Microeconomics II - Winter 2006
Chapter 4
Games with Incomplete Information
Perfect Bayesian and Sequential equilibrium

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Summary

- Examples
- WPBE and Sequential equilibrium
- WPBE and Sequential equilibrium: examples
A  Game B of chapter 2

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
F & \rightarrow & (1,1) \\
A & \rightarrow & (2,1) \\
B & \rightarrow & (0,0) \\
2 & \rightarrow & (1,0) \\
& \rightarrow & (1,0) \\
& \rightarrow & (0,0) \\
& \rightarrow & (1,1) \\
\end{array}
\]
Beer-Quiche.
C Game with a WPBE equilibrium which is not sequential.
D Spence education model (Osborne-Rubinstein’s version).

- A worker (sender) knows her ability $\theta$. The firm (receiver) does not.

- The value of the worker to the firm is $\theta$ and the wage the worker receives is the firm expectation of $\theta$ (competition plus equal expectations).

- Let’s say to make it a “real” game that payoff of employer is $-(w - \theta)^2$ (the expectation of this is maximized at $w = E(\theta)$.)

- The worker sends a signal $e$, the level of education. Her payoff is $w - e/\theta$. There are two types of workers $\theta^L$ and $\theta^H$, with probabilities $p^H$ and $p^L$. 
Let a game
\[ \Gamma = \{ N, \{K_1, ..., K_n\}, R, \{H_1, ..., H_n\}, \{A(x)\}_{x \in K \setminus Z}, \{(\pi_1(z), ..., \pi_n(z))\}_{z \in Z}\} \]

A (Weak) Perfect Bayesian equilibrium (WPBE) is a profile of behavioral strategies such that there exist beliefs with:

- **a** Strategies are *optimal* at *all* information sets, *given the beliefs* (for every node there is a belief \( \mu(x) \geq 0 \), with the requirement \( \sum_{x \in h} \mu(x) = 1 \)).

- **b** Beliefs are *consistent* with the strategies and Bayes rule, wherever possible.

Why *wherever possible*? Because some information sets may not be visited in equilibrium (remember example A).
Formally:

**Definition 1** A behavioral strategy profile $\gamma^* = (\gamma_1^*, \ldots, \gamma_n^*) \in \Psi$ is a weak perfect Bayesian equilibrium for game $\Gamma$ if there exists a system of beliefs $\mu^* = \{(\mu^*(x))_{x \in h}\}_{h \in H}$ such that the assessment $(\gamma^*, \mu^*)$ satisfies the following conditions:

(a) $\forall i \in N, \forall h \in H_i, \forall \gamma_i \in \Psi_i,$

$$\pi_i(\gamma^*|\mu^*, h) \geq \pi_i(\gamma_i, \gamma_{-i}^*|\mu^*, h)$$

(b) $\forall h \in H, \forall x \in h,$

$$\mu^*(x) = \frac{\Pr(x|\gamma^*)}{\Pr(h|\gamma^*)}, \text{ if } \Pr(h|\gamma^*) > 0.$$
Definition 2 Let $\gamma \in \Psi$ be a completely mixed behavioral strategy profile for game $\Gamma$ (that is, $\forall i \in N, \forall h \in H_i, \forall a \in A(h_i), \gamma_i(h)(a) > 0$).

A corresponding assessment $(\mu, \gamma)$ is consistent if $\forall h \in H, \forall x \in h$ we have $\mu(x) = \frac{\Pr(x|\gamma)}{\Pr(h|\gamma)}$.

Definition 3 Let $\gamma \in \Psi$ be any behavioral strategy profile for game $\Gamma$ (not necessarily completely mixed).

A corresponding assessment $(\mu, \gamma)$ is consistent if it is the limit of a sequence of consistent assessments $\{(\mu_k, \gamma_k)\}_{k=1,2,...}$ where $\gamma_k$ is completely mixed for all $k = 1, 2, ...$.
Definition 4 A strategy profile $\gamma^* = (\gamma_1^*, ..., \gamma_n^*) \in \Psi$ is a sequential equilibrium of $\Gamma$ if there exists a system of beliefs $\mu^*$ such that:

\begin{itemize}
    \item[a] $(\gamma^*, \mu^*)$ is a consistent assessment
    \item[b] $\forall i \in N, \forall h \in H_i, \forall \gamma_i \in \Psi_i$
    \[ \pi_i(\gamma^* | \mu^*, h) \geq \pi_i(\gamma_i, \gamma_{-i}^* | \mu^*, h) \]
\end{itemize}

This definition implies a sequential equilibrium is necessarily WPBE.
WPBE and Sequential equilibrium: examples

Game B of chapter 2.

$$\pi_2(a|\mu, h) = 2\mu(A) + \mu(B) > \pi_2(b|\mu, h) = \mu(A) - 2\mu(B)$$

Thus, by requirement (a) of WPBE, player 2 should play $a$ (independently of $\mu$, and the only best response of player 1 is to play $A$.

$(A, a)$ is thus the only WPBE equilibrium, sustained by beliefs $\mu(A) = 1$.

