INTERNATIONAL ECONOMIC REVIEW
Vol. 41, No. 2, May 2000

STOCHASTIC REPLICATOR DYNAMICS*

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This article studies the replicator dynamics in the presence of shocks. I show that under these dynamics, strategies that do not survive the iterated deletion of strictly dominated strategies are eliminated in the long run, even in the presence of nonvanishing perturbations. I also give an example that shows that the stochastic dynamics in this article have equilibrium selection properties that differ from other dynamics in the literature.

1. INTRODUCTION

This article studies a stochastic version of the replicator dynamics. These dynamics model agents with a very low degree of sophistication. My main conclusion is that despite the agents' lack of sophistication, even in the presence of perturbations of several kinds, the dynamics give little weight to strategies that do not survive the iterated deletion of strictly dominated strategies in the long run. I also show that the size of the basin of attraction of an equilibrium under the deterministic version of the dynamics need not determine the equilibrium selected in the stochastic analogue of the dynamics.

When considering the replicator dynamics, it is useful to think of a large population of agents who use pure strategies and are randomly matched to play against each other. The growth rate of the proportion of players using a certain pure strategy is the difference between the expected payoff of that pure strategy, given the proportions of players using every pure strategy, and the average expected payoff in that population. These dynamics can be the result of a process by which agents with very little information learn to play the game or imitate more successful actions. Binmore and Samuelson (1997), Börgers and Sarin (1997), and Schlag (1998) have models where the replicator dynamics are motivated this way. In contrast to other dynamics that have been proposed, like the best-response dynamics of Matsui (1991), the fictitious play of Brown (1951) and Robinson (1951), and the learning papers of Milgrom and Roberts (1990) and Fudenberg and Kreps (1994, 1995), the replicator

* Manuscript received July 1996; revised August 1997.

[1] I owe thanks to Joel Sobel for advice and patience. Thanks are also due to Ken Binmore, Antonio Ciccone, John Conlisk, Valentina Corradi, Gary Cox, Vincent Crawford, Christopher Harris, Takeo Hoshi, Dennis Smallwood, Peyton Young, an editorial board member, and two anonymous referees. The financial support of Spain's Ministry of Education under Grant PB96-0302 and Generalitat of Catalonia under Grant 1997SGR-00138 is gratefully acknowledged. All errors are my own.

dynamics have the characteristic that the strategies whose weight in the population increase need not be best responses and could even be strictly dominated strategies. Still, if selection operates slowly enough or in continuous time, then all limit points of the dynamics are best responses to time averages of past play (see, e.g., Cabrales and Sobel, 1992).

The history of stochastic selection processes is not long, in part because the techniques are relatively new. A seminal article is that by Foster and Young (1990), who develop a model where stochastic perturbations are constantly affecting the replicator dynamics. Kandori et al. (1993) and Young (1993) consider models where the randomness comes from the stochastic arrival of newcomers, who replace some players who leave the population and start by playing something at random. In the three previous models, the processes have ergodic distributions, and the authors arrive at predictions by looking at the limit of these ergodic distributions when the variance of the noise goes to zero. This approach has proven useful because it has been able to select between strict equilibria, something most refinements were unable to do. An exception is Crawford (1995), where it is shown that in some games, strategic uncertainty and adaptive adjustments can give rise to systematic equilibrium selection patterns without having to depend on an ergodic distribution.

I generalize a model developed by Fudenberg and Harris (1992) for symmetric games with two players and two strategies to games with a finite number of players and a finite number of strategies. I define the replicator dynamics in continuous time, and the state variables are points in the simplex. Foster and Young (1990) use a similar model, but they only have one type of shock, while I distinguish between aggregate shocks to payoffs and mutations. There are two types of shocks in my model. First, there are individual, uncorrelated changes of strategy. These result from the the entry of uninformed players. Since I assume that there is no correlation in these changes and the population is very large, these shocks are deterministic shifts to the replicator dynamics. Second, there are aggregate shocks that affect the payoffs to all users of a strategy in the same way. These shocks could be variations of demand in an oligopoly game or changes in sunk costs for an entry game, and so on. These shocks will not average out; they constitute the part of my model that is explicitly stochastic. As a first approximation, they are considered uncorrelated across time. Since the model is formulated in continuous time, Wiener processes are an adequate way to model them.

