7 Appendix B (for online publication)

Proof of Lemma 2: Rearranging terms in (22) and simplifying we get:

$$D_r(K,\gamma) = \left(K\left(\frac{1}{\gamma+1}\right) - \frac{2}{K-1}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma} + \frac{1}{K-1}\frac{1}{(\gamma+1)}\left(\frac{1}{2}\right)^{\gamma-1}, \tag{33}$$

and hence

$$\begin{aligned}&\tfrac{\partial D_r}{\partial K}(K,\gamma) = -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\ &+\left(K\left(\tfrac{-\gamma}{\gamma+1}\right)\tfrac{1}{K+1} + \tfrac{2}{K-1}\tfrac{1}{K+1}\tfrac{\gamma}{\gamma+1} + \left(\tfrac{1}{\gamma+1}\right) + \tfrac{2}{(K-1)^2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{K+1}\right)^{\gamma}.\end{aligned} \tag{34}$$

Now note that the inequality $\partial D_r(K,\gamma)/\partial K > 0$ is equivalent to:

$$\frac{2(K-1)}{K+1}\gamma + (K-1)^2 + 2 > (K+1)\left(\frac{K+1}{2}\right)^{\gamma-1} + \gamma K\frac{(K-1)^2}{K+1},$$

or

$$(K-1)^2\left(1-\frac{\gamma K}{K+1}\right) + 2\left(1+\gamma\frac{K-1}{K+1}\right) > (K+1)^{\gamma}\frac{1}{2^{\gamma-1}},$$

which can be rewritten as

$$(K-1)^2 + 2 + \gamma(K-1)(2-K) > (K+1)^{\gamma}\frac{1}{2^{\gamma-1}}. \tag{35}$$

So, using the identities

$$(K-1)^2 + 2 + (K-1)(2-K) = K+1$$

and

$$(K+1)^{\gamma}\frac{1}{2^{\gamma-1}} = 2\left(\frac{K+1}{2}\right)^{\gamma}$$

we can equivalently write (35) as follows:

$$\frac{K+1}{2} - \left(\frac{K+1}{2}\right)^{\gamma} - \frac{1}{2}(1-\gamma)(K-1)(2-K) > 0.$$

Denote by $\Xi(K,\gamma)$ the term on the left hand side of the previous inequality, conceived as a function of K and γ. Then, to complete the proof, we establish the following property:

$$\forall K > 1, \quad \Xi(K,\gamma) \gtreqless 0 \Leftrightarrow \gamma \lesseqgtr 1. \tag{36}$$

To show this property, note first that $\Xi(K,1) = 0$ for all K, so that $\frac{\partial D_r}{\partial K}(K,\gamma) = 0$ for $\gamma = 1$ and all K. On the other hand,

$$\begin{aligned}\frac{\partial \Xi}{\partial \gamma}(K,\gamma) &= -\left(\frac{K+1}{2}\right)^{\gamma}\ln\frac{K+1}{2} + \frac{1}{2}(K-1)(2-K) \\ &\leq -\ln\frac{K+1}{2} + \frac{1}{2}(K-1)(2-K),\end{aligned}$$

the inequality being strict for all $K > 1$. It is then easy to verify that the terms on the two sides of the above inequality are equal to 0 when $K = 1$ and the term on the right hand side is negative[28] for all $K > 1$, establishing (36) and hence also $\partial D_r(K,\gamma)/\partial K \gtreqless 0 \Leftrightarrow \gamma \lesseqgtr 1$.

