Summary

- Examples ➤
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- Subgame perfect Folk Theorem 1 ➤
- Subgame perfect Folk Theorem 2 ➤
- Subgame perfect Folk Theorem 3 ➤
Examples

A  Game A of chapter 1 repeated (finitely, infinitely) after observing the outcome of all past stages.

\[ P = \{1, \ldots, 18\}, \quad S_i = \mathbb{R}^+, \quad u_i(s) = 2 \sum_{j=1}^{18} \frac{s_j}{18} - s_i \]

B  Game B of chapter 2

Game Γ repeated once after observing the outcome of first stage.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>4,4</td>
<td>1,5</td>
</tr>
<tr>
<td>Y</td>
<td>5,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

C  One-dimensional (in payoffs) game.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>-2,-2</td>
<td>1,1</td>
</tr>
<tr>
<td>M</td>
<td>1,1</td>
<td>-2,-2</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
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Let

\[ G = \{N, \{A_i\}_{i \in N}, \{W_i\}_{i \in N}\} \]

where for all \( i \in N \), \( W_i \) are payoff functions: \( W_i : A_1 \times ... \times A_n \rightarrow \mathbb{R}^+ \).

- \( \Gamma(G) \) is \( G \) repeated (finite or infinite) after observing the outcome of previous repetitions.
- \( \Gamma(G) \) is the **repeated game**
- \( G \) is the **stage game**
- \( A_i \) is the **action set** of player \( i \).
- \( H^{t-1} \) set of all possible histories \( h^{t-1} \) up to time \( t - 1 \),
• A strategy in $\Gamma(G)$ is a function

$$\gamma_i : \bigcup_{i \in \mathbb{N}} H^{t-1} \rightarrow \Delta(A_i)$$

Each $h^{t-1} = ((a_1^1, ..., a_n^1), (a_1^2, ..., a_n^2), ..., (a_1^{t-1}, ..., a_n^{t-1})) = (a_1^1, a_2^1, ..., a_{t-1}^1)$ is composed of the entire sequence of (profiles of) actions for all players up to $t - 1$, and $\gamma_i(h^t) = a_i^t$

To define payoffs, let

$$\pi^\delta_i(h^T) = (1 - \delta) \sum_{t=1}^{T} \delta^{t-1} W_i(a^t)$$

where $T$ can be $\infty$, and for $T$ finite, $\delta = 1$, for simplicity.
There are other criteria for computing payoffs in infinitely repeated games.

The *limit of means*:

\[
\pi_i^\infty(h^T) = \lim_{T \to \infty} \inf \frac{1}{T} \sum_{t=1}^{T} W_i(a^t)
\]

The *overtaking criterion*: A sequence \( h^\infty = (a^1, a^2, ...) \) is preferred to \( \hat{h}^\infty = (\hat{a}^1, \hat{a}^2, ...) \) if

\[
\exists \tau_0 \in \mathbb{N} : \forall \tau > \tau_0, \sum_{t=1}^{\tau} W_i(a^t) > \sum_{t=1}^{\tau} W_i(\hat{a}^t)
\]

Exercise: Think of three different sequences, each one of which is the one most *strictly* preferred under each criterion.
One-stage deviation principle.

**Definition 1** \( \gamma = (\gamma_1, ..., \gamma_n) \in \Psi \) is a subgame-perfect equilibrium of the repeated game \( \Gamma(G) \) if there is no \( i \in N, \gamma'_i \in \Psi_i \) and \( h^{t'} \) such that \( \gamma_i(h^{t'}) \neq \gamma'_i(h^{t'}) \), \( \gamma_i(h^t) = \gamma'_i(h^t) \) \( \forall h^t \neq h^{t'} \) and

\[
\pi_i^\delta(\gamma'_i, \gamma_{-i}|h^{t'}) > \pi_i^\delta(\gamma_i, \gamma_{-i}|h^{t'})
\]

Let $\text{conv} F$ be the convex hull of $F$, or smallest convex set $\hat{F}$ such that $F \subset \hat{F}$. Then,

$$V \equiv \text{conv}\{v \in \mathbb{R}^n | v = W(a), a \in A_1 \times ... A_n\}$$
Convex Hull for payoffs in Game C
Let any $i \in N$, and let $V_i$ be the projection of $V$ on the coordinate $i$. Then:

- $\tilde{v}_i \in V_i$ is the lowest payoff that $i$ can obtain in any Nash equilibrium of the stage game $G_i$.