Section 2 of this article describes the model. In Section 3 I show that strategies that do not survive the iterated deletion of strictly dominated strategies become rare when selection has been operating for a long time. In Section 4 I show that the equilibrium selected by the dynamics used in my article may be different from the one that would be selected by the dynamics used by Kandori et al. (1993). I do this by showing an example where their dynamics (when suitably extended to games with more than two players) select a Pareto-inferior equilibrium, while the ones I use select a Pareto-superior equilibrium when the number of players is large enough. For 2×2 games, the dynamics of Kandori et al. (1993) and the dynamics that I study [which in that class of games are also the dynamics of Fudenberg and Harris (1992)] select the same equilibria, which could lead to the wrong conclusion that the equilibrium-selection properties of both dynamics are the same. The Appendix gathers the proofs.

2. REPLICATOR DYNAMICS WITH AGGREGATE SHOCKS AND MUTATIONS

The game considered here will have finitely many pure strategies and players. There are N players, and the pure strategy set for the ith player is P^i, which has n_i strategies. Player i's payoff function is $u^i : \prod_{k=1}^{N} P^k \to R$. Let S^n denote the standard $n-1$ dimensional simplex, x^i a generic member of S^{n_i} for any player i, and x^{-i} a generic element of $S^{-i} = \prod_{j \neq i} S^{n_j}$. u^i is extended to the space of mixed strategies in the usual way. Thus we represent by $u^i(x^i, x^{-i})$ the payoff to agent i of using strategy x^i when the other players are using the strategy x^{-i}. Any strategy $\alpha \in P^i$ will be identified notationally with the mixed strategy that gives probability one to the pure strategy α.

Suppose that there are N populations of agents, one for each player, and each of them contains a continuum of individuals (if the game were symmetric, we could use just one population for all the players). The biologic interpretation of the replicator dynamics is that they describe the evolution of the proportion of members of each population playing every strategy. Payoffs in this case represent reproductive fitness, or the number of successors for the user of a strategy given the makeup of the population.

A more interesting way to think about this for an economist is given by Schlag (1998). He assumes that agents play a random-matching game in a large population. They learn the actual payoff of another (randomly chosen) agent. If the agents have a rule of action that is "improving" (the expected payoff increases with respect to the present one) for every possible game and state of the population and "unbiased" (depends only on payoffs, not on strategy labels), the rule is going to give a probability of switching to the other agent's strategy that is proportional to the difference in payoffs. This leads to aggregate dynamics that are like the replicator dynamics.

Two other articles relate the replicator dynamics to learning. Börgers and Sarin (1997) show that in the continuous time limit the replicator dynamics are the same as the dynamics arising from the learning model in Cross (1973) [which itself is a special case of Bush and Mosteller's (1951, 1955) stochastic learning theory]. Sarin (1993) axiomatizes Cross's learning dynamics.

The replicator dynamics have a number of interesting properties. This article explores the extension of one of them, namely, the elimination of strategies that do not survive the iterated deletion of strictly dominated strategies, to a context with aggregate stochastic shocks and mutations. This property was first studied in deterministic contexts by Samuelson and Zhang (1992). The deterministic replicator dynamics have other properties that are surveyed in the books of Hofbauer and Sigmund (1984), Cressman (1992), Weibull (1995), Vega-Redondo (1996), Samuelson (1997), and Fudenberg and Levine (1998).

I want to consider now the introduction of shocks to the replicator dynamics. The first type of shock includes that which affect the payoffs of all users of a strategy in the same way. It could be a random change in total demand in an oligopoly game where oligopolists face the same demand curve, a change in the legal system that makes certain strategies more costly, or a change in factor prices that alters the cost of using a technology. I will introduce such shocks by using both the biologic interpretation to the replicator dynamics and the learning interpretation to which

I alluded before. The interest of doing it both ways is that they lead to somewhat different formulations, unlike in the deterministic case.