We conclude, as stated in the proposition, that the minimum of $D_r(K,\gamma)$ is attained at the maximum admissible value of K (i.e. $N-1$) when $\gamma > 1$, while it is attained at the lowest value of K (i.e. $K = 1$)[29] when $\gamma < 1$. This completes the proof of the proposition. ■

Proof of Lemma 3 Note first that the expression of $\partial D_r(K,\gamma)/\partial K$ obtained in (33) can be conveniently rewritten as follows:

$$\begin{aligned}
&\tfrac{\partial D_r}{\partial K}(K,\gamma) = -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\
&+\left(\tfrac{-K^2+K+2}{K-1}\tfrac{1}{K+1}\tfrac{\gamma}{\gamma+1}+\tfrac{1}{\gamma+1}+\tfrac{2}{(K-1)^2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{K+1}\right)^{\gamma} \\
&= -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\
&+\tfrac{1}{\gamma+1}\tfrac{K^2-2K+3}{(K-1)^2}\left(\tfrac{1}{K+1}\right)^{\gamma}-\tfrac{\gamma}{K+1}\left(K^2-K-2\right)\tfrac{1}{K-1}\left(\tfrac{1}{K+1}\right)^{\gamma}
\end{aligned}$$

Differentiating then again with respect to K yields:

$$\begin{aligned}
\frac{\partial^2 D_r(1/2,K,\gamma)}{\partial K^2} &= 2\frac{\left(\frac{1}{2}\right)^{\gamma-1}}{(\gamma+1)(K-1)^3}-\frac{1}{\gamma+1}\frac{\left(\frac{1}{K+1}\right)^{\gamma}}{(K-1)^3(K+1)}\times \\
&\quad \left(K^3\left(-\gamma^2+\gamma\right)+2K^2\left(2\gamma^2-\gamma\right)+5K\left(-\gamma^2+\gamma\right)+4K+2(\gamma-1)^2+2\right) \\
&= \frac{1}{2^{\gamma}(\gamma+1)(K-1)^3}\left(4-\frac{2^{\gamma}}{(K+1)^{\gamma+1}}\left((K-1)^2K\left(\gamma-\gamma^2\right)\right.\right. \\
&\quad \left.\left.+2\gamma^2(K-1)^2+4\gamma(K-1)+4(K+1)\right)\right)
\end{aligned}$$

Hence

$$\frac{\partial^2 D_r(1/2,K,\gamma)}{\partial K^2} < 0$$

if and only if

$$G(K) \equiv \frac{\left(\frac{K+1}{2}\right)^{\gamma+1}}{\frac{(K-1)^2}{8}\left(K\left(\gamma-\gamma^2\right)+2\gamma^2\right)+\gamma\frac{(K-1)}{2}+\frac{(K+1)}{2}} < 1 \tag{37}$$

[28] We have in fact

$$\frac{d\left(-\ln\frac{K+1}{2}+\frac{1}{2}(K-1)(2-K)\right)}{dK} = \frac{K+1-2K^2}{K+1} < \frac{2K(1-K)}{K+1} < 0$$

[29] To complete the argument we verify the claimed continuity property of $D_r(K,\gamma)$, as in (33), at $K = 1$:

$$\begin{aligned}
&\lim_{K\to 1} D_r(K,\gamma) \\
&= \left(\frac{1}{2}\right)^{\gamma}\frac{1}{\gamma+1}-\lim_{K\to 1}\frac{1}{K-1}\frac{2}{\gamma+1}\left[\left(\frac{1}{K+1}\right)^{\gamma}-\left(\frac{1}{2}\right)^{\gamma}\right] \\
&= \left(\frac{1}{2}\right)^{\gamma}\left(\frac{1}{\gamma+1}\right)-\frac{-\gamma 2^{\gamma}}{(\gamma+1)4^{\gamma}} = \left(\frac{1}{2}\right)^{\gamma} = D_r(1,\gamma)
\end{aligned}$$