There is another Nash equilibrium, which is also subgame-perfect $(F, b)$, but not WPBE.

The only WPBE is also sequential, for beliefs $\mu(A) = 1$.

To see this, take a sequence putting probability $(1/k, 1 - 2/k, 1/k)$ respectively on $(F, A, B)$ and $(1 - 1/k, 1/k)$ on $(a, b)$.

This sequence converges to $(A, a)$ and the beliefs associated to it, $\mu^k(A) = \frac{1 - 2/k}{1 - 1/k}$. From this $\lim_{k \to \infty} \mu^k(A) = 1$. 
Beer-Quiche.

There are no separating WPBE equilibria. That is, the Sender-player 1 cannot choose a different action in each information set.

To see this consider the situation where $\gamma^*_s(W) = B, \gamma^*_s(S) = Q$.

Then $\mu(W|B) = 1, \mu(W|Q) = 0$.

Thus, the best response of Receiver-player 2 is:

$\gamma^*_r(B) = D$ (since $\pi_r(D, \gamma^*_s|\mu, B) = 1 > \pi_s(N, \gamma^*_s|\mu, Q) = 0$)

$\gamma^*_r(Q) = N$ (since $\pi_r(D, \gamma^*_s|\mu, Q) = 0 > \pi_s(N, \gamma^*_s|\mu, Q) = -1$).

But then the Sender is not optimizing as $\pi_s(B, \gamma^*_r|W) = 0 < \pi_s(Q, \gamma^*_r|W) = 3$. 
Now consider the situation where $\gamma^*_s(W) = Q, \gamma^*_s(S) = B$.

Then $\mu(W|B) = 0, \mu(W|Q) = 1$.
Thus, the best response of Receiver-player 2 is:
$\gamma^*_r(B) = N$ (since $\pi_r(D, \gamma^*_s|\mu, B) = -1 < \pi_s(N, \gamma^*_s|\mu, Q) = 0$)
$\gamma^*_r(Q) = D$ (since $\pi_r(D, \gamma^*_s|\mu, Q) = 1 > \pi_s(N, \gamma^*_s|\mu, Q) = 0$).
But then the Sender is not optimizing as
$\pi_s(Q, \gamma^*_r|W) = 1 < \pi_s(B, \gamma^*_r|W) = 2$. 
There is a pooling WPBE equilibrium with $\gamma_s^*(W) = B, \gamma_s^*(S) = B$.

Then $\mu(W|B) = 0.1$. Thus, the best response of Receiver is:
$\gamma_r^*(B) = N$ (since $\pi_r(N, \gamma_s^*|\mu, B) = 0 > \pi_s(D, \gamma_s^*|\mu, B) = 1 \times 0.1 - 1 \times 0.9$).
The response after $Q$ depends on beliefs
(since $\pi_r(N, \gamma_s^*|\mu, Q) = 0$ and $\pi_s(D, \gamma_s^*|\mu, Q) = 1 \times \mu(W|Q) - 1 \times \mu(S|Q)$).

In order to show that a pooling equilibrium as above
we need beliefs such that the best response (by Receiver) is such that $B$
is optimal for both types of Sender.
One such response is if $\gamma_r^*(Q) = D$, since then
$\pi_s(Q, \gamma_r^*|W) = 1 < \pi_s(B, \gamma_r^*|W) = 2$
and $\pi_s(Q, \gamma_r^*|S) = 0 < \pi_s(B, \gamma_r^*|S) = 3$.

Some beliefs that would work are $\mu(W|Q) = 1$
as then $\pi_r(N, \gamma_s^*|\mu, Q) = 0 < \pi_s(D, \gamma_s^*|\mu, Q) = 1$. 
There is a pooling equilibrium with $\gamma_s^*(W) = B, \gamma_s^*(S) = Q$.

Then $\mu(W|Q) = 0.1$. Thus, the best response of Receiver-player 2 is: $\gamma_r^*(Q) = N$ (since $\pi_r(N, \gamma_s^*|\mu, Q) = 0 > \pi_s(D, \gamma_s^*|\mu, Q) = 1 \times 0.1 - 1 \times 0.9$). The response after $B$ depends on beliefs (since $\pi_r(N, \gamma_s^*|\mu, B) = 0$ and $\pi_s(D, \gamma_s^*|\mu, B) = 1 \times \mu(W|B) - 1 \times \mu(S|B)$).

In order to show that there is a pooling equilibrium as above we need beliefs such that the best response (by Receiver) is such that $Q$ is optimal for both types of Sender. One such response is if $\gamma_r^*(B) = D$, since then $\pi_s(B, \gamma_r^*|W) = 0 < \pi_s(Q, \gamma_r^*|W) = 3$ and $\pi_s(B, \gamma_r^*|S) = 1 < \pi_s(Q, \gamma_r^*|S) = 2$.

Some beliefs that would work are $\mu(W|B) = 1$, as then $\pi_r(N, \gamma_s^*|\mu, B) = 0 < \pi_s(D, \gamma_s^*|\mu, B) = 1$.
WPBE and Sequential equilibrium: examples

Game with WPBE not sequential

\[(A, b, U)\] is a WPBE equilibrium, as long as \(\mu(a) \geq 2 \times \mu(b) = 2 \times (1 - \mu(a))\).