Let r^i_α be the size of that part of the ith player population that plays strategy α, and let $R^i = \sum_{\alpha=1}^{n_i} r^i_\alpha$. Let x^i_α be the proportion of members of the ith player population using strategy α, that is, $x^i_\alpha = r^i_\alpha / R^i$. Divide time into discrete periods of length τ. At a particular instant in time, a player in the ith population is randomly matched with a player from each of the other $N-1$ populations. Let $\eta_l(t)$ be a collection of d i.i.d. random variables with mean zero, variance τ, and support contained in $[\underline{\eta}, \overline{\eta}]$ and σ^i_α a d-dimensional vector of positive constants.

In the biologic interpretation, individuals play their genetically given strategy, and payoffs are related to reproductive success. Total payoffs for a member of the i player population who is playing pure strategy α in period t are given by $u^i[\alpha, x^{-i}(t)]\tau + \sum_{l=1}^{d} \sigma^i_{\gamma l}\eta_l(t)$. Every period the users of all strategies reproduce after playing the game. Reproduction is asexual, and the offspring inherit the strategies of the parent (i.e., strategies breed true). The number of successors of each individual is given by the sum of the *background* fitness $B^i[t, r(t)]\tau$ (the number of successors independent of the game), plus the payoffs from playing the game. After reproduction, a fraction $D^i[t, r(t)]\tau$ of the users of all strategies (except the newborn) dies in every period.[2] We have then

$$(1) \quad r^i_\alpha(t+\tau) = r^i_\alpha(t)\left\{1 - D^i[t, r(t)]\tau + B^i[t, r(t)]\tau + u^i[\alpha, x^{-i}(t)]\tau + \sum_{l=1}^{d} \sigma^i_{\gamma l}\eta_l(t)\right\}$$

Let W be a d-dimensional Wiener process (these are stochastic processes with continuous sample paths and independent increments with mean zero). By letting the period length τ go to zero, we can obtain the continuous time version of Equation (1):

$$(2) \quad dr^i_\alpha(t) = r^i_\alpha(t)\left(\left\{B^i[t, r(t)] - D^i[t, r(t)] + u^i[\alpha, x^{-i}(t)]\right\} dt + \sum_{l=1}^{d} \sigma^i_{\alpha l} dW_l(t)\right)$$

By Itô's rule, which is the analogue in differential stochastic calculus to the chain rule in ordinary calculus (see, i.e., Karatzas and Shreve, 1991, p. 153), since $x^i_\alpha(t) = r^i_\alpha(t)/R^i(t)$ and $r^i_\alpha(t)$ is given by Equation (2), I obtain:

$$(3) \quad \begin{aligned} dx^i_\alpha(t) = x^i_\alpha(t)\Bigg\{ & u^i[\alpha, x^{-i}(t)]\, dt + \sum_{l=1}^{d} \sigma^i_{\alpha l} dW_l(t) - \sum_{\beta=1}^{n_i} x^i_\beta(t) u^i[\beta, x^{-i}(t)]\, dt \\ & - \sum_{\beta=1}^{n_i}\sum_{l=1}^{d} x^i_\beta(t)\sigma^i_{\beta l} dW_l(t)\Bigg\} \\ & - x^i_\alpha(t)\left[\sum_{\beta=1}^{n_i}\sum_{l=1}^{d} x^i_\beta(t)\sigma^i_{\alpha l}\sigma^i_{\beta l} - \sum_{\beta=1}^{n_i}\sum_{\gamma=1}^{n_i}\sum_{l=1}^{d} x^i_\gamma(t) x^i_\beta(t)\sigma^i_{\beta l}\sigma^i_{\gamma l}\right] dt \end{aligned}$$

[2] Both the death rate and the background fitness depend on the total size of the population. In this way, the model can accommodate scenarios that avoid explosive population growth or extinctions.

There is another way to arrive at the stochastic replicator dynamics that leads to a slightly different formulation.

Assume that agents play the game repeatedly and change their actions only when the action they are currently using does not perform better than a preset standard and that they choose among the alternatives with a probability proportional to their presence in the population. These assumptions were used, in a consumers' choice model, by Smallwood and Conlisk (1979). They are also used to motivate the (deterministic) replicator dynamics by Binmore et al. (1995).