First, we observe that $G(1) = 1$. Thus, to establish (37), it is enough to show that G is decreasing for all $K > 1$. Letting $x \equiv K - 1$ for notational simplicity, $\frac{dG(K)}{dK} < 0$ if, and only if,

$$\frac{\frac{d}{dx}\left(\left(\frac{x}{2}+1\right)^{\gamma+1}\right)}{\left(\frac{x}{2}+1\right)^{\gamma+1}} < \frac{\frac{d}{dx}\left(\gamma x\left(\frac{x}{8}\left(x\left(1-\gamma\right)+1+\gamma\right)+\frac{1}{2}\right)+\frac{x}{2}+1\right)}{\left(\gamma x\left(\frac{x}{8}\left(x\left(1-\gamma\right)+1+\gamma\right)+\frac{1}{2}\right)+\frac{x}{2}+1\right)},$$

or:

$$\frac{\frac{\gamma+1}{2}\left(\frac{x}{2}+1\right)^{\gamma}}{\left(\frac{x}{2}+1\right)^{\gamma+1}} = \frac{\gamma+1}{x+2} < \frac{\frac{1}{2}\gamma+\frac{1}{4}x\gamma+\frac{1}{4}x\gamma^2+\frac{3}{8}x^2\gamma-\frac{3}{8}x^2\gamma^2+\frac{1}{2}}{\frac{x}{2}+\frac{1}{2}x\gamma+\frac{1}{8}x^2\gamma+\frac{1}{8}x^3\gamma+\frac{1}{8}x^2\gamma^2-\frac{1}{8}x^3\gamma^2+\frac{1}{2}}$$

The above inequality is equivalent to the following one:

$$\left(\gamma+\frac{1}{2}x\gamma+\frac{1}{2}x\gamma^2+\frac{3}{4}x^2\gamma-\frac{3}{4}x^2\gamma^2+1\right)(x+2) >$$
$$(\gamma+1)\left(x+\gamma x+\frac{1}{4}x^2\gamma+\frac{1}{4}x^3\gamma+\frac{1}{4}x^2\gamma^2-\frac{1}{4}x^3\gamma^2+1\right),$$

or

$$\begin{array}{c}\gamma x+\frac{1}{2}x^2\gamma+\frac{1}{2}x^2\gamma^2+\frac{3}{4}x^3\gamma-\frac{3}{4}x^3\gamma^2+x\\ +2\gamma+x\gamma+x\gamma^2+\frac{3}{2}x^2\gamma-\frac{3}{2}x^2\gamma^2+2\end{array} >$$
$$\begin{array}{c}x+\gamma x+\frac{1}{4}x^2\gamma+\frac{1}{4}x^3\gamma+\frac{1}{4}x^2\gamma^2-\frac{1}{4}x^3\gamma^2+1+\\ \gamma x+\gamma^2 x+\frac{1}{4}x^2\gamma^2+\frac{1}{4}x^3\gamma^2+\frac{1}{4}x^2\gamma^3-\frac{1}{4}x^3\gamma^3+\gamma\end{array},$$

or

$$\frac{7}{4}x^2\gamma+\frac{1}{2}x^3\gamma+\gamma+1+\frac{1}{4}x^3\gamma^3 > \frac{1}{4}x^2\gamma^3+\frac{3}{4}x^3\gamma^2+\frac{3}{2}x^2\gamma^2.$$

That is

$$\frac{x^2\gamma}{2}\left(\frac{7}{2}-\frac{\gamma^2}{2}-3\gamma\right)+\frac{1}{4}x^3\gamma\left(2+\gamma^2-3\gamma\right)+\gamma+1>0.$$

the above inequality being always true if $\gamma < 1$, which completes the proof. ■

Proof of Proposition 3: From (23) and (33) we get:

$$\begin{aligned} & D_c(K,\gamma) - D_r(K,\gamma) \qquad (38)\\ = & \left(\frac{1}{2}\right)^{\gamma}\left(\frac{1}{K}\right)^{\gamma-1} - K\left(\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma} + \frac{2}{K-1}\frac{1}{\gamma+1}\left(\left(\frac{1}{K+1}\right)^{\gamma} - \left(\frac{1}{2}\right)^{\gamma}\right)\end{aligned}$$

As shown in Propositions 1 and 2, when $\gamma < 1$ and N is even, the optimal structure both for the ring and the completely connected structures has all components of size $K + 1 = 2$. As we noticed, when $K = 1$ the pattern of exposure is identical for the ring and the completely connected structure, hence the value of the above expression equals zero in that case, as can be verified.[30]

