2003). It could also explain the decline in social capital documented by Putnam (2000). Finally, we discuss aspects of social welfare and public policies. As expected, given the external effects, the investments are sub-optimal and therefore the public agents should be interested in subsidizing it. We uncover the conditions under which it is better to dedicate the first unit of subsidy to the socialization effort rather than the productive effort.

2 The game

$N = \{1, \ldots, n\}$ is a set of players.

We consider a two-stage game of network formation and R&D investment.

Stage one. Each player i selects a number $k_i > 0$. Let $\mathbf{k} = (k_1, ..., k_n)$ be a choice profile, and $\overline{\mathbf{k}} = \sum_{i \in N} k_i/n$ its average coordinate. Then, i and j interact with probability:

$$g_{ij}(\mathbf{k}) = g_{ji}(\mathbf{k}) = \frac{k_i k_j}{n\overline{\mathbf{k}}}. \tag{1}$$

The case $i = j$ corresponds to an unmatched player. The expected connectivity of i is k_i. Let $\mathbf{G}(k) = [g_{ij}(\mathbf{k})]_{i,j \in N}$. This is the $n-$symmetric adjacency matrix for the expected realization of the network with random links (1).

[explain the Poisson meeting process].

Interim stage. Players know $\mathbf{k}$ and the profile of realized idiosyncratic shocks $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, where the ε_is are i.i.d. random variables with common cumulative distribution function $F(\cdot)$ on $[\underline{\varepsilon}, \overline{\varepsilon}]$, expected value ε, and variance σ_ε^2.

Stage two. Each player i selects a number $s_i > 0$. Let $\mathbf{s} = (s_1, ..., s_n)$ be a choice profile. Let $p_{ij} = g_{ij}$ if $i \neq j$, and $p_{ii} = g_{ii}/2$. Player i's utility is:

$$u_i(\mathbf{k}, \mathbf{s}; \varepsilon) = [b + \varepsilon_i + \alpha \sum_{j=1}^{n} p_{ij} s_j] s_i - \frac{1}{2} s_i^2 - \frac{1}{2} k_i^2$$

where $b > 0$ and $\alpha \geq 0$.

QQQWe restrict our analysis to the set $\Omega = \{k | \alpha k_i^2 < n\overline{k}, k_i \geq 0, \forall i \in N\}$QQQ

3 Equilibrium characterization for large economies

3.1 The $\epsilon-$ equilibrium

DEFINITION 1 *An $\epsilon-$equilibrium is a set of mappings $\mathbf{s}^*(\mathbf{k}) = (s_1(\mathbf{k}), ..., s_n(\mathbf{k}))$ and a profile $\mathbf{k}^*$ such that:*

(*a*) *given an outcome $\mathbf{k}$ for the first-stage game, $\mathbf{s}^*(\mathbf{k})$ is a Nash equilibrium of second-stage game with payoff functions $u_i(\mathbf{k}, \cdot; \varepsilon)$, for all i;*

(*b*) *given a set of mappings $\mathbf{s}^*(\mathbf{k})$, $\mathbf{k}^*$ is an $\epsilon-$Nash equilibrium for the first-stage game with payoff functions $v_i(\cdot) = \mathbb{E}_F u_i(\cdot, \mathbf{s}^*(\cdot); \varepsilon)$, that is, for all player i and $\mathbf{k}'$, we have $|v_i(\mathbf{k}^*) - v_i(\mathbf{k}')| < \epsilon$.*

In words, SPNE with almost best reply in first stage.

We now characterize the $\epsilon-$equilibria of the game.

Given a profile $\mathbf{k} = (k_1, ..., k_n)$, let:

$$\lambda(\mathbf{k}) = \frac{\alpha\overline{\mathbf{k}}}{\overline{\mathbf{k}} - \alpha\overline{\mathbf{k}^2}},$$

where $\overline{\mathbf{k}^2} = \sum_{i\in N} k_i^2/n$. For a given average coordinate $\overline{\mathbf{k}}$, the parameter $\lambda(\mathbf{k})$ increases with the variance of the coordinates of $\mathbf{k}$, with a rate of increase monotone in α. Also, for a given variance of these coordinates, $\lambda(\mathbf{k})$ decreases with the average coordinate $\overline{\mathbf{k}}$, again with a rate monotone in α.

Define the following symmetric matrix:

$$\mathbf{M}(\mathbf{k}) = [\mathbf{I} - \alpha\mathbf{G}(\mathbf{k})]^{-1} = \sum_{p=0}^{+\infty} \alpha^p \mathbf{G}^p(\mathbf{k}).$$

The coefficient $m_{ij}(\mathbf{k})$ counts the total number of direct and indirect paths in the expected network $\mathbf{G}(\mathbf{k})$, where paths of lenght p are weighted by the decaying factor α^p. We show that:[4]

$$m_{ij}(\mathbf{k}) = \begin{cases} \lambda(\mathbf{k}) g_{ij}(\mathbf{k}), \text{ if } i \neq j \\ 1 + \lambda(k) g_{ii}(\mathbf{k}), \text{ if } i = j \end{cases}.$$

Define $\beta_i(\mathbf{k}) = m_{i1}(\mathbf{k}) + ... + m_{in}(\mathbf{k})$. This is the sum of all paths stemming from i in the expected network where links are independently and randomly drawn with probability (1). It is readily checked that:

$$\beta_i(\mathbf{k}) = 1 + \lambda(\mathbf{k}) k_i.$$

[4]See Lemma in appendix.

The vector $\beta(\mathbf{k})$ is a measure of centrality in the random graph $\mathbf{G}(\mathbf{k})$, reminiscent of the Bonacich centrality measure for fixed networks. Note that, for a given value of $\lambda(\mathbf{k})$, $\beta_i(\mathbf{k})$ increases with k_i.