Divide again time into discrete periods of length τ. Total payoffs now are $u^i[\alpha, x^{-i}(t)]\tau + \sum_{l=1}^{d} \sigma_{\gamma l}^i \eta_l(t)$ plus an idiosyncratic uniformly distributed random shock with support $[-A/2, A/2]$. Agents change their strategies when total payoff is less than a certain acceptable level, which we normalize to 0. Let's assume that $A, \underline{\eta}, \overline{\eta}$ are such that

$$\max_{i,\alpha,\beta} u^i(\alpha, \beta) + d \max_{i,\alpha,l} \sigma_{\alpha l}^i \overline{\eta} \leq \frac{A}{2} \qquad \text{and} \qquad \min_{i,\alpha,\beta} u^i(\alpha, \beta) + d \max_{i,\alpha,l} \sigma_{\alpha l}^i \overline{\eta} \geq -\frac{A}{2}$$

With these constraints, any strategy at any time can either give a payoff above the acceptable level or fail to do so with positive probability. If the performance of a strategy is adequate, agents keep using it. If it is not, they choose strategy γ in the next period with probability $x_\gamma^i(t)$. The probability that strategy α fails for a player i is equal to

$$p_\alpha^i(t) = \frac{-u^i[\alpha, x^{-i}(t)]\tau - \sum_{l=1}^{d} \sigma_{\alpha l}^i \eta_l(t)}{A}$$

Let's assume that the proportion of the population that experiences a payoff below the satisfaction level is exactly $p_\alpha^i(t)$ and the proportion of them who switch to strategy γ is exactly $x_\gamma^i(t)$. The dynamics that result for the population shares are

$$x_\alpha^i(t+\tau) = x_\alpha^i(t)[1 - p_\alpha^i(t)] + \sum_{\gamma=1}^{n_i} x_\gamma^i(t) p_\gamma^i(t) \tag{4}$$

We can rewrite Equation (1) in the following way:

$$\begin{aligned} x_\alpha^i(t+\tau) = x_\alpha^i(t) + \frac{x_\alpha^i(t)}{A} \Bigg\{ & u^i[\alpha, x^{-i}(t)]\tau + \sum_{l=1}^{d} \sigma_{\alpha l}^i \eta_l(t) - u_i[x^i(t), x^{-i}(t)]\tau \\ & - \sum_{\beta=1}^{n_i} \sum_{l=1}^{d} x_\beta^i(t) \sigma_{\beta l}^i \eta_l(t) \Bigg\} \end{aligned} \tag{5}$$

By letting the period length τ go to zero, we can obtain the continuous time version:

$$\begin{aligned} dx_\alpha^i(t) = x_\alpha^i(t) \Bigg\{ & u^i[\alpha, x^{-i}(t)]\, dt + \sum_{l=1}^{d} \sigma_{\alpha l}^i dW_l(t) - \sum_{\beta=1}^{n_i} x_\beta^i(t) u^i[\beta, x^{-i}(t)]\, dt \\ & - \sum_{\beta=1}^{n_i} \sum_{l=1}^{d} x_\beta^i(t) \sigma_{\beta l}^i dW_l(t) \Bigg\} \end{aligned} \tag{6}$$

Although we assume that the agents take decisions about changing strategies based on their *expected* payoff $u^i[\alpha, x^{-i}(t)]$, we also could have assumed that they observe their actual payoff $u^i(\alpha, k)$, since averaging over the (infinite) population would have led to Equation (5) as well. Thus the equilibrium-selection properties of Section 4 are independent of this assumption. This contrasts with finite population models. Robson and Vega-Redondo (1996) show that whether one observes the expected payoff, as in Kandori et al. (1993), or the actual payoff can affect equilibrium selection in that type of model.

Equations (3) and (6) differ by the term in the second line of Equation (3). This term does not have a qualitative effect in the results of this article (in fact, it would only change a bound on a variance in the main result), but it reminds us that in stochastic models like this, modeling decisions that are innocuous in deterministic models, such as interpreting payoffs as number of successors or as a tool for players to decide if the present strategy is adequate, can have implications. I will use Equation (3) to be consistent with Fudenberg and Harris (1992).