Consider now the case $\gamma > 1$, for which $K = N - 1$ (i.e. minimal segmentation) is optimal for both structures. Evaluating (38) at this value of K we find:

[30]Strictly speaking, we can show that that its limit for $K \to 1$ equals zero.

$$D_c(N-1,\gamma) - D_r(N-1,\gamma) =$$
$$= \left[\left(\frac{1}{N}\right)^{\gamma} - \left(\frac{1}{2}\right)^{\gamma}\right]\left[\frac{2}{N-2}\frac{1}{\gamma+1} - \frac{N-1}{1+\gamma}\right] + \left(\frac{1}{2}\right)^{\gamma}\left[\left(\frac{1}{N}\right)^{\gamma-1} - \frac{N-1}{1+\gamma}\right]$$

Since $2 \leq (N-1)(N-2)$ for $N \geq 3$, we have that for all[31] $N > 1 + (1+\gamma)^{\frac{1}{\gamma}}$ the desired conclusion follows:

$$D_c(K,\gamma) - D_r(K,\gamma) < 0.$$

This completes the proof. ■

Proof of Proposition 4: From (23), we can write:

$$D_c(K,\gamma,\gamma',p) = p\,K\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,K\left(\frac{1}{2K}\right)^{\gamma'}$$

Hence

$$\frac{\partial D_c}{\partial K}(K,\gamma,\gamma',p) = -p\,(\gamma-1)\left(\frac{1}{2K}\right)^{\gamma} - (1-p)\left(\gamma'-1\right)\left(\frac{1}{2K}\right)^{\gamma'}. \tag{39}$$

and $\frac{\partial D_c}{\partial K} > 0$ is equivalent to

$$(1-p)\left(1-\gamma'\right)\left(\frac{1}{2K}\right)^{\gamma'} > p\,(\gamma-1)\left(\frac{1}{2K}\right)^{\gamma},$$

or, since $\gamma > 1$ and $\gamma' < 1$,

$$K > \frac{1}{2}\left(\frac{p\,(\gamma-1)}{(1-p)\,(1-\gamma')}\right)^{\frac{1}{\gamma-\gamma'}}.$$

This implies that $D_c(K,\gamma,\gamma',p)$ is minimized at the point

$$\hat{K}(p) = \frac{1}{2}\left(\frac{p\,(\gamma-1)}{(1-p)\,(1-\gamma')}\right)^{\frac{1}{\gamma-\gamma'}}$$

provided this point is admissible, i.e. $\hat{K}(p) \in [1, N-1]$.

Compute next the second derivative of $D_c(\cdot)$:

$$\begin{aligned}
\frac{\partial^2 D_c}{\partial K^2}(K,\gamma,\gamma',p) &= p\,(\gamma-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\left(\gamma'-1\right)\frac{\gamma'}{K}\left(\frac{1}{2K}\right)^{\gamma'} \\
&\geq p\,(\gamma-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\left(\gamma'-1\right)\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma'} \\
&= -\frac{\gamma}{K}\frac{\partial D_c}{\partial K}(K,\gamma,\gamma',p)
\end{aligned}$$

Thus $\frac{\partial^2 D_c}{\partial K^2}(1/2,K,\gamma,\gamma',p) > 0$ for all feasible $K < \hat{K}(p)$, i.e. the function $D_c(\cdot)$ is convex in this range.

[31] Note that $(1+\gamma) < (N-1)^{\gamma-1}(N-1) < (N)^{\gamma-1}(N-1)$

The optimal degree of segmentation for the completely connected structure is obtained as a solution of problem 21. Denote by $(K_i^*)_{i=1}^C$ a vector of component sizes that solves this optimization problem. We will show that there exists some appropriate range $[p_0, p_1]$ such that if $p \in [p_0, p_1]$, the optimal component sizes are such that $K_i^* = K_j^* = K^*$ for all $i, j = 1, 2, \ldots, C$ and some common K^* with $2 \leq K^* \leq N - 2$.

