7 Appendix B (for online publication)

Proof of Lemma 2: Rearranging terms in (22) and simplifying we get:

\[ D_r(K, \gamma) = \left( K \left( \frac{1}{\gamma + 1} \right) - \frac{2}{K - 1} \frac{1}{\gamma + 1} \right) \left( \frac{1}{K + 1} \right)^\gamma + \frac{1}{K - 1} \frac{1}{\gamma + 1} \left( \frac{1}{2} \right)^{\gamma^{-1}}, \quad (33) \]

and hence

\[ \frac{\partial D_r}{\partial K}(K, \gamma) = -\frac{1}{(K-1)^2} \frac{1}{\gamma+1} \left( \frac{1}{2} \right)^{\gamma^{-1}} + \left( K \left( \frac{-\gamma}{\gamma + 1} \right) K + 1 \right) \frac{1}{K + 1} + \frac{2}{K - 1} K + 1 \frac{\gamma}{\gamma + 1} + \frac{2}{(K-1)^2} \frac{1}{\gamma+1} \left( \frac{1}{K + 1} \right)^\gamma. \quad (34) \]

Now note that the inequality \( \frac{\partial D_r}{\partial K}(K, \gamma)/\partial K > 0 \) is equivalent to:

\[ \frac{2(K-1)}{K+1} \gamma + (K-1)^2 + 2 > (K+1) \left( \frac{K+1}{2} \right)^{\gamma^{-1}} + \gamma K \frac{(K-1)^2}{K+1}, \]

or

\[ (K-1)^2 \left( 1 - \frac{\gamma K}{K+1} \right) + 2 \left( 1 + \gamma \frac{K-1}{K+1} \right) > (K+1)^{\gamma} \frac{1}{2^{\gamma-1}}, \]

which can be rewritten as

\[ (K-1)^2 + 2 + \gamma (K-1)(2-K) > (K+1)^{\gamma} \frac{1}{2^{\gamma-1}}. \quad (35) \]

So, using the identities

\[ (K-1)^2 + 2 + (K-1)(2-K) = K+1 \]

and

\[ (K+1)^{\gamma} \frac{1}{2^{\gamma-1}} = 2 \left( \frac{K+1}{2} \right)^{\gamma} \]

we can equivalently write (35) as follows:

\[ \frac{K+1}{2} - \left( \frac{K+1}{2} \right)^{\gamma} - \frac{1}{2} (1 - \gamma) (K-1)(2-K) > 0. \]

Denote by \( \Xi(K, \gamma) \) the term on the left hand side of the previous inequality, conceived as a function of \( K \) and \( \gamma \). Then, to complete the proof, we establish the following property:

\[ \forall K > 1, \quad \Xi(K, \gamma) \geq 0 \iff \gamma \leq 1. \quad (36) \]

To show this property, note first that \( \Xi(K, 1) = 0 \) for all \( K \), so that \( \frac{\partial \Xi}{\partial K}(K, \gamma) = 0 \) for \( \gamma = 1 \) and all \( K \). On the other hand,

\[ \frac{\partial \Xi}{\partial \gamma}(K, \gamma) = - \left( \frac{K+1}{2} \right)^{\gamma} \ln \frac{K+1}{2} + \frac{1}{2} (K-1)(2-K) \leq - \ln \frac{K+1}{2} + \frac{1}{2} (K-1)(2-K), \]
the inequality being strict for all \( K > 1 \). It is then easy to verify that the terms on the two sides of the above inequality are equal to 0 when \( K = 1 \) and the term on the right hand side is negative for all \( K > 1 \), establishing (36) and hence also \( \partial D_r(K, \gamma) / \partial K \gtrless 0 \Leftrightarrow \gamma \gtrless 1 \).

We conclude, as stated in the proposition, that the minimum of \( D_r(K, \gamma) \) is attained at the maximum admissible value of \( K \) (i.e. \( N - 1 \)) when \( \gamma > 1 \), while it is attained at the lowest value of \( K \) (i.e. \( K = 1 \)) when \( \gamma < 1 \). This completes the proof of the proposition.

\[ \text{Proof of Lemma 3} \]

Note first that the expression of \( \partial D_r(K, \gamma) / \partial K \) obtained in (33) can be conveniently rewritten as follows:

\[
\frac{\partial D_r}{\partial K}(K, \gamma) = -\frac{1}{(K-1)^2} \left( \frac{1}{2} \right)^{\gamma-1} - \frac{1}{(K-1)^2} \left( \frac{1}{2} \right)^{\gamma-1} + \frac{1}{(K-1)^2} \gamma + \frac{1}{(K-1)^2} \gamma + \frac{2}{(K-1)^2} \gamma + \frac{1}{(K-1)^2} \gamma \right)
\]