THEOREM 1 *Suppose that $2\alpha(b+\varepsilon) < 1$. For any $\epsilon > 0$ there is a population size n^* such that, for any $n > n^*$, the game has a unique pure strategy $\epsilon-$equilibrium. This equilibrium is symmetric and given by:*

$$\begin{aligned} \mathbf{s}^*(k,...,k) &= b\beta(k,...,k) + \mathbf{M}(k,...,k) \cdot \varepsilon \\ k^* &= \frac{1}{2\alpha}\left(1 - \sqrt{1 - 4(b+\varepsilon)^2\alpha^2}\right) \end{aligned}$$

Denote by $s^*(k)$ any coordinate of $\mathbf{s}^*(k,...,k)$, that gives the individual strategy at the second-stage game.

The support of ε has to be such that $\mathbf{s}^*(\mathbf{k}) \geq \mathbf{0}$ for all $\mathbf{k}$. For instance, a sufficient condition is that the support of ε is positive, that is, $\underline{\varepsilon} \geq 0$.

From the first-stage perspective, $\mathbf{s}^*(k,...,k)$ is a random vector with a deterministic component, given by $b\beta(k,...,k)$, and a stochastic component, given by the random vector $\mathbf{M}(k,...,k) \cdot \varepsilon$. The deterministic component is the same for all players. The realizations of the stochastic component, though, vary across players with the ε_is. As a result, the own-investment equilibrium efforts that are actually undertaken are generally heterogeneous across players. This heterogeneity, though, can be correlated across individuals. This is because each coordinate of the random vector is a mixture of all the ε_is. The resulting pattern of correlation depends on the network of connections through $\mathbf{M}(k,...,k)$. In fact, these correlations reflect the spillovers coming from the socialization efforts, and thus vary with these efforts and the resulting network.

We can also compute the *social multiplier* of a given player i, that is, the impulse response of the aggregate outcome to the shock ε_i. We have:

$$\frac{\partial}{\partial \varepsilon_i} \sum_j s_j^*(k) = \sum_j m_{ij}(k) = \beta_i(k),$$

which gives yet another interpretation to the Bonacich centrality.

The socialization effort is identical for all individuals in the $\epsilon-$equilibrium. At equilibrium, the marginal cost of an extra unit of effort, k_i has to equal the marginal benefit from this extra effort. As societies becomes sufficiently large this marginal benefit is equal to $b^2\lambda(\mathbf{k})$. The variable $\lambda(\mathbf{k})$ summarizes the direct and indirect effects of spillovers on the choice of socialization efforts. There is an obvious effect from the spillovers exerted by direct contacts in the network. Since the direct contacts themselves experience an influence from their own direct contacts, this also affects socialization indirectly. Obviously, this can proceed inductively (but with decaying strength) to generate effects that span the whole network.

Denote by $\mathbf{1}$ the vector of ones. In the sequel, when $\mathbf{k}$ has identical coordinates, that is, $\mathbf{k} = k\mathbf{1}$, we use indistinguishly $\mathbf{k}$ or k.

Example 1 *Suppose that the ε_is follow a truncated normal distribution on the positive real line with expected value ε. Then, from the first-stage perspective (prior to the interim stage), the second-stage equilibrium vector $\mathbf{s}^*$ follows a truncated multivariate Normal process on the positive orthant.*[5] *The vector of expected values is $\mu(k) = (b+\varepsilon)\,\beta(k)$. The variance-covariance matrix is $\mathbf{\Omega}(k) = \sigma_\varepsilon^2 \mathbf{M}^2(k)$, with:*

$$\omega_{ij}(k) = \sigma_\varepsilon^2 \begin{cases} \lambda(k) g_{ij}(k)[2 + \lambda(k)\frac{\overline{k^2}}{\overline{k}}], \text{ if } i \neq j \\ 1 + \lambda(k) g_{ii}(k)[2 + \lambda(k)\frac{\overline{k^2}}{\overline{k}}], \text{ if } i = j \end{cases}.$$

In particular, the aggregate equilibrium profile follows a (univariate) truncated normal process:

$$\sum_j s_j^*(k) \sim N(\mu_a, \sigma_a^2),$$

where

$$\begin{cases} \mu_a = n(b+\varepsilon)\left[1 + \lambda(k)\overline{k}\right] \\ \sigma_a^2 = n\sigma_\varepsilon^2 \left[1 + 2\lambda(k)\overline{k} + \lambda^2(k)\overline{k^2}\right] \end{cases}$$

QQQinsert figure for the case $n = 2$ showing ellipse contours of the quantiles of this distribution on the two-dimensional space [use Mathematica and the add-on package Statistics 'MultinormalDistribution loaded from Statistics].QQQ

From now on, we assume throughout that $2\alpha(b+\varepsilon) < 1$.

3.2 Exact equilibria

Theorem 2 *There is a population size n^* such that for any $n > n^*$, there exists a pure strategy equilibrium. In all pure strategy equilibria, the strategies for all players $(k_i, s_i(k))$ converge to $(k^*, s^*(k))$ as n goes to infinity.*

[5] See, e.g., Horrace (2005) and references therein.

4 Socialization and investment: the response to incentives

4.1 Relative response of s and k to a change in $(\alpha, b, \varepsilon, \sigma_\varepsilon)$

PROPOSITION **1** *Let the exogenous payoff parameters* $(\alpha, b, \varepsilon, \sigma_\varepsilon)$ *be scaled by a common multiplicative factor* $1/\sqrt{2\alpha(b+\varepsilon)} > \delta \geq 1$. *Then, at the unique* $\epsilon-$ *equilibrium,* k^* *increases more than* $\mathbb{E}_F s^*(k^*)$ *in percentage terms.*

Strengthening the marginal return of all efforts tends to increase directly own investment s_i, for every individual: There are extra effects that come from the synergistic payoff.