Choose p_0 such that $\hat{K}(p_0) = \frac{N}{2} - 1$. Such a choice is feasible and unique since by A.3 $N > 4$, $\hat{K}(\cdot)$ is increasing in p, $\hat{K}(0) = 0$, and $\hat{K}(p) \to \infty$ as $p \to 1$. Next we show that, for all $p \geq p_0$, whenever $C \geq 2$,the vector $(K_i^*)_{i=1}^C$ solving problem 21 satisfies:

$$\forall i, j = 1, 2, \ldots, C, \quad K_i^* = K_j^* \leq \hat{K}(p) \tag{40}$$

Let K_i^* and K_j^* stand for any two component sizes that are part of the solution to the optimization problem. First note that, since $\hat{K}(p) \geq N/2 - 1$, if $K_i^* > \hat{K}(p)$ then we must have that $K_j^* < \hat{K}(p)$. But such asymmetric arrangement cannot be part of a solution to problem 21 because $D_c(\cdot, \gamma, \gamma', p)$ is increasing at K_i^* and decreasing at K_j^*. Hence a sufficiently small increase of K_j and a decrease of K_i, which keeps $K_i + K_j$ unchanged, is feasible and allows to decrease the expected mass of defaults. The only possibility, therefore, is that $K_j^* \leq \hat{K}(p)$ and $K_j^* \leq \hat{K}(p)$.

To complete the argument and establish (40), suppose that at an optimum we have $K_i^* \neq K_j^*$ for at least two components i, j. Since, as shown in the previous paragraph, neither K_i^* nor K_j^* can exceed $\hat{K}(p)$, both K_i^*, K_j^* lie in the convex part of the function $D_c(\cdot, \gamma, \gamma', p)$. It follows, therefore, that if we replace these two (dissimilar) components with two components of equal size $\frac{1}{2}(K_i^* + K_j^*)$, feasibility is still satisfied and the overall expected mass of defaults is reduced, contradicting that the two heterogeneous components of size K_i^*, K_j^* belongs to an optimum configuration.

We have thus shown that, when $p \geq p_0$, if at the optimum we have $C \geq 2$, the unique optimal configuration involves a uniform segmentation in components of common size $K^*(p) \leq \hat{K}(p)$. It remains then to show that at the optimum we indeed have $C \geq 2$. At $p = p_0$ the optimum exhibits two components, $C = 2$, since the optimal component size $\hat{K}(p_0) = N/2 - 1$ is feasible. Since $\hat{K}(p)$ is increasing and continuous in p and $D_c(K, \gamma, \gamma', p)$ is continuous in K, by continuity there exists some p_1, with $p_0 < p_1 < 1$, such that for all $p \in (p_0, p_1)$ the expected mass of defaults in a structure with two components, both of size $N/2 - 1$, is still smaller than that in a single component of size N. That is, at the optimum $C \geq 2$.

Since $N/2 - 1 > 1$, this completes the proof that the optimal component size $K^* + 1$ is "intermediate," i.e. satisfies $1 < K^* < N - 1$. ■

Proof of Proposition 5: For the probability distribution of the b shock stated in the claim, the expected mass of firms not directly hit by a b shock who default in a completely connected component of size K when a b shock hits the component is:

$$D_c(K, \gamma, p) = (1 - p)K + pK \left(\frac{1}{2K}\right)^{\gamma}. \tag{41}$$

Differentiating the above expression with respect to K yields:

$$\frac{\partial D_c(K,\gamma,p)}{\partial K} = (1-p) - (\gamma-1)\,p\,\left(\frac{1}{2K}\right)^{\gamma},$$

which is negative for all K as long as (25) is satisfied. This establishes that the optimal degree of segmentation for the complete structure is minimal, that is obtains at $K = N-1$.