Differentiating again with respect to \( K \) yields:

\[
\frac{\partial^2 D_r}{\partial K^2}(1/2, K, \gamma) = 2 \left( \frac{1}{2} \right)^{\gamma-1} - \frac{1}{(K-1)^3} \gamma + \frac{1}{(K-1)^3} \gamma + \frac{2}{(K-1)^3} \gamma + \frac{1}{(K-1)^3} \gamma + \frac{2}{(K-1)^3} \gamma + \frac{1}{(K-1)^3} \gamma \right)
\]

Hence

\[
\frac{\partial^2 D_r}{\partial K^2}(1/2, K, \gamma) < 0
\]

if and only if

\[
G(K) = \frac{(K+1)^{\gamma+1}}{8 (K-1)^3} (K (\gamma - 2) + 2\gamma^2) + \gamma (K-1)^2 + \gamma (K+1) < 1 \quad (37)
\]

\[28\text{We have in fact}\]

\[
\frac{d}{dK} \left( K \frac{K+1}{2} + \frac{1}{2} (K-1) (2-K) \right) = \frac{K+1-2K^2}{K+1} < \frac{2K (1-K)}{K+1} < 0
\]

\[29\text{To complete the argument we verify the claimed continuity property of } D_r(K, \gamma) \text{, as in (33), at } K = 1:\]

\[
\lim_{K \to 1} D_r(K, \gamma) = \left( \frac{1}{2} \right)^{\gamma} \frac{1}{\gamma+1} - \lim_{K \to 1} \frac{1}{K-1} \gamma + \frac{2}{(K+1)^3} \gamma = \left( \frac{1}{2} \right)^{\gamma} - \frac{-\gamma^2}{(\gamma+1)^4} = \left( \frac{1}{2} \right)^{\gamma} = D_r(1, \gamma)
\]
First, we observe that $G(1) = 1$. Thus, to establish (37), it is enough to show that $G$ is decreasing for all $K > 1$. Letting $x \equiv K - 1$ for notational simplicity, $\frac{dG(K)}{dK} < 0$ if, and only if,

$$\frac{d}{dx} \left( \frac{(\frac{x}{2} + 1)^{\gamma+1}}{x} \right) < \frac{d}{dx} \left( \gamma x \left( \frac{x}{2} (x(1-\gamma) + 1 + \gamma) + \frac{1}{2} + \frac{x}{2} \right) \right),$$

or:

$$\frac{2^{\gamma+1} (\frac{x}{2} + 1)^{\gamma+1}}{(\frac{x}{2} + 1)^{\gamma+1}} = \frac{\gamma + 1}{x + 2} < \frac{\frac{1}{2} + \frac{1}{4}x^{2} + \frac{1}{2}x^{2} + \frac{3}{8}x^{2} - \frac{3}{8}x^{2} + \frac{3}{8}x^{2} + \frac{1}{2}}{\frac{x}{2} + \frac{x}{2}x^{2} + \frac{x}{2}x^{2} - \frac{x}{2}x^{2} + \frac{1}{2}}$$

The above inequality is equivalent to the following one:

$$\left( \frac{\gamma}{2} + \frac{1}{2}x^{2} + \frac{1}{2}x^{2} + \frac{3}{4}x^{2} - \frac{3}{4}x^{2} + 1 \right) (x + 2) > (\gamma + 1) \left( x + \gamma x + \frac{1}{4}x^{2} + \frac{1}{4}x^{2} + \frac{3}{4}x^{2} + 1 \right),$$