- $\hat{v}_i = V_i$ is defined:
  
  $\hat{v}_i = \min_{\alpha_i \in \Delta(A_i)} \max_{\alpha_{-i} \in \Delta(A_{-i})} W_i(\alpha_i, \alpha_{-i})$.

- $v_i^* = \max_{\alpha \in \Delta(A)} W_i(\alpha)$. 
Minmax payoffs for player 1 in Game B
Minmax payoffs for player 1 in Game C

\[ s_2 = \text{Prob}(R) \]

\[ u(T, s_2) \]
\[ u(M, s_2) \]
\[ u(B, s_2) \]
Theorem 2 (Friedman 1971) Let \( v \in V \) with \( v_i > \tilde{v}_i \) for all \( i \in N \). There exists \( \tilde{\delta} < 1 \) such that if \( 1 > \delta > \tilde{\delta} \), there exists a subgame-perfect equilibrium of the repeated game \( \Gamma(G) \) whose payoffs for each player \( i \in N \) coincide with \( v_i \).

**Proof.** Suppose there exists a pure \( a \in A \) such that \( W(a) = v \). Denote \( \tilde{\alpha}^j \) an action profile such that \( W_j(\tilde{\alpha}^j) = \tilde{v}_j \). Then let the strategy profile \( \gamma \) as follows:

\[
\begin{align*}
\gamma_i(h^{t-1}) &= a_i \text{ if } \forall \tau \leq t - 1, \text{ there is no unilateral deviation.} \\
\gamma_i(h^{t-1}) &= \tilde{\alpha}_i^j, \text{ otherwise, with } j \text{ being the first unilateral deviator.}
\end{align*}
\]
Suppose first that $h^t$ is such that no player has ever deviated unilaterally. Then the payoff for player $i$ if choosing an alternative action $a'_i$ rather than $a_i$ is bounded above by

$$(1 - \delta^{t-1}) v_i + (1 - \delta) \delta^{t-1} v^*_i + \delta^t \tilde{v}_i$$

the payoff for keeping the same strategy is

$$(1 - \delta^{t-1}) v_i + (1 - \delta) \delta^{t-1} v_i + \delta^t v_i$$

The difference between these two amounts is:

$$\delta^{t-1} ((1 - \delta)(v^*_i - v_i) + \delta(\tilde{v}_i - v_i))$$

and this is smaller than 0 for $\delta$ close to 1, since $\tilde{v}_i - v_i < 0$. 
Suppose, on the other hand that $h^t$ is such that some player has deviated unilaterally at some $\tau < t$.

Then, a deviation at $t$ cannot possibly change future behavior (so its profitability or not is independent of the future), and it cannot increase profits at $t$, since the actions form an equilibrium of the stage game.

Finally, let $\delta_i$ such that

$$\left( (1 - \delta_i)(v_i^* - v_i) + \delta_i(\tilde{v}_i - v_i) \right) < 0$$

That is,

$$\delta_i > \frac{v_i^* - v_i}{v_i^* - \tilde{v}_i}$$

Obviously, it must be true that for $\delta > \delta_i$

$$\left( (1 - \delta)(v_i^* - v_i) + \delta(\tilde{v}_i - v_i) \right) < 0$$

Thus, if we define $\bar{\delta}$ as $\max_{i \in N}\{\delta_i\}$, the result follows.
Repeated games are not always “nice.”
Theorem 3 (Fudenberg and Maskin 1986) Suppose that the dimension of $V = n$. Then for any $v \in V$ with $v_i > \hat{v}_i$ for all $i \in N$, there exists $\delta < 1$ such that if $1 > \delta > \bar{\delta}$, there exists a subgame-perfect equilibrium of the repeated game $\Gamma(G)$ whose payoffs for each player $i \in N$ coincide with $v_i$.