4.2 Reaching the giant component: phase transition

PROPOSITION **2** *Let the exogenous payoff parameters* $(\alpha, b, \varepsilon, \sigma_\varepsilon)$ *be scaled by a common multiplicative factor* $1/\sqrt{2\alpha(b+\varepsilon)} > \delta \geq 1$. *There exists a threshold* $\overline{\alpha}$ *such that, for* $\alpha < \overline{\alpha}$, *when* δ *reaches a threshold value of* δ^*, *the* $\epsilon-$*equilibrium network jumps discontinuously from a fragmented graph (with several disconnected components) to a highly connected graph (with a single giant component).*

Phase transition Symmetric equilibrium yields to an Erdös-Rényi random graph where each link follows a binomial process of common parameter k^*/n given by Theorem 1. As n increases, the following holds almost surely: when $k^* > 1$ (resp. $k^* < 1$), a giant component arises encompassing a non-negligible fraction of nodes (resp. amost all nodes are part of small trees of negligible size). The case $k^* = 1$ corresponds to a discontinuity in network topology.

We are looking here at the possibility that as δ varies we go discontinuously from a state where there are several disconnected components to one with a single giant component in the network. The transition happens when $k = 1$, thus we will look for values of the parameter δ (and obviously, others) such as k goes from below the value of 1 to above that value.

5 Policies: how should you spend your first dollar?

THEOREM **3** *When* $\alpha^2\sigma_\varepsilon^2 > 3/4$, *the first unit of subsidy is always optimally allocated to socialization effort,* k_i. *When* $\alpha^2\sigma_\varepsilon^2 < 3/4$ *the first unit of subsidy is optimally allocated to socialization effort* k_i *if and only if the expected marginal return to own investment,* $b+\varepsilon$, *is low enough.*

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Appendix

Proof of Theorem 1: We proceed by backwards induction. We first solve for the second-stage game.

Lemma 1 *When $p_{ii} < 1/2\alpha$, the unique interior Nash equilibrium in pure strategies of the second-stage game is*

$$\mathbf{s}^*(\mathbf{k}) = b\beta(\mathbf{k}){+}\mathbf{M}(\mathbf{k}){\cdot}\varepsilon. \tag{2}$$

Proof. Fix $\mathbf{k}$ and ε, a realization of the ε_is. Provided payoffs are concave in own actions, the interior equilibria solve $\partial u_i(\mathbf{s}, \mathbf{k}; \varepsilon)/\partial s_i = 0$, for all $i = 1, ..., n$, which is equivalent to $[\mathbf{I}{-}\alpha\mathbf{G}(\mathbf{k})] \cdot \mathbf{s} = b\mathbf{1} + \varepsilon$.This system of linear equations has a unique generic solution, given by (2). The condition for concavity in own actions, $\partial^2 u_i(\mathbf{s}, \mathbf{k}; \varepsilon)/\partial s_i^2 < 0$, is equivalent to $p_{ii} < 1/2\alpha$. ∎

We now solve for the first-stage game. Given (2), the expected payoffs are:

$$\begin{aligned} v_i(\mathbf{k}) &= \mathbb{E}_F u_i(\mathbf{k}, \mathbf{s}^*(\mathbf{k}); \varepsilon) = (b+\varepsilon)\mathbb{E}_F(s_i) + \alpha\mathbb{E}_F(\sum_{j=1}^n p_{ij}s_i s_j) - \frac{1}{2}\mathbb{E}_F(s_i^2) - \frac{1}{2}k_i^2 \\ &= (b+\varepsilon)^2\beta_i(\mathbf{k}) + \alpha\sum_j p_{ij}\omega_{ij} - \frac{1}{2}\omega_{ii} - \frac{1}{2}k_i^2. \end{aligned}$$

In the sequel, we omit the parameter $\mathbf{k}$ when there is no risk of confusion. Also, we use the shortcut $b' = b + \varepsilon$.

The first-order conditions for the Nash equilibrium of the first-stage game lead to the following equation:

$$k_i = b'^2\frac{\partial\beta_i}{\partial k_i} + \alpha\sum_j\left[p_{ij}\frac{\partial\omega_{ij}}{\partial k_i} + \frac{\partial p_{ij}}{\partial k_i}\omega_{ij}\right] - \frac{1}{2}\frac{\partial\omega_{ii}}{\partial k_i}, \tag{3}$$

where:

