Networks - Fall 2005
Chapter 2
Play on networks 2: Strategic complements
Ballester, Calvó-Armengol and Zenou 2005

October 31, 2005
Summary

• Introduction
• Nash equilibrium in pure strategies.
• Example
• Interpretation: Counting path length
• Policy: The Key Player
• Generalization of above set-up.
Introduction (1/4)

- Let network $g$ with $g_{ij} \in \{0, 1\}$.

- For all $i \in N$, action $x_i \geq 0$.

- $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = g_{ij} b''(x_i + \bar{x}_i) \leq 0$ in Bramoullé-Kranton.

- $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = g_{ij} \lambda \geq 0$ here. Local strategic complements.

- Linear-quadratic utilities

$$u_i(x_1, ..., x_n; g) = \alpha x_i - \frac{1}{2} x_i^2 + \lambda \sum_{j \in N} g_{ij} x_i x_j; \lambda \geq 0, \alpha > 0.$$
• With $\lambda = 0$, no interdependence and $x^*_i = \alpha$.

• With $\lambda > 0$, interdependence.

• FOC:

$$\frac{\partial u_i}{\partial x_i} = \alpha - x_i + \lambda \sum_{j \in N} g_{ij} x_j = 0.$$ 

• FOC ($x_i - \lambda \sum_{j \in N} g_{ij} x_j = \alpha$) in general gives a system of equations

$$[I - \lambda G] \vec{x} = \alpha \vec{1}.$$ 

• Determinant of $[I - \lambda G]$ is a polynomial in $\lambda$, thus generically invertible matrix.
• We study this more in depth later.

• Now, suppose you have a regular network, where for all \( i \in N, \sum_{j \in N} g_{ij} = k \).

• Then an equilibrium exists with \( x_i = x \) for all \( i \in N \). We must have \( \alpha - x + \lambda k x = 0 \), thus \( x^* = \frac{\alpha}{1 - \lambda k} \) (assuming \( \lambda k < 1 \)).

• For \( \lambda > 0 \), \( x^*(\lambda) \) is increasing in \( \lambda \) (when equilibrium exists).

• In general, outcome will depend on the network, when there is heterogeneity.
Remark 1  We show here there is a generically unique Nash equilibrium in pure strategies.

• Notice that \( u_i(x_1, \ldots, x_n; g) \) is such that \( \frac{\partial^2 u_i}{\partial x_i^2} = -1 < 0 \). This implies:

• \( x^* \) is a Nash equilibrium iff for all \( i \in N \) either

1. (a) \( x_i^* = 0 \) and \( \frac{\partial u_i}{\partial x_i}(0, x_{-i}^*) \leq 0 \)

(b) \( x_i^* > 0 \) and \( \frac{\partial u_i}{\partial x_i}(x^*) = 0 \).

• But notice that if \( x_i^* = 0 \), \( \frac{\partial u_i}{\partial x_i}(0, x_{-i}^*) = \alpha + \lambda \sum_{j \in N} g_{ij} x_j^* > 0 \).

• Thus only (b) is relevant and \( x^* \) is a Nash equilibrium iff:

\[
[I - \lambda G] \overrightarrow{x^*} = \alpha \overrightarrow{1}, \quad \text{and} \quad x_i^* > 0 \quad \text{for all} \ i \in N.
\]
Nash equilibrium in pure strategies. (2/5)

- Solution of former equation exists and is unique iff \( \det [I - \lambda G] \neq 0 \).

- There exists a finite number of values of \( \lambda \) such that \([I - \lambda G]\) is degenerate, and it has Lebesgue measure zero, thus generically unique Nash equilibrium.

- When a solution exists, is it necessarily in \( \mathbb{R}^+ \)?

- Debreu and Herstein (1953), the matrix \([I - \lambda G]^{-1} = M(g, \lambda)\) is well-defined and non-negative iff \( \lambda \) is smaller than the largest eigenvalue of \( G \).

- If \( \lambda \) is small enough

\[
[I - \lambda G]^{-1} = \sum_{k \geq 0} \lambda^k G^k
\]
Nash equilibrium in pure strategies. (3/5)

- To see this diagonalize $G = P^{-1} \begin{bmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n \end{bmatrix} P$.

- Thus $\lambda^k G^k = P^{-1} \begin{bmatrix} (\lambda \mu_1)^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\lambda \mu_n)^k \end{bmatrix} P$.