Next, using (33) and (19), noting that $\bar{L} > \frac{1}{H} = K+1$ for all K, we obtain the following expression for the expected mass of defaults in the case of the ring structure:

$$\begin{aligned} D_r(K,\gamma,p) &= (1-p)\left(K - \left(K - \frac{2}{K+1}\right)\frac{K+1}{\bar{L}}\right) \\ &+ p\left[\left(\frac{K}{\gamma+1} - \frac{2}{K-1}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma}\right] + p\left[\frac{1}{K-1}\frac{1}{2^{\gamma-1}}\frac{1}{\gamma+1}\right] \end{aligned}$$

It suffices then to show that the expected mass of defaults is smaller for the ring than for the completely connected structure when $K = N-1$: $D_c(N-1,\gamma,p) > D_r(N-1,\gamma,p)$ or, substituting the above expressions:

$$\begin{aligned} (1-p)\,(N-1) + p\,(N-1)\,\left(\tfrac{1}{2(N-1)}\right)^{\gamma} &> (1-p)\left(N-1-\tfrac{N^2-N-2}{2N-1}\right) + \\ p\left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} &+ p\left[\tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1}\right], \end{aligned}$$

which can be rewritten as

$$\begin{gathered} \left(\tfrac{1-p}{p}\right)\tfrac{N^2-N-2}{2N-1} > \\ \left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + \tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1} - (N-1)\,\left(\tfrac{1}{2(N-1)}\right)^{\gamma}. \end{gathered}$$

Using (25) the above inequality holds for an open interval of values of p if

$$\begin{gathered} (\gamma-1)\,\left(\tfrac{1}{2(N-1)}\right)^{\gamma}\tfrac{N^2-N-2}{2N-1} > \\ \left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + \tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1} - (N-1)\,\left(\tfrac{1}{2(N-1)}\right)^{\gamma} \end{gathered}$$

or

$$\begin{gathered} (\gamma-1)\,\left(\tfrac{1}{2(N-1)}\right)^{\gamma} 2\left(\tfrac{(N-1)^2+N-3}{2(2(N-1)+1)}1\right) + (N-1)\,\left(\tfrac{1}{2(N-1)}\right)^{\gamma} \\ -2\left(\tfrac{(N-1)}{2}\left(\tfrac{1}{\gamma+1}\right) - \tfrac{1}{\gamma+1}\left(\tfrac{1}{N-2}\right)\right)\left(\tfrac{1}{N}\right)^{\gamma} - \left(\tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{(\gamma+1)}\right) > 0 \end{gathered}$$

Noticing that by A.3 and (24) we have $N \geq 5$ and this in turn implies

$$\frac{(N-1)^2+N-3}{4\,(N-1)+2} \geq \frac{N-1}{4},$$

a sufficient condition for the above inequality to hold is that:

$$\begin{aligned} &(\gamma-1)\,\left(\frac{1}{2\,(N-1)}\right)^{\gamma} 2\left(\frac{N-1}{4}\right) + \frac{2}{\gamma+1}\frac{1}{N-2}\left(\frac{1}{N}\right)^{\gamma} \\ &+ (N-1)\,\left(\frac{1}{2\,(N-1)}\right)^{\gamma} - \left(\frac{N-1}{\gamma+1}\right)\left(\frac{1}{N-1}\right)^{\gamma} - \left(\frac{1}{N-2}\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) \\ = \;& \frac{N-2}{(N-1)^{\gamma-1}}\left((\gamma-1)\,\frac{1}{2^{\gamma+1}} + \frac{1}{2^{\gamma}} - \frac{1}{\gamma+1}\right) + \frac{2}{\gamma+1}\left(\frac{1}{N}\right)^{\gamma} - \left(\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) > 0. \end{aligned}$$

Since $\gamma \in (1,2)$, this inequality is in turn satisfied if the following hold:

$$\left[\frac{N-2}{(N-1)^{\gamma-1}} + \left(\frac{1}{N}\right)^{\gamma}\right]\left(\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}\right) - \left(\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) > 0$$

or

$$(N-1)^{2-\gamma} - \left(\frac{1}{(N-1)^{\gamma-1}} - \frac{1}{N^{\gamma}}\right) > \frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}}$$

which is implied by the inequality

$$(N-1)^{2-\gamma} - \left(\frac{1}{4^{\gamma-1}} - \frac{1}{5^{\gamma}}\right) > \frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}}$$

that is in turn equivalent to (24). This completes the proof of the proposition. ■