One feature of the replicator dynamics is that if a strategy disappears or is never in the population, it will never reappear again. This is true because one cannot imitate a strategy that nobody is using or inherit it in a biologic context. And it will be true independently of the payoff of that strategy.

I want to incorporate in the model the possibility that strategies that are not used by anybody in a given period start to be used in later periods, while retaining the assumption that the agents are not sophisticated. For this reason, I assume that new players replace part of the population at all times and that some of them adopt strategies in a random way that is independent of the actions of both old players and other new players. I model the effect of these new players in the dynamics as a deterministic shock that modifies the transition rates for all periods. The aggregate effect of the newcomers that take actions at random is modeled in a deterministic fashion because their actions are assumed to be uncorrelated across individuals, and the population is so large that we can invoke the law of large numbers to assume that the average of these actions is not random. By the next time these new players can change their strategies, they start behaving like other members of the population. By analogy with the biologic literature, I call these newcomers *mutants* and their actions *mutations*.

Adding the mutations to Equation (2) as in Fudenberg and Harris (1992), we get

$$\begin{aligned} dr^i_\alpha(t) &= r^i_\alpha(t)\left\{u^i[\alpha, x^{-i}(t)]\,dt + \sum_{l=1}^{d} \sigma^i_{\alpha l}\, dW_l(t)\right\} \\ &\quad + \sum_{\beta=1}^{n_i}[\lambda^i_{\alpha\beta} r^i_\beta(t) - \lambda^i_{\beta\alpha} r^i_\alpha(t)]\,dt \end{aligned} \tag{7}$$

$\lambda^i_{\alpha\beta}$ is the rate at which members of population i that are using strategy β will be replaced in a given period by players who choose strategy α. I will call the $\lambda^i_{\alpha\beta}$'s *mutation rates*. Notice that $\lambda^i_{\alpha\alpha}$ can have any value without affecting the dynamics. Choose, for example, $\lambda^i_{\alpha\alpha} = \min_{\beta\neq\alpha} \lambda^i_{\alpha\beta}$.

Applying Itô's rule,

$$\text{(8)} \quad dx_\alpha^i(t) = x_\alpha^i(t)\Bigg\{u^i[\alpha, x^{-i}(t)]\,dt + \sum_{l=1}^{d}\sigma_{\alpha l}^i dW_l(t) - \sum_{\beta=1}^{n_i} x_\beta^i(t)u^i[\beta, x^{-i}(t)]\,dt - \sum_{\beta=1}^{n_i}\sum_{l=1}^{d} x_\beta^i(t)\sigma_{\beta l}^i dW_l(t)\Bigg\}$$
$$- x_\alpha^i(t)\left[\sum_{\beta=1}^{n_i}\sum_{l=1}^{d} x_\beta^i(t)\sigma_{\alpha l}^i\sigma_{\beta l}^i - \sum_{\beta=1}^{n_i}\sum_{\gamma=1}^{n_i}\sum_{l=1}^{d} x_\gamma^i(t)x_\beta^i(t)\sigma_{\beta l}^i\sigma_{\gamma l}^i\right]dt$$
$$+ \sum_{\beta=1}^{n_i}[\lambda_{\alpha\beta}^i x_\beta^i(t) - \lambda_{\beta\alpha}^i x_\alpha^i(t)]\,dt$$
$$-x_\alpha^i(t)\left[\sum_{\gamma=1}^{n_i}\sum_{\beta=1}^{n_i}\lambda_{\gamma\beta}^i x_\beta^i(t) - \sum_{\gamma=1}^{n_i}\sum_{\beta=1}^{n_i}\lambda_{\beta\gamma}^i x_\gamma^i(t)\right]dt$$