or

$$\gamma x + \frac{1}{2}x^{2} + \frac{1}{2}x^{2} + \frac{3}{4}x^{2} + 3x^{2} - \frac{3}{8}x^{2} + x + 2 \gamma + x^{2} + x^{2} + \frac{1}{2}x^{2} - \frac{3}{2}x^{2} + 2 \gamma + x^{2} + x^{2} + \frac{1}{4}x^{2} + \frac{1}{2}x^{2} + \frac{3}{4}x^{2} - \frac{1}{4}x^{2} + 1 + x + \gamma x + \frac{1}{4}x^{2} + 1 + y x + \gamma x + \frac{1}{4}x^{2} + \frac{1}{2}x^{2} + \frac{1}{4}x^{2} + 1 + \gamma + \gamma x + \gamma x + \frac{1}{2}x^{2} + \frac{1}{4}x^{2} + \frac{3}{4}x^{2} + \gamma x + \gamma x + \gamma x + \frac{1}{4}x^{2} + \frac{1}{2}x^{2} + \frac{3}{4}x^{2} + \gamma x + \gamma x + \gamma x + \gamma + 1 > 0.$$
\[ D_c(N - 1, \gamma) - D_r(N - 1, \gamma) = \]
\[ = \left( \left( \frac{1}{N} \right)^\gamma - \left( \frac{1}{2} \right)^\gamma \right) \left( \frac{2}{N - 2} \frac{1}{\gamma + 1} - \frac{N - 1}{1 + \gamma} \right) + \left( \frac{1}{2} \right)^\gamma \left[ \left( \frac{1}{N} \right)^{-1} - \frac{N - 1}{1 + \gamma} \right] \]

Since \( 2 \leq (N - 1)(N - 2) \) for \( N \geq 3 \), we have that for all\(^{31} \) \( N > 1 + (1 + \gamma)^\frac{1}{\gamma} \) the desired conclusion follows:
\[ D_c(K, \gamma) - D_r(K, \gamma) < 0. \]

This completes the proof. \( \blacksquare \)

**Proof of Proposition 4:** From (23), we can write:
\[ D_c(K, \gamma, \gamma', p) = p K \left( \frac{1}{2K} \right)^\gamma + (1 - p) K \left( \frac{1}{2K} \right)^{\gamma'} \]

Hence
\[ \frac{\partial D_c}{\partial K}(K, \gamma, \gamma', p) = -p (\gamma - 1) \left( \frac{1}{2K} \right)^\gamma - (1 - p) (\gamma' - 1) \left( \frac{1}{2K} \right)^{\gamma'} \]. \hspace{1cm} (39)

and \( \frac{\partial D_c}{\partial K} > 0 \) is equivalent to
\[ (1 - p) (1 - \gamma') \left( \frac{1}{2K} \right)^{\gamma'} > p (\gamma - 1) \left( \frac{1}{2K} \right)^\gamma, \]
or, since \( \gamma > 1 \) and \( \gamma' < 1 \),
\[ K > \frac{1}{2} \left( \frac{p (\gamma - 1)}{(1 - p)(1 - \gamma')} \right)^{\frac{1}{\gamma - \gamma'}}. \]

This implies that \( D_c(K, \gamma, \gamma', p) \) is minimized at the point
\[ \hat{K}(p) = \frac{1}{2} \left( \frac{p (\gamma - 1)}{(1 - p)(1 - \gamma')} \right)^{\frac{1}{\gamma - \gamma'}} \]

provided this point is admissible, i.e. \( \hat{K}(p) \in [1, N - 1] \).

Compute next the second derivative of \( D_c(\cdot) \):
\[ \frac{\partial^2 D_c}{\partial K^2}(K, \gamma, \gamma', p) = p (\gamma - 1) \frac{\gamma}{K} \left( \frac{1}{2K} \right)^\gamma + (1 - p) \frac{\gamma'}{K} \left( \frac{1}{2K} \right)^{\gamma'} \]
\[ \geq p (\gamma - 1) \frac{\gamma}{K} \left( \frac{1}{2K} \right)^\gamma + (1 - p) \frac{\gamma'}{K} \left( \frac{1}{2K} \right)^{\gamma'} \]
\[ = -\frac{\gamma}{K} \frac{\partial D_c}{\partial K}(K, \gamma, \gamma', p) \]

Thus \( \frac{\partial^2 D_c}{\partial K^2}(1/2, K, \gamma, \gamma', p) > 0 \) for all feasible \( K < \hat{K}(p) \), i.e. the function \( D_c(\cdot) \) is convex in this range.

\(^{31}\)Note that \( (1 + \gamma) < (N - 1)^{\gamma - 1} (N - 1) < (N)^{\gamma - 1} (N - 1) \)
The optimal degree of segmentation for the completely connected structure is obtained as a solution of problem 21. Denote by \((K_i^*)_{i=1}^C\) a vector of component sizes that solves this optimization problem. We will show that there exists some appropriate range \([p_0, p_1]\) such that if \(p \in [p_0, p_1]\), the optimal component sizes are such that \(K_i^* = K_j^* = K^*\) for all \(i, j = 1, 2, \ldots, C\) and some common \(K^*\) with \(2 \leq K^* \leq N - 2\).