- So if $\lambda \max_i \{\mu_i\} < 1$, $\sum_{k \geq 0} \lambda^k G^k$ converges and
  \[
  \bar{x}^* = \alpha [I - \lambda G]^{-1} \bar{1}
  \]

- Summarizing the above we have:

Proposition 2 Let $\mu_1(g)$ be the largest positive eigenvalue of $G$. If $\lambda \mu_1(g) < 1$, the game has a unique interior pure strategy equilibrium given by
  \[
  \frac{x_i^*}{\alpha} = m_{i1}(g, \lambda) + \ldots + m_{in}(g, \lambda)
  \]
Nash equilibrium in pure strategies. (4/5)

\[ M(g, \lambda) = \left[ m_{ij}(g, \lambda) \right] = [I - \lambda G]^{-1} = \sum_{k \geq 0} \lambda^k G^k. \]

Notice differences with previous model:

1. Equilibrium unique with complement - multiplicity with substitutes.

2. Equilibrium interior with complement - interior equilibria unstable with substitutes.
Suppose a 3 person network, with 1 connected to 2 and 3.

\[ G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow G^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad G^3 = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \]

• By induction

\[ G^{2p} = \begin{bmatrix} 2^p & 0 & 0 \\ 0 & 2^{p-1} & 2^{p-1} \\ 0 & 2^{p-1} & 2^{p-1} \end{bmatrix}, \quad G^{2p+1} = \begin{bmatrix} 0 & 2^p & 2^p \\ 2^p & 0 & 0 \\ 2^p & 0 & 0 \end{bmatrix} \]

• \[ x^*_1 = \sum_{p=0}^{\infty} \left[ \lambda 2^p 2^p + \lambda 2^{p+1} 2^p + \lambda 2^{p+1} 2^p \right] = \frac{1}{1-2\lambda^2} + \frac{2\lambda}{1-2\lambda^2} = \frac{1+2\lambda}{1-2\lambda^2} \]

• \[ x^*_2 = x^*_3 = \sum_{p=0}^{\infty} \left[ \lambda 2^{p+1} 2^p + \lambda 2^p 2^{p-1} + \lambda 2^p 2^{p-1} \right] = \frac{1+\lambda}{1-2\lambda^2}. \]
• Condition for existence $1 - 2\lambda^2 > 0$, $\lambda < 1/\sqrt{2}$.

• In general for a star with $n$ nodes, largest eigenvalue of $G = \sqrt{n - 1}$. 
Interpretation: Counting path length (1/3)

• How many paths are there (in example) starting at node $i$ between individuals $i$ and $j$ with length 2 (not repeating traveled through nodes)?

• Between 1&1 - 2, between 1&2 or 1&3 - 0.

• Between 2&1 - 0, between 2&2 or 2&3 -1.

• Between 3&1 - 0, between 3&2 or 3&3 -1.

• Notice that $G^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

• This is general. For $G^k = \left[ g_{i,j}^{[k]} \right]$ counts total number of paths in $g$ of length $k$ starting at node $i$ between individuals $i$ and $j$. 
Interpretation: Counting path length (2/3)

- Now $\sum_{k \geq 0} \lambda^k g_{ij}^{[k]}$ is the total number of paths in $g$ of all lengths between individuals $i$ and $j$ but discounting paths of length $k$ by $\lambda^k$.

- Remember $m_{ij}(g, \lambda) = \sum_{k \geq 0} \lambda^k g_{ij}^{[k]}$.

**Definition 3** Bonacich (1987). Take network $g$ and parameter $\lambda$ small enough. The network centrality of individual $i$ in $g$ of parameter $\lambda$ is

$$b_i(g, \lambda) \equiv \sum_{j=1}^{n} m_{ij}(g, \lambda) = m_{ii}(g, \lambda) + \sum_{j \neq i} m_{ij}(g, \lambda)$$

- Since $\frac{x_i^*}{\alpha} = \sum_{j=1}^{n} m_{ij}(g, \lambda) = b_i(g, \lambda)$, the equilibrium action is proportional so Bonacich centrality.
In first place one must propose a planner’s objective.

1. \( F(g; \lambda, \alpha) = \sum_{j=1}^{n} x_j^* = \alpha \sum_{j=1}^{n} b_i(g, \lambda) \). This may be the measure if the network is simply a “factor of production” of a “good” or a “bad” (the model was originally created to study crime.)

2. \( G(g; \lambda, \alpha) = \sum_{j=1}^{n} u_j(x^*; g) \). This is more useful if we think of a “public good” setup.

For the second measure notice that by FOC \( \alpha - x_i^* + \lambda \sum_{j \in N} g_{ij} x_j^* = 0 \). Thus

\[
u_j(x^*; g) = x_i^* \left( \alpha - \frac{1}{2} x_i^* + \lambda \sum_{j \in N} g_{ij} x_j^* \right) = x_i^* \left( 0 + \frac{1}{2} x_i^* \right) = \frac{1}{2} x_i^*^2 \]
And thus

\[ G(g; \lambda, \alpha) = \frac{1}{2} b_i(g, \lambda)^2. \]

**PLANNER’S TOOLS-THE KEY PLAYER**

- Classical public economics tools (tax subsidy) modify: \( \lambda, \alpha \).
- To the extent she can control it \( \rightarrow \) Modify \( g \)
  - Reshuffle network.
  - Eliminate link(s).