$$
\begin{aligned}
\frac{\partial\lambda}{\partial k_i} &= \frac{\alpha^2}{n}\frac{2k_i-\overline{k^2}}{\left(\overline{k}-\alpha\overline{k^2}\right)^2} \\
\frac{\partial\beta_i}{\partial k_i} &= \lambda(k)+k_i\frac{\partial\lambda}{\partial k_i} \\
\frac{\partial p_{ij}}{\partial k_i} &= \frac{\partial g_{ij}}{\partial k_i}=\frac{1}{n}\left[\frac{k_j}{\overline{k}}-\frac{k_ik_j}{n\overline{k}^2}\right], \text{ if } i\neq j \\
\frac{\partial p_{ii}}{\partial k_i} &= \frac{1}{2}\frac{\partial g_{ii}}{\partial k_i}=\frac{1}{2n}\left[\frac{k_i}{\overline{k}}-\frac{k_i^2}{n\overline{k}^2}\right] \\
\frac{1}{\sigma_\varepsilon^2}\frac{\partial\omega_{ij}}{\partial k_i} &= \left(2+\lambda\frac{\overline{k^2}}{\overline{k}}\right)\left[\frac{\partial\lambda}{\partial k_i}g_{ij}+\frac{\partial g_{ij}}{\partial k_i}\lambda\right]+\lambda g_{ij}\left[\frac{\partial\lambda}{\partial k_i}\frac{\overline{k^2}}{\overline{k}}+\frac{1}{n}\left(2\lambda\frac{k_i}{\overline{k}}-\lambda\frac{\overline{k^2}}{\overline{k}^2}\right)\right] \\
\frac{1}{\sigma_\varepsilon^2}\frac{\partial\omega_{ii}}{\partial k_i} &= \left(2+\lambda\frac{\overline{k^2}}{\overline{k}}\right)\left[\frac{\partial\lambda}{\partial k_i}g_{ii}+\frac{\partial g_{ii}}{\partial k_i}\lambda\right]+\lambda g_{ii}\left[\frac{\partial\lambda}{\partial k_i}\frac{\overline{k^2}}{\overline{k}}+\frac{1}{n}\left(2\lambda\frac{k_i}{\overline{k}}-\lambda\frac{\overline{k^2}}{\overline{k}^2}\right)\right].
\end{aligned}
$$

When $\mathbf{k}$ is symmetric, that is, $k_1=...=k_n=\overline{k}=ng_{ij}=k$, and $\overline{k^2}=\overline{k}^2=k^2$, these expressions become:

$$
\begin{aligned}
\lambda &= \frac{\alpha}{1-\alpha k} \\
\frac{\partial\lambda}{\partial k_i} &= \frac{1}{nk}(2-k)\lambda^2 \\
\frac{\partial\beta_i}{\partial k_i} &= \lambda+\frac{1}{n}(2-k)\lambda^2 \\
\frac{\partial p_{ij}}{\partial k_i} &= \frac{\partial g_{ij}}{\partial k_i}=\frac{1}{n}\left(1-\frac{1}{n}\right), \text{ if } i\neq j \\
\frac{\partial p_{ii}}{\partial k_i} &= \frac{1}{2}\frac{\partial g_{ii}}{\partial k_i}=\frac{1}{2n}\left(1-\frac{1}{n}\right) \\
\frac{1}{\sigma_\varepsilon^2}\frac{\partial\omega_{ij}}{\partial k_i} &= 2\frac{\lambda^2}{n^2}(2-k)(1+\lambda k)+\frac{\lambda^2k}{n}+\frac{2\lambda}{n}\left(1-\frac{1}{n}\right),
\end{aligned}
\tag{4}
$$

and (3) can now be written as follows:

$$
\begin{aligned}
k &= b'^2\lambda+\frac{b'^2}{n}(2-k)\lambda^2 \\
&\quad +\frac{\lambda}{n}\sigma_\varepsilon^2\left[2\frac{\lambda^2}{n^2}(2-k)(1+\lambda k)+\frac{\lambda^2k}{n}+\frac{2\lambda}{n}\left(1-\frac{1}{n}\right)\right]\left[\alpha\frac{k}{n}\left(n-\frac{1}{2}\right)-\frac{1}{2}\right] \\
&\quad +\alpha\sigma_\varepsilon^2\frac{1}{n}\left(1-\frac{1}{n}\right)\left[\frac{1}{2}+\left(n-\frac{1}{2}\right)\left[\lambda\frac{k}{n}(2+\lambda k)\right]\right]
\end{aligned}
\tag{5}
$$

Lemma 2 *Let $k^*(n)$ be a solution to (5). Then, $k^*(n) \in O(1)$ when $n \to +\infty$.*

Proof. Suppose not. Then, $k^*(n) \in O(n^p)$ when $n \to +\infty$, for some $p > 0$. This implies that $\lim_{n\to+\infty} \lambda(k^*(n))k^*(n) = -1$, and $\lim_{n\to+\infty} \lambda(k^*(n)) = 0$, and then:

$$\begin{aligned}
\lim_{n\to+\infty} (b'^2\lambda(k^*(n)) + \frac{b'^2}{n}(2-k^*(n))\lambda(k^*(n))^2) &= 0 \qquad (6)\\
\lim_{n\to+\infty} \frac{\lambda(k^*(n))}{n}\sigma_\varepsilon^2 \left[2\frac{\lambda(k^*(n))^2}{n^2}(2-k^*(n))\left(1+\lambda(k^*(n))k^*(n)\right)+\right. & \qquad (7)\\
\left.\frac{\lambda(k^*(n))^2k^*(n)}{n} + \frac{2\lambda(k^*(n))}{n}\left(1-\frac{1}{n}\right)\right]\left[\alpha\frac{k^*(n)}{n}\left(n-\frac{1}{2}\right)-\frac{1}{2}\right] &= 0\\
\lim_{n\to+\infty} \alpha\sigma_\varepsilon^2\frac{1}{n}\left(1-\frac{1}{n}\right)\left[\frac{1}{2}+\left(n-\frac{1}{2}\right)\left[\lambda\frac{k_n^*}{n}\left(2+\lambda(k_n^*)k_n^*\right)\right]\right] &= 0
\end{aligned}$$

Therefore, the right-hand side of (5) evaluated at $k^*(n)$ tends to zero when $n \to +\infty$, while the left-hand side tends to infinity, which is a contradiction. ▮

Lemma 3 *Let $k^*(n)$ be a solution to (5). Then, there exists an n^* such that, for all $n > n^*$, the second-order conditions associated to (5) hold if and only if $\partial k^*(n)/\partial\alpha > 0$.*