Notice that under that condition, this equilibrium satisfies the requirement (a) of the definition,

\[
\pi_1(A, \gamma_{-1}) = 1 > \pi_1(B, \gamma_{-1}) = 0, \pi_1(A, \gamma_{-1}) = 1 > \pi_1(C, \gamma_{-1}) = 0,
\]

and \(\pi_2(a, \gamma_{-2} | \mu) = \mu(B) \times 0 + \mu(C) \times 0 \leq \pi_2(b, \gamma_{-2} | \mu) = \mu(B) \times 0 + \mu(C) \times 1\)

and \(\pi_3(U, \gamma_{-3} | \mu) = \mu(a) \times 1 + \mu(b) \times 0 \geq \pi_3(V, \gamma_{-3} | \mu) = \mu(a) \times 0 + \mu(b) \times 2\)

(since \(\mu(a) \geq 2 \times \mu(b)\)).

These beliefs also satisfy requirement (b) because given \(\gamma_1(A) = 1\) any beliefs satisfy the definition.
WPBE and Sequential equilibrium: examples
(7/10)

\((A, b, U)\) is NOT a sequential equilibrium. The reason is that beliefs with \(\mu(a) \geq 2 \times \mu(b) = 2 \times (1 - \mu(a))\) cannot be part of a consistent assessment.

Let any beliefs \(\mu(a), \mu(b)\) be part of a consistent assessment where \(\gamma = (A, b, U)\). Let also \((\gamma_{1}^{k}, \gamma_{2}^{k}, \gamma_{3}^{k})\), be the sequence that converges to \(\gamma\). Then, in a consistent assessment

\[
\mu^{k}(a) = \frac{\gamma_{1}^{k}(B) \times \gamma_{2}^{k}(a)}{\gamma_{1}^{k}(B) \times \gamma_{2}^{k}(a) + \gamma_{1}^{k}(B) \times \gamma_{2}^{k}(b)} = \frac{\gamma_{2}^{k}(a)}{\gamma_{2}^{k}(a) + \gamma_{2}^{k}(b)} = \gamma_{2}^{k}(a);
\]

and

\[
\mu^{k}(b) = \gamma_{2}^{k}(b).
\]

Thus, since we know that \(\lim_{k \to \infty} \gamma_{2}^{k}(a) = 0\) we must have in a consistent assessment that \(\mu(a) = 0 < 2(1 - \mu(a))\).
Spence education model (Osborne and Rubinstein’s version).

**Pooling equilibrium.** \( e_L = e_H = e^* \).

In this case, necessarily, \( \mu(\theta^H|e^*) = p^H \), thus \( w(e^*) = p^H \theta^H + p^L \theta^L \). For this to be an equilibrium we need that for all alternative \( e \),

\[
    w(e) - e/\theta^i \leq w(e^*) - e^*/\theta^i \quad \text{for} \quad i = H, L.
\]

The easiest way to achieve this is if the firm believes that all deviations come from \( \theta^L \). Thus \( \mu(\theta^H|e) = 0 \), and \( w(e) = \theta^L \) if \( e \neq e^* \). Thus, best possible deviation is if \( e = 0 \) (the salary is equal for all \( e \neq e^* \) and the cost is lowest at \( e = 0 \)). Then \( w(0) \leq w(e^*) - e^*/\theta^i \) or \( i = H, L \) if \( \theta^L \leq p^H \theta^H + p^L \theta^L - e^*/\theta^L \), that is, if \( e^* \leq \theta^L \frac{p^H}{\theta^H - \theta^L} \).
WPBE and Sequential equilibrium: examples

Separating equilibrium. \( e_L = 0 \neq e_H = e^* \).

In this case, we must have necessarily \( e_L = 0 \).

Suppose not. Then \( e_L > 0 \). In as separating equilibrium \( w(e_L) = \theta^L \).
Furthermore, the wage for \( w(0) = \mu(\theta^H|0)\theta^H + \mu(\theta^L|0)\theta^L \geq \theta^L \).
But the cost of education is 0, so that the payoff under \( e = 0 \) is \( \theta^L \),
whereas under \( e_L \) it is \( \theta^L - e_L < \theta^L \), a contradiction.

In order for neither worker wanting to choose a different \( e \), it is easiest to assume \( \mu(\theta^H|e) = 0 \) if \( e \neq e^* \).

Then, the best possible deviation for \( \theta^H \) is \( e = 0 \)
(same wage and more cost otherwise)
and the best possible deviation for \( \theta^L \) is \( e^* \)
(same wage as with \( e = 0 \) and more cost otherwise).
To have that \( e_L = 0 \neq e_H = e^* \) are optimal now only requires that:

\[
\theta^L \geq \theta^H - e^*/\theta^L \quad \text{and} \quad \theta^L \leq \theta^H - e^*/\theta^H
\]

This is equivalent to

\[
(\theta^H - \theta^L)\theta^H \geq e^* \geq (\theta^H - \theta^L)\theta^L
\]
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