Proof. Suppose there exists a pure $a \in A$ such that $W(a) = v$. Suppose also, there is a pure $\hat{a}^j$ for all $j \in N$ such that $W_j(\hat{a}^j) = \hat{v}_j$.

Choose a vector $v' \in int(V)$ and $\varepsilon > 0$ such that for all $i \in N$

$$\hat{v}_i < v'_i < v_i$$

and the vector

$$v'(i) = (v'_1 + \varepsilon, ..., v'_{i-1} + \varepsilon, v'_i, v'_{i-1} + \varepsilon, ..., v'_n) \in V$$

The full dimension of $V$ guarantees $v'(i)$ exists.
Assume also that there is a pure action profile $a(i)$ for all $i \in N$ such that $W_j(a(i)) = v(i)_j$.

Let $w_i^j = W_i(\hat{a}^j)$ the payoff of $i$ when minmaxing $j$. Choose $T$ such that for all $i$

$$v_i^* + T\hat{v}_i < \min_{a \in A} W_i(a) + Tv_i'$$

This $T$ guarantees that, if $\delta$ is close to 1, deviating once (and getting $v_i^*$) and then being minmaxed $T$ periods is worse than getting the worst possible thing once and then getting $v_i'$ for $T$ periods.
Now let the strategy profile $\gamma$ as follows:

**Phase I** For histories $h^t \in \text{Phase I}$, $\gamma_i(h^t) = a_i$. $h^0 \in \text{Phase I}$, and $h^t \in \text{Phase I}$ unless a unilateral deviation from $a_j$. If such a deviation by player $j$ arises at $t$, $h^{t+1} \in \text{Phase II}_j$

**Phase II}_j For histories $h^t \in \text{Phase II}_j$, $\gamma_i(h^t) = \hat{a}_i^j$. After the first period $\tau$ such that $h^\tau \in \text{Phase II}_j$ the histories $h^t \in \text{Phase II}_j$ for $t \in [\tau, \tau + T - 1]$ unless an unilateral deviation from $\gamma_i(h^t) = \hat{a}_i^j$. If such a deviation by player $i$ arises at $t \in [\tau, \tau + T]$, $h^{t+1} \in \text{Phase II}_i$, otherwise $h^{\tau+T} \in \text{Phase III}_j$

**Phase III}_j For histories $h^t \in \text{Phase III}_j$, $\gamma_i(h^t) = a(j)_i$. After the first period $\tau$ such that $h^\tau \in \text{Phase III}_j$ the histories $h^t \in \text{Phase III}_j$ unless an unilateral deviation from $\gamma_i(h^t) = a(j)_i$. If such a deviation by player $i$ arises at $t$, $h^{t+1} \in \text{Phase II}_i$, otherwise $h^t \in \text{Phase III}_j$ for all $t \geq \tau$. 
To show this strategy profile $\gamma$ is a subgame-perfect equilibrium, by the one-stage deviation principle, it suffices to show that no player $i \in N$ can gain after any history $h^t$ by choosing $a_i \neq \gamma_i(h^t)$ and conforming to $\gamma_i(h^s)$ for $s > t$.

**Deviation in Phase I** The payoff from deviating once is bounded above by:

$$(1 - \delta)v_i^* + \delta(1 - \delta^T)\hat{v}_i + \delta^{T+1}v_i'$$

The payoff from not deviating is $v_i$. Since $\hat{v}_i < v_i' < v_i$, the payoff from not deviating is bigger for $\delta$ close enough to 1.
Deviation in Phase III

The payoff from deviating once for $i \neq j$ is bounded above by:

$$(1 - \delta)v_i^* + \delta(1 - \delta^T)\hat{v}_i + \delta^{T+1}v_i'$$

The payoff from not deviating is $v_i' + \epsilon$. Since $\hat{v}_i < v_i' < v_i' + \epsilon$, the payoff from not deviating is bigger for $\delta$ close enough to 1.