Proof. By the implicit function theorem, we have:

$$\frac{\partial^2 v_i}{\partial k_i^2} = -\frac{\partial^2 v_i/\partial k_i\partial\alpha}{\partial k_i/\partial\alpha}.$$

Therefore, the second-order condition $\partial^2 v_i/\partial k_i^2 \le 0$ holds if and only if $\partial^2 v_i/\partial k_i\partial\alpha$ and $\partial k_i/\partial\alpha$ are of the same sign. We have:

$$\frac{1}{\sigma_\varepsilon^2}\frac{\partial^2 v_i}{\partial k_i\partial\alpha} = \frac{\partial m_{ii}}{\partial k_i} + \frac{1}{\sigma_\varepsilon^2}\sum_j\left[p_{ij}\frac{\partial\omega_{ij}}{\partial k_i} + \frac{\partial p_{ij}}{\partial k_i}\omega_{ij}\right].$$

Let $k^*(n)$ be a solution to (5). We know that $k^*(n) \in O(1)$, when $n \to +\infty$. In particular, this implies that $\lambda(k^*(n)) \in O(1)$. Using (4) we then get, when $n \to +\infty$:

$$\begin{aligned}
\frac{\partial m_{ii}}{\partial k_i} &= \frac{\lambda(k^*(n))}{n} + o(\frac{1}{n})\\
\frac{1}{\sigma_\varepsilon^2}\left[p_{ii}\frac{\partial\omega_{ii}}{\partial k_i} + \frac{\partial p_{ii}}{\partial k_i}\omega_{ii}\right] &= \frac{1}{2n} + o(\frac{1}{n})\\
\frac{1}{\sigma_\varepsilon^2}\sum_{j\neq i}\left[p_{ij}\frac{\partial\omega_{ij}}{\partial k_i} + \frac{\partial p_{ij}}{\partial k_i}\omega_{ij}\right] &= \frac{2\lambda(k^*(n))}{n}(2+\lambda(k^*(n))k^*(n))k^*(n) + o(\frac{1}{n})
\end{aligned}$$

Therefore, when $n \to +\infty$, we have

$$\frac{1}{\sigma_\varepsilon^2}\frac{\partial^2 Eu_i}{\partial k_i \partial \alpha} = \frac{\lambda(k^*(n))}{n} + \frac{1}{2n} + \frac{2\lambda(k^*(n))}{n}(2+\lambda(k^*(n))k^*(n))k^*(n) + o(\frac{1}{n}),$$

which implies that $\partial^2 Eu_i/\partial k_i \partial \alpha > 0$ for n high enough. Therefore, as $n \to +\infty$, the second-order condition associated to (5) holds at $k^*(n)$ if and only if $\partial k^*(n)/\partial \alpha > 0$. ∎

Lemma 4 *For all $\epsilon > 0$, there exists n^* such that, for all $n > n^*$, the two-stage game has a unique symmetric $\epsilon-$equilibrium $(\mathbf{s}^*(k), k^*)$ where $\mathbf{s}^*(k)$ is given in (2) while :*

$$k^* = \frac{1}{2\alpha}\left(1 - \sqrt{1-4(b+\varepsilon)^2\alpha^2}\right).$$

Proof. By Lemma 2, any solution $k^*(n)$ to (5) is such that $k^*(n) \in O(1)$ when n gets larger. This implies that:

$$\begin{aligned}
\lim_{n\to+\infty}\left(\frac{b^2}{n}(2-k^*(n))\lambda(k^*(n))^2\right) &= 0 \quad (8)\\
\lim_{n\to+\infty}\frac{\lambda(k^*(n))}{n}\sigma_\varepsilon^2\left[2\frac{\lambda(k^*(n))^2}{n^2}(2-k^*(n))\left(1+\lambda(k^*(n))k^*(n)\right)\right. & \quad (9)\\
\left.+\frac{\lambda(k^*(n))^2k^*(n)}{n}+\frac{2\lambda(k^*(n))}{n}\left(1-\frac{1}{n}\right)\right]\left[\alpha\frac{k^*(n)}{n}\left(n-\frac{1}{2}\right)-\frac{1}{2}\right] &= 0\\
\lim_{n\to+\infty}\alpha\sigma_\varepsilon^2\frac{1}{n}\left(1-\frac{1}{n}\right)\left[\frac{1}{2}+\left(n-\frac{1}{2}\right)\left[\lambda(k^*(n))\frac{k^*(n)}{n}\left(2+\lambda(k^*(n))k^*(n)\right)\right]\right] &= 0
\end{aligned}$$

Let $k^* = \lim_{n\to+\infty} k^*(n)$, which is well defined. Taking limits in (5) we deduce that:

$$k^* = (b+\varepsilon)^2\lambda(k^*). \quad (10)$$

So, the equilibrium candidates must solve:

$$k^* - \alpha k^{*2} - (b+\varepsilon)^2\alpha = 0 \quad (11)$$

There are two roots, which are:

$$k^{**} = \frac{1}{2\alpha}\left(1+\sqrt{1-4(b+\varepsilon)^2\alpha^2}\right) \text{ and } k^* = \frac{1}{2\alpha}\left(1-\sqrt{1-4(b+\varepsilon)^2\alpha^2}\right)$$

It is easy to check that:

$$\frac{\partial k^{**}}{\partial \alpha} > 0 \text{ and } \frac{\partial k^*}{\partial \alpha} < 0$$

Using Lemma 3, the only root that satisfies the second-order condition is k^*. We now check that the condition $p_{ii}(k) < 1/2\alpha$ in Lemma 1 holds for $k = k^*$. Given that $p_{ii}(k^*) = k^*/2n$, this is equivalent to $1 - \sqrt{1 - 4(b + \varepsilon)^2\alpha^2} < 2n$, which is always true.

