Building socio-economic Networks: How many conferences should you attend?

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January 2006
Summary

- Introduction
- The game
- Equilibrium (large economies)
- Response to incentives
- Policies: how should you spend your first dollar?
- A couple of extensions
● Spillovers between different agents generate incentives for “linking.”
  ● Research and development.
  ● Labor Market Information.
  ● Friendships and “Social Capital.”

● If linking is done “non-cooperatively,” inefficiencies arise (overlinking - underwork), so role for policy.
• Prior work:
  
  
  
  
  • Networks (theory): Jackson (2005), Goyal and Moraga (2001).
  
  • Networks (empirics): Pammolli and Riccaboni (2001), Owen-Smith et al. (2004).
• They do not look very much at endogenous and costly network formation.

• When they do, they simplify away the game after forming the network.

• Reason: Analytical intractability.
We analyze a network formation game in two stages:

- First - Socialization effort.
- Second - Productive effort.

The key simplification is: undirected socialization.

- Each link created with probability equal to product of socialization efforts.
- Thus random network.

Strategy space much simpler (one dimensional for each player - rather than $n - 1$-dimensional), so equilibrium is a smaller-sized fixed point.
• Equilibrium: for “large” groups - unique and symmetric.

• An increase in the returns to “success” makes socialization effort relatively stronger.
  
  • An explanation for the explotion of R&D collaboration.
  
  • Perhaps also for the decrease in social capital.

• Public policy: where should you put your first euro?
**The game**

Let $N = \{1, \ldots, n\}$ be a set of players.

We consider a two-stage game:

**Stage one:** Players select $k_i > 0$. $i$ and $j$ interact with probability:

$$g_{ij}(k) = g_{ji}(k) = \frac{k_i k_j}{\sum_{l \in N} k_l} = \frac{k_i k_j}{n \langle k \rangle}.$$

**Interim stage:** Learn $k$ and i.i.d. shocks on $[\varepsilon, \bar{\varepsilon}]$, expected value $\varepsilon$, and variance $\sigma_\varepsilon^2$.

**Stage two:** Players select $s_i > 0$. Let $p_{ij} = g_{ij}$ if $i \neq j$, and $p_{ii} = g_{ii}/2$.

**Player $i$’s utility:**

$$u_i(s, k) = [b + \varepsilon_i + \alpha \sum_j p_{ij} s_j]s_i - \frac{1}{2} s_i^2 - \frac{1}{2} k_i^2$$

where $b > 0$ and $\alpha \geq 0$. 
Productive effort

Let $G(k) = [g_{ij}(k)]_{i,j\in N}$ be matrix of random links. Define:

$$\lambda(k) = \frac{\alpha \langle k \rangle}{\langle k \rangle - \alpha \langle k^2 \rangle}.$$

**Lemma 1** When $p_{ii} < 1/2\alpha$, the unique interior Nash equilibrium in pure strategies of the second-stage game is:

$$s^*(k) = b\beta(k) + M(k) \cdot \varepsilon \quad (1)$$

$$M(k) = [I - \alpha G(k)]^{-1} = \sum_{p=0}^{+\infty} \alpha^p G^p(k).$$
• \( m_{ij}(k) \) counts the total number of direct and indirect paths in the expected network \( G(k) \), where paths of length \( p \) are weighted by the decaying factor \( \alpha^p \).

\[
m_{ij}(k) = \begin{cases} 
\lambda(k)g_{ij}(k), & \text{if } i \neq j \\
1 + \lambda(k)g_{ii}(k), & \text{if } i = j 
\end{cases}
\]

• Define \( \beta_i(k) = m_{i1}(k) + ... + m_{in}(k) \). This is the sum of all paths stemming from \( i \) in the expected network where links are independently and randomly drawn with probability \( (g_{ij}(k)) \).

• \( \beta_i(k) = 1 + \lambda(k)k_i \) is a measure of centrality in the random graph \( G(k) \), reminiscent of the Bonacich centrality measure for fixed networks.
Socialization effort

The expected payoffs are:

\[Eu_i(k) = (b + \varepsilon)E(s_i) + \alpha E(\sum_j p_{ij}s_is_j) - \frac{1}{2}E(s_i^2) - \frac{1}{2}k_i^2\]

\[= (b + \varepsilon)^2 \beta_i + \alpha \sum_j p_{ij} \omega_{ij} - \frac{1}{2} \omega_{ii} - \frac{1}{2}k_i^2\]