Choose \(p_0\) such that \(\hat{K}(p_0) = \frac{N}{2} - 1\). Such a choice is feasible and unique since by A.3 \(N > 4\), \(\hat{K}(\cdot)\) is increasing in \(p\), \(\hat{K}(0) = 0\), and \(\hat{K}(p) \to \infty\) as \(p \to 1\). Next we show that, for all \(p \geq p_0\), whenever \(C \geq 2\), the vector \((K_i^*)_{i=1}^C\) solving problem 21 satisfies:

\[
\forall i, j = 1, 2, \ldots, C, \quad K_i^* = K_j^* \leq \hat{K}(p) \tag{40}
\]

Let \(K_i^*\) and \(K_j^*\) stand for any two component sizes that are part of the solution to the optimization problem. First note that, since \(\hat{K}(p) \geq N/2 - 1\), if \(K_i^* \geq \hat{K}(p)\) then we must have that \(K_j^* < \hat{K}(p)\). But such asymmetric arrangement cannot be part of a solution to problem 21 because \(D_c(\cdot, \gamma, \gamma', p)\) is increasing at \(K_i^*\) and decreasing at \(K_j^*\). Hence a sufficiently small increase of \(K_j^*\) and a decrease of \(K_i^*\), which keeps \(K_i + K_j\) unchanged, is feasible and allows to decrease the expected mass of defaults. The only possibility, therefore, is that \(K_j^* \leq \hat{K}(p)\) and \(K_j^* \leq \hat{K}(p)\).

To complete the argument and establish (40), suppose that at an optimum we have \(K_i^* \neq K_j^*\) for at least two components \(i, j\). Since, as shown in the previous paragraph, neither \(K_i^*\) nor \(K_j^*\) can exceed \(\hat{K}(p)\), both \(K_i^*, K_j^*\) lie in the convex part of the function \(D_c(\cdot, \gamma, \gamma', p)\). It follows, therefore, that if we replace these two (dissimilar) components with two components of equal size \(\frac{1}{2}(K_i^* + K_j^*)\), feasibility is still satisfied and the overall expected mass of defaults is reduced, contradicting that the two heterogeneous components of size \(K_i^*, K_j^*\) belongs to an optimum configuration.

We have thus shown that, when \(p \geq p_0\), if at the optimum we have \(C \geq 2\), the unique optimal configuration involves a uniform segmentation in components of common size \(K^*(p) \leq \hat{K}(p)\). It remains then to show that at the optimum we indeed have \(C \geq 2\).

At \(p = p_0\) the optimum exhibits two components, \(C = 2\), since the optimal component size \(\hat{K}(p_0) = N/2 - 1\) is feasible. Since \(\hat{K}(p)\) is increasing and continuous in \(p\) and \(D_c(K, \gamma, \gamma', p)\) is continuous in \(K\), by continuity there exists some \(p_1\), with \(p_0 < p_1 < 1\), such that for all \(p \in (p_0, p_1)\) the expected mass of defaults in a structure with two components, both of size \(N/2 - 1\), is still smaller than that in a single component of size \(N\). That is, at the optimum \(C \geq 2\).

Since \(N/2 - 1 > 1\), this completes the proof that the optimal component size \(K^* + 1\) is “intermediate,” i.e. satisfies \(1 < K^* < N - 1\).

**Proof of Proposition 5:** For the probability distribution of the \(b\) shock stated in the claim, the expected mass of firms not directly hit by a \(b\) shock who default in a completely connected component of size \(K\) when a \(b\) shock hits the component is:

\[
D_c(K, \gamma, p) = (1 - p)K + pK \left(\frac{1}{2K}\right)^\gamma. \tag{41}
\]
Differentiating the above expression with respect to $K$ yields:

$$\frac{\partial D_r(K, \gamma, p)}{\partial K} = (1 - p) - (\gamma - 1) p \left( 1 - \frac{1}{2K} \right)^\gamma,$$

which is negative for all $K$ as long as (25) is satisfied. This establishes that the optimal degree of segmentation for the complete structure is minimal, that is obtains at $K = N - 1$.