Finally, we show that there is no asymmetric $\epsilon-$Nash equilibrium of the first-stage game when the second-stage strategies are $\mathbf{s}^*(k)$ given in (2). Let $\mathbf{k}^*(n) = (k_1^*(n), ..., k_n^*(n))$ be a profile that solves the system of equations given by (3), $i = 1, .., n$. Following the same reasoning than Lemma 2, it is readily checked that the first-order conditions for player i becomes:

$$k_i = b^2 \frac{\partial \beta_i}{\partial k_i},$$

when n gets larger. Furthermore, for n large, we have

$$\lim_{n \to +\infty} \frac{\partial \lambda(\mathbf{k}^*(n))}{\partial k_i} = 0,$$

and, thus:

$$\lim_{n \to +\infty} \frac{\partial \beta_i(\mathbf{k}^*(n))}{\partial k_i} = \lambda(\mathbf{k}^*(n)) = \frac{\alpha \overline{\mathbf{k}^*(n)}}{\overline{\mathbf{k}^*(n)} - \alpha \overline{\mathbf{k}^*(n)^2}}$$

which does not depend on i. Therefore, $\lim_{n \to +\infty} k_i^*(n) \equiv k_i^* = b^2 \lambda(\mathbf{k}^*(n))$, and $k_1^* = ... = k_n^* = \overline{k}$. ▮

Proof of Theorem 2: In process. ▮

Proof of Proposition 1: Let's scale up the exogenous payoff parameters $(\alpha, b, \varepsilon, \sigma_\varepsilon)$ by a common multiplicative factor $\delta \geq 1$. Let $\epsilon > 0$. Take n large enough such that Theorem 1 for this ϵ. Let $(k^*, s^*(k))$ be the unique $\epsilon-$equilibrium of the game, and let $\mathbb{E}_F s^* = s^*(k^*)$ be the expected players' equilibrium play at the second-stage game. We compute the elasticity η^* that keeps track of the relative changes on k^* and $\mathbb{E}_F s^*$ under the $\delta-$rescaling of the parameters, that is:

$$\eta^* = \frac{k^*}{\mathbb{E}_F s^*} \frac{\frac{\partial \mathbb{E}_F s^*}{\partial \delta}}{\frac{\partial k^*}{\partial \delta}}.$$

From now on, we omit the superscript *.

We know from (2) that $\mathbb{E}_F s(\delta) = \delta(b+\varepsilon)\beta(k(\delta))$. By definition, $\beta(k(\delta)) = 1+\lambda(k(\delta))k(\delta)$, and by (10) we have $k(\delta) = \delta^2(b + \varepsilon)^2\lambda(k(\delta))$. Altogether, this implies that:

$$\mathbb{E}_F s(\delta) = \delta(b + \varepsilon) + \frac{k(\delta)^2}{\delta(b + \varepsilon)}. \tag{12}$$

From now on, we set $b' = b + \varepsilon$.

Using (12) we then compute the elasticity and get:obtain:

$$\eta = \frac{k}{\mathbb{E}_F s} \frac{b' - \frac{k^2}{\delta^2 b'}}{\frac{\partial k}{\partial \delta}} + \frac{2k^2}{\delta b' \mathbb{E}_F s}.$$

Using the expression for k in Theorem 1, we get:

$$\frac{\partial k}{\partial \delta} = -\frac{k}{\delta} + 4\alpha b'^2 \delta^2 \frac{1}{1 - 2\alpha\delta k}.$$

Therefore, $\partial k/\partial\delta > k/\delta$ if and only if $2\alpha b'^2\delta^3 - k + 2\alpha\delta k > 0$. But, we know from (11) that $k(\delta)$ solves $k - \alpha\delta k^2 - \alpha b'^2\delta^3 = 0$. Therefore, the previous inequality is equivalent to $k(\delta) > 0$, which is true as long as the solution to (11) is well-defined, that is, $1 \leq \delta < \overline{\delta}$, with $\overline{\delta} = 1/\sqrt{2\alpha b'} > 1$.

Suppose that $1 \leq \delta < \overline{\delta}$. Then, the elasticity is such that $\eta < 1$ if and only if the following holds:

$$\frac{k}{\mathbb{E}_F s}\left(b' - \frac{k^2}{\delta^2 b'}\right) < \left(1 - \frac{2k^2}{\delta b' \mathbb{E}_F s}\right)\frac{\partial k}{\partial \delta},$$

where $\partial k/\partial\delta > k/\delta > 0$. A sufficient condition is:

$$\frac{k}{\mathbb{E}_F s}\left(b' - \frac{k^2}{\delta^2 b'}\right) \leq \left(1 - \frac{2k^2}{\delta b' \mathbb{E}_F s}\right)\frac{k}{\delta}.$$

Rearranging terms this is equivalent to:

$$\delta^2 b'^2 + k^2 \leq \delta b' \mathbb{E}_F s,$$

which, from (12), holds with equality. ∎

Proof of Proposition 2: By Theorem 1, at the unique symmetric $\epsilon-$equilibrium, all players undertake the same socialization. The random network of links thus corresponds with a canonical binomial Erdös-Rényi network where each link if formed with the common probability $k^*(\delta)/n$. For these networks, a giant component emerges abruptly with $k^*(\delta)$ when $k^*(\delta) = 1$, which, after some algebra, is equivalent to:

$$\alpha(b + \varepsilon)\delta^3 + \alpha\delta - 1 = 0.$$

This is a polynomial in δ of degree 3. Denote it $P(\delta)$.