But notice that by relabeling the summation indices we obtain

$$\sum_{\gamma=1}^{n_1}\sum_{\beta=1}^{n_i}\lambda_{\gamma\beta}^i x_\beta^i(t) - \sum_{\gamma=1}^{n_i}\sum_{\beta=1}^{n_i}\lambda_{\beta\gamma}^i x_\gamma^i(t) = \sum_{\gamma=1}^{n_1}\sum_{\beta=1}^{n_i}\lambda_{\gamma\beta}^i x_\beta^i(t) - \sum_{\beta=1}^{n_i}\sum_{\gamma=1}^{n_i}\lambda_{\gamma\beta}^i x_\beta^i(t) = 0$$

This implies that we can rewrite Equation (8) as

$$\text{(9)} \quad dx_\alpha^i(t) = x_\alpha^i(t)\Bigg\{u^i[\alpha, x^{-i}(t)]\,dt + \sum_{l=1}^{d}\sigma_{\alpha l}^i dW_l(t) - \sum_{\beta=1}^{n_i} x_\beta^i(t)u^i[\beta, x^{-i}(t)]\,dt - \sum_{\beta=1}^{n_i}\sum_{l=1}^{d} x_\beta^i(t)\sigma_{\beta l}^i dW_l(t)\Bigg\}$$
$$- x_\alpha^i(t)\left[\sum_{\beta=1}^{n_i}\sum_{l=1}^{d} x_\beta^i(t)\sigma_{\alpha l}^i\sigma_{\beta l}^i - \sum_{\beta=1}^{n_i}\sum_{\gamma=1}^{n_i}\sum_{l=1}^{d} x_\gamma^i(t)x_\beta^i(t)\sigma_{\beta l}^i\sigma_{\gamma l}^i\right]dt$$
$$+ \sum_{\beta=1}^{n_i}[\lambda_{\alpha\beta}^i x_\beta^i(t) - \lambda_{\beta\alpha}^i x_\alpha^i(t)]\,dt$$

The mutation term in Equation (9), which I use for simplicity and consistence with Fudenberg and Harris (1992), is the first type of mutation term that Boylan (1994, on p. 15, attributed to Burger, 1989) describes. Boylan (1994) proposes other types of mutations, but for the purposes of this article, the more general type would give the same results. However, mutations can, in general, be quite important for equilibrium selection. Binmore et al. (1995) and Binmore and Samuelson (1996) show that depending on the exact form of the evolutionary drift/mutation, even equilibria that are not subgame perfect can be asymptotically stable.

3. STRATEGIES THAT SURVIVE THE ITERATED DELETION OF STRICTLY DOMINATED STRATEGIES

One of the first questions to arise when considering selection dynamics that come from less than rational behavior is whether the outcomes generated resemble the ones predicted from a rationality perspective so that as-if-rational type arguments can be made. I will need some definitions for this discussion.

Strategy $x' \in S^{n_i}$ is *strictly dominated* in $H_i \subset S^{n_i}$ relative to $H_{-i} \subset \Pi_{j \neq i} S^{n_j}$ if there exists $x \in H_i$ such that $u^i(x, y) > u^i(x', y)$ for all $y \in H_{-i}$. Let $D_i(H_i, H_{-i})$ be the set of mixed strategies in H_i that are not strictly dominated in H_i relative to H_{-i}. Let N sequences of sets be defined as follows:

$$H_i^{(0)} = S^{n_i} \quad H_{-i}^{(0)} = \Pi_{j \neq i} S_j^{n_j} \qquad i = 1, \ldots, N$$

$$H_i^{(n)} = D_i(H_i^{(n-1)}, H_{-i}^{(n-1)}) \quad H_{-i}^{(n)} = \Pi_{j \neq i} H_j^{(n)} \qquad \text{for } n > 0, i = 1, \ldots, N$$

Then $x' \in S^{n_i}$ survives *strict iterated admissibility* (SIA) if $x' \in H_i^{\infty}$, where $H_i^{\infty} = \bigcap_{n=0}^{\infty} H_i^{(n)}$.