Next, using (33) and (19), noting that $\bar{L} > \frac{1}{N} = K + 1$ for all $K$, we obtain the following expression for the expected mass of defaults in the case of the ring structure:

$$D_r(K, \gamma, p) = (1 - p) \left( K - \left( K - \frac{2}{K+1} \right) \frac{K+1}{L} \right) + p \left[ \left( \frac{K}{\gamma+1} - \frac{2}{K-1} \frac{1}{\gamma+1} \right) \left( \frac{1}{K+1} \right)^\gamma + \frac{1}{K-1} \frac{1}{2^{\gamma - 1}} \frac{1}{\gamma+1} \right].$$

It suffices then to show that the expected mass of defaults is smaller for the ring than for the completely connected structure when $K = N - 1$: $D_r(N - 1, \gamma, p) > D_r(N - 1, \gamma, p)$ or, substituting the above expressions:

$$(1 - p) (N - 1) + p (N - 1) \left( \frac{1}{2(N-1)} \right)^\gamma > (1 - p) \left( N - 1 - \frac{N^2 - N - 2}{2N - 1} \right) + p \left( \frac{N-1}{\gamma+1} - \frac{2}{N-2} \frac{1}{\gamma+1} \right) \left( \frac{1}{N} \right)^\gamma + p \left[ \frac{1}{N-2} \frac{1}{2^{\gamma - 1}} \frac{1}{\gamma+1} \right],$$

which can be rewritten as

$$\left( \frac{1-p}{1} \right) \frac{N^2 - N - 2}{2N - 1} > \left( \frac{N-1}{\gamma+1} - \frac{2}{N-2} \frac{1}{\gamma+1} \right) \left( \frac{1}{N} \right)^\gamma + \frac{1}{N-2} \frac{1}{2^{\gamma - 1}} \frac{1}{\gamma+1} - (N - 1) \left( \frac{1}{2(N-1)} \right)^\gamma.$$

Using (25) the above inequality holds for an open interval of values of $p$ if

$$\left( \frac{N-1}{\gamma+1} - \frac{2}{N-2} \frac{1}{\gamma+1} \right) \left( \frac{1}{N} \right)^\gamma + \frac{1}{N-2} \frac{1}{2^{\gamma - 1}} \frac{1}{\gamma+1} - (N - 1) \left( \frac{1}{2(N-1)} \right)^\gamma$$

or

$$\left( \frac{N-1}{\gamma+1} - \frac{2}{N-2} \frac{1}{\gamma+1} \right) \left( \frac{1}{N} \right)^\gamma > 2 \left( \frac{(N-1)^2 + N - 3}{2(N-1)(N-1+1)} \right) + (N - 1) \left( \frac{1}{2(N-1)} \right)^\gamma - 2 \left( \frac{1}{N-2} \frac{1}{2^{\gamma - 1}} \frac{1}{\gamma+1} \right) \left( \frac{1}{N-1} \right)^\gamma,$$

Noticing that by A.3 and (24) we have $N \geq 5$ and this in turn implies

$$\frac{(N - 1)^2 + N - 3}{4(N - 1) + 2} \geq \frac{N - 1}{4},$$

a sufficient condition for the above inequality to hold is that:

$$\left( \frac{N-1}{\gamma+1} - \frac{2}{N-2} \frac{1}{\gamma+1} \right) \left( \frac{1}{N} \right)^\gamma > 2 \left( \frac{1}{2(N-1)} \right)^\gamma + \frac{1}{\gamma + 1} \frac{N - 2}{N - 1} \left( \frac{1}{N-1} \right)^\gamma.$$

$$= \frac{N - 2}{(N - 1)^{\gamma-1}} \left( \frac{1}{\gamma - 1} \frac{1}{2^{\gamma + 1}} + \frac{1}{\gamma} \frac{1}{\gamma + 1} \right) + \frac{2}{\gamma + 1} \left( \frac{1}{N} \right)^\gamma - \left( \frac{1}{2^{\gamma - 1}} \frac{1}{\gamma + 1} \right) > 0.$$
Since $\gamma \in (1, 2)$, this inequality is in turn satisfied if the following hold:

\[
\left\lfloor \frac{N - 2}{(N - 1)^{\gamma - 1}} + \left( \frac{1}{N} \right)^{\gamma} \right\rfloor \left( \frac{\gamma + 1}{2^{\gamma + 1}} - \frac{1}{\gamma + 1} \right) - \left( \frac{1}{2^{\gamma - 1} (\gamma + 1)} \right) > 0
\]

or

\[
(N - 1)^{2 - \gamma} - \left( \frac{1}{(N - 1)^{\gamma - 1}} - \frac{1}{N^\gamma} \right) > \frac{1}{2^{\gamma - 1} (\gamma + 1)}
\]

which is implied by the inequality

\[
(N - 1)^{2 - \gamma} - \left( \frac{1}{4^{\gamma - 1}} - \frac{1}{5^\gamma} \right) > \frac{1}{2^{\gamma - 1} (\gamma + 1)}
\]

that is in turn equivalent to (24). This completes the proof of the proposition. ■