The payoff from deviating once for Player $j$ is bounded above by:

$$(1 - \delta)v_j^* + \delta(1 - \delta^T)\hat{v}_j + \delta^{T+1}v_j'$$

The payoff from not deviating is $v_j'$. The inequality

$v_i^* + TV_i < \min_{a \in A} W_i(a) + Tv_i'$

guarantees that not deviating is optimal.
Deviation in Phase II\(_j\)  The payoff from not deviating for \(i \neq j\) when \(T'\) periods in the Phase remain is:

\[
(1 - \delta^{T'} ) w^j_i + \delta^{T'} (v^l_i + \varepsilon)
\]

If this player deviates she gets at most

\[
(1 - \delta) v^*_i + \delta (1 - \delta^T) \hat{v}_i + \delta^{T+1} v^l_i
\]

Since \(v^l_i + \varepsilon > v^l_i\) not deviating is optimal for \(\delta\) high enough.

The payoff from not deviating for player \(j\) when \(T'\) periods in the Phase remain is:

\[
(1 - \delta^{T'} \hat{v}_j + \delta^{T'} v^l_i
\]

If this player deviates she gets at most

\[
(1 - \delta) \hat{v}_j + \delta (1 - \delta^T) \hat{v}_j + \delta^{T'} v^l_i
\]

Obviously not deviating is optimal (here notice that deviating is pointless as there is no possible immediate gain when being minmaxed and it prolongs punishment).
Theorem 4 (Benoit and Krishna 1985) Suppose that for all \( i \in N \), there is a Nash equilibrium of the stage game \( G, \bar{a}^i \) such that \( W_i(\bar{a}^i) > W_i(\tilde{a}^i) \), and that the dimension of \( V = n \). Then for any \( v \in V \) with \( v_i > \tilde{v}_i \) for all \( i \in N \), and for all \( \varepsilon > 0 \), there is a \( T^* \) such for \( T > T^* \) there exists a subgame-perfect equilibrium of the repeated game \( \Gamma^T(G) \) whose payoffs for each player \( i \in N \) \( v'_i \) are such that \( |v_i - v'_i| < \varepsilon \).

Proof. Assume, as usual that there is \( a \in A \) with \( W(a) = v \), and also that \( v_i > \tilde{v}_i \) for all \( i \in N \) (the general case is similar to the previous theorem).
Consider a terminal path \((a^{T-n+1}, a^{T-n+2}, \ldots, a^T)\) with \(a^{T-n+i} = \bar{a}^i\) for \(i \in N\). Since

\[ a^i W_i(\bar{a}^i) \text{ is the worst NE payoff.} \]

\[ \bar{a}^i \text{ is a NE with } W_i(\bar{a}^i) > W_i(\tilde{a}^i) \]

The average payoff in this path is strictly bigger for any \(i \in N\) than that from the constant path \((\tilde{a}^i, \tilde{a}^i, \ldots, \tilde{a}^i)\) in that period.

Let \(\mu_i > 0\), be this difference in payoffs, and \(\mu = \min_{i \in N} \mu_i\)

Now let \(q\) paths like that one. Comparing those \(q\) paths with \(q\) constant paths \((\tilde{a}^i, \tilde{a}^i, \ldots, \tilde{a}^i)\) the difference in payoffs is at least \(q\mu\).
Both paths can be part of subgame-perfect equilibria.

Let now strategies:

I $\gamma_i(h^{t-1}) = a_i$ if $\forall t \leq T - qn$, and for all $\tau \leq t - 1$ there was no unilateral deviation from $a_j$ in $\tau$.

II $\gamma_i(h^{t-1}) = \tilde{a}_i^j$ if $\forall t > T - qn$, and for all $\tau \leq t - qn$ there was no unilateral deviation from $a_j$ in $\tau$. $\tilde{a}_i^j$ is chosen so that $j = n - [T - t]_n$

III $\gamma_i(h^{t-1}) = \tilde{a}_i^j$ otherwise, where $j$ is the first player to unilaterally deviate from $a_j$ in $\tau \leq T - qn$.

For sufficiently high $q$ the strategies are best responses to one another at all $h^t$ (check) if $T^* > qn$. $q$ is independent of $T^*$. So just choose $T > T^*$ and the result follows. •
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Chapter 5
Repeated Games - Folk Theorems

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