Obtaining the equilibrium profile \(k^*\) is messy. The first-order conditions are:

\[k_i = (b + \varepsilon)^2 \frac{\partial \beta_i}{\partial k_i} + \alpha \sum_j \left[ p_{ij} \frac{\partial \omega_{ij}}{\partial k_i} + \frac{\partial p_{ij}}{\partial k_i} \omega_{ij} \right] - \frac{1}{2} \frac{\partial \omega_{ii}}{\partial k_i} \quad (2)\]

where:
\[
\frac{\partial \lambda}{\partial k_i} = \frac{\alpha^2}{n} \frac{2k_i - \langle k^2 \rangle}{\left( \langle k \rangle - \alpha \langle k^2 \rangle \right)^2} \\
\frac{\partial \beta_i}{\partial k_i} = \lambda(k) + k_i \frac{\partial \lambda}{\partial k_i} \\
\frac{\partial p_{ij}}{\partial k_i} = \frac{\partial g_{ij}}{\partial k_i} = \frac{1}{n} \left[ \frac{k_j}{\langle k \rangle} - \frac{k_i k_j}{n \langle k \rangle^2} \right], \text{ if } i \neq j \\
\frac{\partial p_{ii}}{\partial k_i} = \frac{1}{2} \frac{\partial g_{ii}}{\partial k_i} = \frac{1}{2n} \left[ \frac{k_i}{\langle k \rangle} - \frac{k_i^2}{n \langle k \rangle^2} \right] \\
\frac{1}{\sigma^2} \frac{\partial \omega_{ij}}{\partial k_i} = \left( 2 + \lambda \frac{\langle k^2 \rangle}{\langle k \rangle} \right) \left[ \frac{\partial \lambda}{\partial k_i} g_{ij} + \frac{\partial g_{ij}}{\partial k_i} \lambda \right] \\
+ \lambda g_{ij} \left[ \frac{\partial \lambda}{\partial k_i} \frac{\langle k^2 \rangle}{\langle k \rangle} + \frac{1}{n} \left[ 2\lambda \frac{k_i}{\langle k \rangle} - \lambda \frac{\langle k^2 \rangle}{\langle k \rangle^2} \right] \right]
\]
Equilibrium (large economies) (5/8)

\[
\frac{1}{\sigma^2} \frac{\partial \omega_{ii}}{\partial k_i} = \left( 2 + \lambda \frac{k^2}{\langle k \rangle} \right) \left[ \frac{\partial \lambda}{\partial k_i} g_{ii} + \frac{\partial g_{ii}}{\partial k_i} \lambda \right] + \lambda g_{ii} \left[ \frac{\partial \lambda}{\partial k_i} \frac{\langle k^2 \rangle}{\langle k \rangle} + \frac{1}{n} \left[ 2 \lambda \frac{k_i}{\langle k \rangle} - \lambda \frac{\langle k^2 \rangle}{\langle k \rangle^2} \right] \right]
\]

In a symmetric equilibrium:

\[
k = (b + \varepsilon)^2 \lambda + \frac{(b + \varepsilon)^2}{n} (2 - k) \lambda^2 + \lambda \sigma^2 \left[ \frac{2 \lambda^2}{n^2} (2 - k) (1 + \lambda k) + \frac{\lambda^2 k}{n} + \frac{2 \lambda}{n} \left( 1 - \frac{1}{n} \right) \right] \left[ \frac{k}{n} \left( n - \frac{1}{2} \right) - \frac{1}{2} \right] + \alpha \sigma^2 \left[ \frac{1}{n} \left( 1 - \frac{1}{n} \right) \left[ \frac{1}{2} + \left( n - \frac{1}{2} \right) \left[ \frac{k}{n} (2 + \lambda k) \right] \right] \right]
\]
Lemma 2 For $n$ large, $k^*$ is $O(n^0)$.

Proof. When $k^*$ is $O(n^p)$ for $p > 0$, $\lim_{n \to +\infty} \lambda k = -1$, and $\lim_{n \to +\infty} \lambda = 0$. Thus:

$$\lim_{n \to +\infty} (((b + \varepsilon)^2 \lambda + \frac{(b + \varepsilon)^2}{n} (2 - k) \lambda^2) = 0$$

$$\lim_{n \to +\infty} \frac{\lambda}{\sigma^2 \varepsilon} \left[ \frac{2 \lambda^2}{n^2} (2 - k) (1 + \lambda k) + \frac{\lambda^2 k}{n} + \frac{2 \lambda}{n} \left(1 - \frac{1}{n}\right) \right] \left[ \frac{\lambda}{n} \left(n - \frac{1}{2}\right) - \frac{1}{2} \right] = 0$$

$$\lim_{n \to +\infty} \alpha \sigma^2 \frac{1}{n} \left(1 - \frac{1}{n}\right) \left[ \frac{1}{2} + \left(n - \frac{1}{2}\right) \left[ \frac{k}{n} (2 + \lambda k) \right] \right] = 0$$