**Further details of the proof of Proposition 9:**

Noting that $\alpha' - \alpha = (1 - \theta)^2 (\beta - 1) / \beta$, the exposure matrix for the star structures can be conveniently rewritten as follows:

\[
\bar{A} = \begin{pmatrix}
\alpha' & (1 - \alpha') / \beta & (1 - \alpha') / \beta & \cdots & (1 - \alpha') / \beta \\
1 - \alpha' & \alpha & (\alpha' - \alpha) / (\beta - 1) & \cdots & (\alpha' - \alpha) / (\beta - 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 - \alpha' & (\alpha' - \alpha) / (\beta - 1) & (\alpha' - \alpha) / (\beta - 1) & \cdots & \alpha
\end{pmatrix},
\]

Recall that $\alpha = 1/2$ while $\alpha'$ is determined, together with $\theta$, by (29) and (30). Its properties are characterized below:

**Lemma 4** For all $\beta > 2$, the solution of (29) and (30) is unique and given by continuous, monotonically increasing functions $\theta(\beta)$ and $\alpha'(\beta)$, such that

\[
5/9 \leq \alpha'(\beta) < 2 - \sqrt{2} \quad (42)
\]

**Proof of Lemma 4:**

It can be easily verified that for all $\beta > 2$ there is only one admissible (i.e., lying between 0 and 1) solution of (29), given by

\[
\frac{2 + \sqrt{2\beta^2 - 2\beta}}{2\beta + 2}.
\]

This expression defines the function $\theta(\beta)$, which is increasing if and only if the following inequality is satisfied:

\[
\frac{(4\beta - 2)}{2\sqrt{2\beta^2 - 2\beta}} (2\beta + 2) > \left( 4 + 2\sqrt{2\beta^2 - 2\beta} \right),
\]

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which is equivalent to
\[
2\beta^2 + \beta - 1 > 2\sqrt{2\beta^2 - 2\beta + 2\beta^2} - 2\beta
\]
or
\[
9\beta^2 - 6\beta + 1 > 8\beta^2 - 8\beta
\]
always satisfied for \(\beta > 2\). The minimal value of \(\theta\) in this range is then \(\theta(2) = 2/3\), while the maximum is \(\lim_{\beta \to \infty} \theta(\beta) = 1/\sqrt{2}\).

Also, \(\alpha'(\beta)\) is also increasing in \(\beta\)
\[
\frac{d\alpha'}{d\beta} = 2(2\theta - 1) \frac{d\theta}{d\beta}.
\]
Hence its minimum value is \(\alpha'(\beta) = 5/9\) and its maximum is \(2 - \sqrt{2}\).

The precise expression of \(G_{\text{star}}(L) = (G^s_{\text{star}}(L) + G^l_{\text{star}}(L)) / 2\) is obtained from that of \(G^s_{\text{star}}(L)\) and \(G^l_{\text{star}}(L)\) and is given by:

\[
G_{\text{star}}(L) = \begin{cases} 
0 & \text{for } L \leq 2 \\
\frac{1}{2} & \text{for } 2 < L \leq \frac{1}{1-\alpha'} \\
\frac{1}{2} + \frac{1}{2}\beta & \text{for } \frac{1}{1-\alpha'} < L \leq \frac{\beta}{\alpha'} \\
\frac{1}{2} + \frac{1}{2}\beta & \text{for } \frac{\beta}{\alpha'} < L \leq \min \left\{ \frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'} \right\} \\
\frac{\beta}{2} + \beta & \text{for } \min \left\{ \frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'} \right\} < L \leq \max \left\{ \frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'} \right\} \\
\frac{1+\beta}{2} + \beta & \text{for } \min \left\{ \frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'} \right\} < L \leq \max \left\{ \frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'} \right\} \\
2\beta & \text{for } L > \max \left\{ \frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'} \right\}
\end{cases}
\]
since again it can be verified, given the previous lemma, that
\[
2 < \frac{1}{1-\alpha'} < \frac{\beta}{\alpha'} < \min \left\{ \frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'} \right\}.
\]