Recall that $1 \leq \delta < \overline{\delta}$, with $\overline{\delta} = 1/\sqrt{2\alpha(b+\varepsilon)} > 1$. We have $P(1) = \alpha(b+\varepsilon) + \alpha - 1 < 0 \Leftrightarrow \alpha(1 + b + \varepsilon) < 1$, while $P(\overline{\delta}) > 0$ is equivalent to $1 + 4\alpha^2 + 2\alpha > 8\alpha(b+\varepsilon)$. So, all we need to find is values of α and $b + \varepsilon$ such that the following three conditions hold:

1. $2\alpha(b+\varepsilon) < 1$.

2. $\alpha(1+b+\varepsilon) < 1$.

3. $1+4\alpha^2+2\alpha > 8\alpha(b+\varepsilon)$.

Fix $b+\varepsilon$. When $\alpha = 0$, the three conditions hold trivially. By continuity, there exists an $\overline{\alpha}$, which depends on the value of $b+\varepsilon$, such that the conditions hold for $\alpha < \overline{\alpha}(b+\varepsilon)$.

For instance, suppose that $b+\varepsilon < 1$. Then, condition 2 implies condition 1, and a sufficient condition for 3 is $4\alpha^2 - 6\alpha + 1 > 0$, true when $0 \leq \alpha < (3-\sqrt{5})/4$. Then, when $b+\varepsilon < 1$, a sufficient condition for 1 through 3 is $0 \leq \alpha < (3-\sqrt{5})/4$. ∎

Proof of Theorem 3: We assume that the technology for producing k and s can be represented by $L_k = \sqrt{k}$ and $L_s = \sqrt{s}$. The planner subsidizes a fraction of the cost of the inputs. We denote by $1-\theta$ (resp. $1-\tau$) the fraction of own-effort (resp. socialization effort) being subsidized. The corresponding subsidized utilities are:

$$u_i^{\theta,\tau}(\mathbf{k},\mathbf{s};\varepsilon) = [b+\varepsilon_i + \alpha \sum_{j=1}^{n} p_{ij}s_j]s_i - \frac{1}{2}\theta s_i^2 - \frac{1}{2}\tau k_i^2.$$

Let $\epsilon > 0$ and suppose that n is high enough. At the unique $\epsilon-$equilibrium, the second-stage strategy is given by Theorem 1 where the exogenous parameters are $(\alpha/\theta, b/\theta, \varepsilon/\theta, \sigma_\varepsilon^2/\theta^2)$, while the first-stage strategy is

$$k^* = \frac{\theta}{2\alpha}\left[1 - \sqrt{1 - 4\frac{(b+\varepsilon)^2\alpha^2}{\theta^4\tau}}\right]. \tag{13}$$

In particular, the expected payoffs at the $\epsilon-$equilibrium are:

$$\mathbb{E}_F u_i^{\theta,\tau}(k) = \frac{(b+\varepsilon)^2}{\theta^2} - \frac{1}{2}\frac{\sigma_\varepsilon^2}{\theta^2} + \frac{1}{2}\tau k^2.$$

We now compute and compare the effect of a first unit of subsidy on s (by decreasing θ) and on k (by decreasing τ). More precisely, denote by $dT = 1$ the first unit of subsidy, where $T = (1-\theta)s^2 + (1-\tau)k^2$.

In what follows, we set $b' = b+\varepsilon$.

Suppose first that s, and only s, is subsidized, that is, $d\tau = 0$. All the change in dT corresponds to a change $d\theta$ in θ. Then:

$$\left.\frac{\partial \mathbb{E}_F u_i^{\theta,\tau}}{\partial T}\right|_{d\tau=0, d\theta>0} = \frac{\frac{\partial \mathbb{E}_F u_i^{\theta,\tau}(k)}{\partial \theta}}{2(1-\theta)s\frac{\partial s}{\partial \theta} + 2(1-\tau)k\frac{\partial k}{\partial \theta} - s^2}.$$

Starting from a situation with no subsidy, that is, $\theta = \tau = 1$, this becomes (with trivial notations, and after some algebra):

$$\frac{\partial \mathbb{E}_F u_i^{d\theta>0}}{\partial T} = -\frac{1}{s^2}\frac{\partial \mathbb{E}_F u_i}{\partial \theta} = -\frac{\alpha^2 b'^2}{k^2}\left[-2b'^2 + \sigma_\varepsilon^2 + k\left.\frac{\partial k}{\partial \theta}\right|_{\theta=\tau=1}\right]. \tag{14}$$

Suppose now that k, and only k, is subsidized, that is, $d\theta = 0$. Now, all the change in dT corresponds to a change $d\tau$ in τ. We have:

$$\left.\frac{\partial \mathbb{E}_F u_i^{\theta,\tau}}{\partial T}\right|_{d\tau>0, d\theta=0} = \frac{\partial \mathbb{E}_F u_i^{\theta,\tau}(k)}{2(1-\tau)k\frac{\partial k}{\partial \tau} - k^2}.$$

Again, starting from a situation with no subsidy where $\theta = \tau = 1$, we have:

$$\frac{\partial \mathbb{E}_F u_i^{d\tau>0}}{\partial T} = -\frac{1}{k^2}\frac{\partial \mathbb{E}_F u_i}{\partial \tau} = -\frac{1}{2} - \frac{1}{k}\left.\frac{\partial k}{\partial \tau}\right|_{\theta=\tau=1}. \tag{15}$$

We now compute (14) and (15).