**Definition 4** Node \( i \) is a Key Player iff

\[
i \in \arg \max_{j \in N} \left\{ \sum_{k=1}^{n} b_k(g, \lambda) - \sum_{k \neq j} b_k(g^{-j}, \lambda) \right\}
\]
• Notice that
\[
\sum_{k=1}^{n} b_k(g, \lambda) - \sum_{k \neq j} b_k(g^{-j}, \lambda) = \underbrace{b_i(g)}_{\text{i's direct contribution}} + \sum_{k \neq j} \left( b_k(g, \lambda) - b_k(g^{-j}, \lambda) \right).
\]

• Thus Key Player need not be the player with highest centrality, since indirect contribution also matters.

• Example:

**Proposition 5** Node $i$ is a Key Player iff

\[
i \in \arg \max_{j \in N} \left\{ \frac{b_j(g, \lambda)^2}{m_{jj}(g)} \right\}
\]

To show this we first prove:
Lemma 6 $m_{ij}(g) \cdot m_{ik}(g) = m_{ii}(g) \left[ m_{jk}(g) - m_{jk}(g^{-1}) \right]$

Proof. $m_{ii}(g) = \sum_{p \geq 0} \lambda^p g_{ii}^{[p]}$

$m_{jk}(g) - m_{jk}(g^{-1}) = \sum_{p \geq 2} \lambda^p \left[ g_{jk}^{[p]} - g_{jk}^{[p]} \right]$

Thus $

B = \sum_{p=2}^{\infty} \lambda^p \left[ \sum_{r+s=p \atop r \geq 0, s \geq 2} g_{ii}^{[r]} \cdot g_{j(i)k}^{[s]} \right]$
Notice that \( \left( \sum_{p\geq1} \lambda^p x^p \right) \left( \sum_{p\geq1} \lambda^p y^p \right) = \sum_{p\geq2} \lambda^p \left( \sum_{r+s=p} x^r y^s \right) \)

Thus
\[
\sum_{p\geq2} \lambda^p \sum_{r^i+s^i=p} g_{ji}^{[r^i]} \cdot g_{ik}^{[s^i]} = \left( \sum_{p\geq1} \lambda^p g_{ji}^{[p]} \right) \left( \sum_{p\geq1} \lambda^p g_{ji}^{[p]} \right)
\]

Now to prove the proposition. By lemma:
\[
\sum_{k\neq j} \left( b_k(g, \lambda) - b_k(g^{-j}, \lambda) \right) = \sum_{j\neq i} \sum_k \left[ m_{jk}(g) - m_{jk}(g^{-1}) \right]
\]
\[
= \sum_{j\neq i} \sum_k \frac{m_{ij}(g) \cdot m_{ik}(g)}{m_{ii}(g)}
\]
\[
= \sum_{j\neq i} \frac{m_{ij}(g)}{m_{ii}(g)} \sum_k m_{ik}(g)
\]
\[
= \sum_{j\neq i} \frac{m_{ij}(g)}{m_{ii}(g)} \sum_k m_{ik}(g)_{b_i(g,\lambda)}
\]
Thus:

\[
b_i(g) + \sum_{k \neq j} \left( b_k(g, \lambda) - b_k(g^{-j}, \lambda) \right) = b_i(g) \left[ 1 + \sum_{j \neq i} \frac{m_{ij}(g)}{m_{ii}(g)} \right]
\]

\[
= b_i(g) \left[ \frac{m_{ii}(g) + \sum_{j \neq i} m_{ij}(g)}{m_{ii}(g)} \right]
\]

\[
= \frac{b_i(g)^2}{m_{ii}(g)}
\]

- Note that \( \frac{b_i(g)^2}{m_{ii}(g)} = b_i(g) \left[ 1 + \sum_{j \neq i} \frac{m_{ij}(g)}{m_{ii}(g)} \right] \),

- Thus what matters is not only centrality, but also the composition of the contribution.

- If the relative weight of outer paths to self loops is larger, more likely to be Key Player.
Generalization of above set-up.

Let

\[ u_i(x_1, \ldots, x_n; g) = \alpha x_i + \sum_{j \in N} \sigma_{ij} x_i x_j; \lambda \geq 0, \alpha > 0. \]

\[ \sigma = \min_{ij \in g} \sigma_{ij}; \quad \bar{\sigma} = \max_{ij \in g} \sigma_{ij}; \quad \frac{\partial^2 u_i}{\partial x_i^2} = \sigma_{ii} < 0 \]

Conditions: \( \sigma_{ii} = \sigma < \min\{0, \bar{\sigma}\} \), concavity on myself is highest.

In Bramoullé-Kranton:

\[ \frac{\partial^2 u_i}{\partial x_i^2} = b''(x_i + \bar{x}_i) = \frac{\partial^2 u_i}{\partial x_i \partial x_j}; \text{ if } g_{ij} \neq 0. \]
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