Right hand side tends to zero, but left hand side goes to infinity. \[\square\]
Proposition 3  For $n$ large, there is a unique symmetric subgame perfect equilibrium. The actions $s^*(k)$ at the second stage are given in Lemma 1 while the equilibrium value of $k$ in the first stage tends to

$$\lim_{n \to +\infty} k \equiv k^* = \frac{1}{2\alpha} \left(1 - \sqrt{1 - 4(b + \varepsilon)^2\alpha^2}\right)$$

Proof. By the previous lemma, $k^*$ is $O(n^0)$. This implies that in the limit we have to satisfy:

$$k = (b + \varepsilon)^2 \lambda$$

So the equilibrium candidate must solve:

$$k - \alpha k^2 - (b + \varepsilon)^2 \alpha = 0$$

Lemma 4  For $n$ large, there are no asymmetric equilibria.
The equilibrium approximation for \((\alpha, b + \varepsilon, \sigma_\varepsilon) = (1, 0.1^{0.5}, 1)\)

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Response to incentives (1/2)

Relative response of $s$ and $k$ to a change in $(\alpha, b + \varepsilon, \sigma_\varepsilon)$

Proposition 5 Let $(\alpha, b + \varepsilon, \sigma_\varepsilon)$ be scaled by factor $1/\sqrt{2\alpha(b + \varepsilon)} > \delta \geq 1$. Then, equilibrium $k^*$ increases more than $s^*(k^*)$ in percentage terms.

- Number of Mergers and Acquisitions and R&D Collaborations per Month in the pharmaceutical industry. One-Year Moving Averages (Pammolli and Riccaboni, 2001).
Response to incentives (2/2)

Reaching the giant component: phase transition.

**Proposition 6** Let \((α, b + ε, σ_ε)\) be scaled by \(1/\sqrt{2α(b + ε)} > δ ≥ 1\). There exists a threshold \(\bar{α}\) such that, for \(α < \bar{α}\), when δ reaches a threshold value of \(δ^*\), the equilibrium network jumps from a fragmented graph to a highly connected graph (single giant component).

Symmetric equilibrium is Erdős-Rényi random graph (each link is binomial parameter \(k^*/n\). The transition happens when \(k = 1\), i.e. when the following holds:

1. \(2α(b + ε) < 1\).  2. \(α < \frac{1}{1+b+ε}\).  3. \(1 + 4α^2 + 2α > 8α(b + ε)\).

For example, when \(b + ε < 1\), this happens when \(α < \left(3 - \sqrt{5}\right)/4\). 


Policies: how should you spend your first dollar?

(1/2)

Relative impact of \( k \) and \( s \) subsidy

The technology for producing \( k \) and \( s \) is: \( L_k = \frac{1}{2}\sqrt{k} \) and \( L_s = \frac{1}{2}\sqrt{s} \).

Subsidies are a fraction of the cost of the labor input ((\( 1 - \theta \), \( 1 - \tau \))):

\[
\begin{align*}
  u_i &= \left( b + \varepsilon_i + \alpha \sum_j p_{ij}s_j \right) s_i - \frac{1}{2} \theta s_i^2 - \frac{1}{2} \tau k_i^2
\end{align*}
\]

In second stage we have: \( s_i^*(\alpha, \beta, \sigma^2) \).

In first stage: \( \tau k = \frac{(b+\varepsilon)^2}{\theta^2} \lambda(\theta) \), which implies that \( k^* = \frac{\theta}{2\alpha} \left[ 1 - \sqrt{1 - 4 \frac{(b+\varepsilon)^2 \alpha^2}{\theta^4 \tau}} \right] \), so that in particular

\[
E u_i(k) = \frac{(b + \varepsilon)^2}{\theta^2} - \frac{1}{2} \sigma^2 + \frac{1}{2} \tau k^2
\]

Now we will show the effect of the first unit of subsidy, on \( k \) and on \( s \).
That is, we compute $\frac{\partial E u_i(k)}{\partial T}$, where $T = (1 - \theta)s^2 + (1 - \tau)k^2$
for $d\tau > 0, d\theta = 0$ and for $d\tau = 0, d\theta > 0$ and compare.

**Theorem 7** When $\alpha^2\sigma^2 > 3/4$, the first unit of subsidy is always optimally allocated to socialization effort, $k_i$. When $\alpha^2\sigma^2 < 3/4$ the first unit of subsidy is optimally allocated to socialization effort $k_i$ if and only if the expected marginal return to own investment, $b + \varepsilon$, is low enough.
A couple of extensions

1. Decisions taken simultaneously - No qualitative changes.

2. Heterogeneity - \( b = (b_1, ..., b_n) \)

   (a) A mean-preserving spread of \( b \) leads to a mean-preserving spread of both \( s \) and \( k \), and a shift upwards in the mean.

   (b) \( k_i \) is \( i \)'s expected connectivity. So, we can map distribution of fundamentals into distribution of connectivity (beyond Erdös-Renyi).
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