From (13), we obtain:

$$\left.\frac{\partial k}{\partial \theta}\right|_{\theta=\tau=1} = k - \frac{4\alpha b'^2}{\sqrt{1-4b'^2\alpha^2}},$$

and

$$\left.\frac{\partial k}{\partial \tau}\right|_{\theta=\tau=1} = -\frac{\alpha b'^2}{1-2\alpha k}.$$

Let:

$$\Delta = \frac{\partial \mathbb{E}_F u_i^{d\theta>0}}{\partial T} - \frac{\partial \mathbb{E}_F u_i^{d\tau>0}}{\partial T} = 2\frac{\alpha^2 b'^4}{k^2} - \frac{\alpha^2 b'^2}{k^2}\sigma^2 - \alpha^2 b'^2 + \frac{4\alpha^3 b'^4}{k\,(1-2\alpha k)} + \frac{1}{2} - \frac{1}{k}\frac{\alpha b'^2}{1-2\alpha k}. \tag{16}$$

Then, a first unit of subsidy to k has a higher overall impact than a first unit of subsidy to s if and only if $\Delta < 0$.

Noting that $k = \alpha k^2 + \alpha b'^2$ from (11), and that $2\alpha k < 1$ from (13), we have:

$$-\alpha^2 b'^2 + \frac{4\alpha^3 b'^4}{k\,(1-2\alpha k)} = \frac{\alpha^2 b'^2}{k\,(1-2\alpha k)}\left[k + 2\alpha b'^2\right] < 0,$$

and:

$$\frac{1}{2} - \frac{1}{k}\frac{\alpha b'^2}{1-2\alpha k} = \frac{k - 2\alpha k^2 - 2\alpha b'^2}{2k(1-2\alpha k)} = \frac{-k}{2k(1-2\alpha k)}.$$

Therefore, we can write:

$$\Delta = 2\frac{\alpha^2 b'^4}{k^2} - \frac{\alpha^2 b'^2}{k^2}\sigma_\varepsilon^2 + \frac{\alpha^2 b'^2 (k + 2\alpha b'^2)}{k\,(1-2\alpha k)} - \frac{k}{2k(1-2\alpha k)}.$$

Then, with some algebra:

$$2k^2(1-2\alpha k)\Delta = \left(2\alpha b'^2 - k\right)\left[2\alpha b'^2 + k - 2\alpha^2 b'^2 k\right] - 2\alpha^2 b'^2 (1-2\alpha k)\sigma_\varepsilon^2.$$

Let $x = 4\alpha^2 b'^2 \in [0,1]$. Then $2\alpha k = 1 - \sqrt{1-x}$ and $\Delta < 0$ can be rewritten as:

$$g_{\alpha,\sigma_\varepsilon}(x) = \frac{1}{2\alpha^2}\left(x - 1 + \sqrt{1-x}\right)\left(\frac{x}{2} + 1 - \sqrt{1-x} + \frac{x}{2}\sqrt{1-x}\right) - x\sqrt{1-x}\sigma_\varepsilon^2 < 0.$$

Note that $g_{\alpha,\sigma_\varepsilon}(0) = g_{\alpha,\sigma_\varepsilon}(1) = 0$.

Let $y = \sqrt{1-x}$, so that $x = 1-y^2$. Define $h(y) = 8\alpha^2\sqrt{1-x}\, g'_{\alpha,\sigma}(x)$ on $[0,1]$. Derivating $g_{\alpha,\sigma_\varepsilon}(x)$, substituting by y and rearranging terms gives:

$$h(y) = (1-y)\left(-3 + 5y + 5y^2 + 5y^3\right) - \left(3y^2 - 1\right)\gamma,$$

where $\gamma = 4\alpha^2\sigma_\varepsilon^2 \geq 0$. This is a polynomial of degree four on $[0,1]$. We can deduce the sign of $g'_{\alpha,\sigma_\varepsilon}$ from that of h on $[0,1]$.

Derivating twice we have $h'(y) = -20y^3 - 6\gamma y + 8$, and $h''(y) = -60y^2 - 6\gamma \leq 0$, $\forall y \in [0,1], \lambda \geq 0$. Hence, h' decreases in $[0,1]$, with $h'(0) = 8 > 0$ and $h'(0) = -12 - 6\gamma < 0$. Therefore, h' has a unique root on $(0,1)$, denoted by y', h increases in $(0,y')$ and decreases on $(y',1)$, with $h(0) = \gamma - 3$ and $h(1) = -2\gamma < 0$.

Note that $g'_{\alpha,\sigma}(0) = h(1)/8\alpha^2 < 0$, and:

$$\lim_{x\uparrow 1} g'_{\alpha,\sigma_\varepsilon}(x) = \begin{cases} +\infty \text{ if } \gamma > 3 \\ -\infty \text{ if } 0 \leq \gamma < 3 \end{cases}.$$

Suppose, first, that $\gamma > 3$, that is, $\alpha^2\sigma_\varepsilon^2 > 3/4$. Then, h has a unique root in $(0,1)$, denoted by y^*, implying that $g'_{\alpha,\sigma_\varepsilon}(x)$ has also a unique root on $(0,1)$, given by $x^* = 1 - y^{*2}$. Therefore, $g_{\alpha,\sigma_\varepsilon}$ decreases on $(0,x^*)$ and increases on $(x^*,1)$. Given that $g_{\alpha,\sigma_\varepsilon}(0) = g_{\alpha,\sigma_\varepsilon}(1) = 0$, we deduce that $g_{\alpha,\sigma_\varepsilon}(x) < 0$ on $(0,1)$.

Suppose, now, that $\gamma < 3$, that is, $\alpha^2\sigma_\varepsilon^2 < 3/4$. Then, h has either zero or two roots in $(0,1)$. Therefore, there exists an $x' \leq 1$ such that $g_{\alpha,\sigma_\varepsilon}(x) < 0$ if and only if $0 \leq x \leq x'$. ∎