Further details of the proof of Proposition 9:

Noting that $\alpha' - \alpha = (1-\theta)^2(\beta-1)/\beta$, the exposure matrix for the star structures can be conveniently rewritten as follows:

$$\widetilde{A} = \begin{pmatrix} \alpha' & (1-\alpha')/\beta & (1-\alpha')/\beta & \cdots & (1-\alpha')/\beta \\ 1-\alpha' & \alpha & (\alpha'-\alpha)/(\beta-1) & \cdots & (\alpha'-\alpha)/(\beta-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-\alpha' & (\alpha'-\alpha)/(\beta-1) & (\alpha'-\alpha)/(\beta-1) & \cdots & \alpha \end{pmatrix},$$

Recall that $\alpha = 1/2$ while α' is determined, together with θ, by (29) and (30). Its properties are characterized below:

Lemma 4 *For all $\beta > 2$, the solution of (29) and (30) is unique and given by continuous, monotonically increasing functions $\theta(\beta)$ and $\alpha'(\beta)$, such that*

$$5/9 \leq \alpha'(\beta) < 2 - \sqrt[2]{2} \tag{42}$$

Proof of Lemma 4:

It can be easily verified that for all $\beta > 2$ there is only one admissible (i.e., lying between 0 and 1) solution of (29), given by

$$\frac{2+\sqrt{2\beta^2 - 2\beta}}{2\beta+2}.$$

This expression defines the function $\theta(\beta)$, which is increasing if and only if the following inequality is satisfied:

$$\frac{(4\beta-2)}{2\sqrt{2\beta^2-2\beta}}(2\beta+2) > \left(4 + 2\sqrt{2\beta^2-2\beta}\right),$$

which is equivalent to

$$2\beta^2 + \beta - 1 > 2\sqrt{2\beta^2 - 2\beta} + 2\beta^2 - 2\beta$$

or

$$9\beta^2 - 6\beta + 1 > 8\beta^2 - 8\beta$$

always satisfied for $\beta > 2$. The minimal value of θ in this range is then $\theta(2) = 2/3$, while the maximum is $\lim_{\beta\to\infty} \theta(\beta) = 1/\sqrt[2]{2}$.

Also, $\alpha'(\beta)$ is also increasing in β

$$\frac{d\alpha'}{d\beta} = 2(2\theta - 1)\frac{d\theta}{d\beta}.$$

Hence its minimum value is $\alpha'(\beta) = 5/9$ and its maximum is $2 - \sqrt[2]{2}$. ■

The precise expression of $G_{star}(L) = \left(G^s_{star}(L) + G^\ell_{star}(L)\right)/2$ is obtained from that of $G^s_{star}(L)$ and $G^\ell_{star}(L)$ and is given by:

$$G_{star}(L) = \begin{cases} 0 & \text{for } L \leq 2 \\ \frac{1}{2} & \text{for } 2 < L \leq \frac{1}{1-\alpha'} \\ \frac{1}{2} + \frac{1}{2}\beta & \text{for } \frac{1}{1-\alpha'} < L \leq \frac{\beta}{\alpha'} \\ \frac{1}{2} + \frac{1}{2}2\beta & \text{for } \frac{\beta}{\alpha'} < L \leq \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ \begin{array}{c} \frac{\beta}{2} + \beta \text{ if } \frac{\beta-1}{\alpha'-1/2} \leq \frac{\beta^2}{1-\alpha'} \\ \frac{1+\beta}{2} + \beta \text{ otherwise} \end{array} & \text{for } \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} < L \leq \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ 2\beta & \text{for } L > \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \end{cases}, \tag{43}$$

since again it can be verified, given the previous lemma, that

$$2 < \frac{1}{1-\alpha'} < \frac{\beta}{\alpha'} < \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\}.$$