Strategies that do not survive SIA are not justifiable for a rational player, so if a nonnegligible part of the population plays them a nonvanishing proportion of the time, the dynamics cannot be thought of as behaving in a way that mimics the traditional economic notion of rationality. The usual justification for strong rationality assumptions is that in the long run, behavior is close to rational due to unspecified selection processes. It is interesting, then, to find whether the replicator dynamics eliminate all but admissible strategies in the long run. This is true for continuous time replicator dynamics, as shown by Samuelson and Zhang (1992), but not for the discrete time case, as shown in Dekel and Scotchmer (1992). Nevertheless, Cabrales and Sobel (1992) show that the result can be partially recovered and give sufficient conditions for discrete time dynamics to avoid in the limit strategies that do not survive SIA. The question now is whether a similar result is true for a model such as the one proposed above.

The payoff function with respect to which I consider the strict domination is the average (over the aggregate noise, the noise from mutations, and the frequencies with which actions are adopted among i's opponents) payoff function $u^i(\alpha, x)$. Total payoff, which includes the aggregate shocks, can be different from $u^i(\alpha, x)$, although, on average, they coincide. I will show that the elimination of non-SIA strategies by the replicator dynamics is maintained even when transitory payoff perturbations and mutations are added to the model.

Proposition 1A demonstrates that if there are no mutation rates and the dominating strategies are initially present in the population, the weight of non-SIA strategies converges to zero as time goes to infinity. Proposition 1B demonstrates that if mutation rates are small and selection has been operating for a long time, the probability that nonnegligible proportions of the population are playing a non-SIA strategy is small. I cannot say that the weight of a non-SIA strategy will be small with probability one because it could happen that a streak of good luck makes the proportion of users of a generally bad strategy grow for a while.

These results do not depend on the existence of an ergodic distribution, and it is not necessary for variances to be infinitely small. This is interesting because many other results in the literature of stochastic dynamics concern the limit of the ergodic distribution as the variance of the aggregate shocks goes to zero [see, for example, the review of stochastic dynamics in Vega-Redondo (1996) or Fudenberg and Levine (1998)].

Let r be any $n_i \times 1$ vector and

$$V_r^i(t) = \Pi_{\alpha=1}^{n_i}[x_\alpha^i(t)]^{r_\alpha}$$

Let $\overline{\lambda} = \max_{\alpha,\beta,i}(\lambda_{\alpha\beta}^i)$, and $\underline{\lambda} = \min_{\alpha,\beta,i}(\lambda_{\alpha\beta}^i)$.

Suppose that r is a strategy vector for player i. If r is a non-SIA strategy and V_r^i is zero, at least one of the pure strategies that have positive weight under r has to be zero.

PROPOSITION 1A. *Let strategy $p \in S^{n_i}$ fail strict iterated admissibility. If $\lambda_{\alpha\beta}^i = 0$ for all α, β and if $x_\alpha^i(0) > 0$ for all α, then there is $\bar{\sigma}_p > 0$ such that if $\max_{\alpha,i,l}\{\sigma_{\alpha l}^i\} < \bar{\sigma}_p$,*

$$\lim_{t\to\infty} V_p^i(t) = 0 \quad \text{a.s.}$$

PROPOSITION 1B. *Let pure strategy $\gamma \in S^{n_i}$, fail strict iterated admissibility. If $\overline{\lambda}/\underline{\lambda}$ is bounded, as we let $\overline{\lambda} \to 0$, there is $\bar{\sigma}_p > 0$ such that if $\max_{\alpha,i,l}\{\sigma_{\alpha l}^i\} < \bar{\sigma}_p$,*

$$\lim_{\lambda\to 0}\left\{\limsup_{t\to\infty} E[x_\gamma^i(t)]\right\} = 0$$

Although strategy γ in Proposition 1B is a pure strategy, it can be dominated by any mixed strategy. Proposition 1B implies that the probability that the weight of a non-SIA strategy is larger than any given positive number K, which may be as small as we want, will be very close to zero when selection has been operating for a long enough time provided that the variance of the noise is below a certain bound and the mutation rates are both small and not orders of magnitude apart from one another.

The proof of Proposition 1A is based on two main facts. One is that the deterministic part of the flow tends to make V_p^i small (when the variances of the shocks are small). The other is that the stochastic shocks average out to zero by the law of large numbers.

The only moderately mysterious part of Proposition 1A (which recurs in Proposition 1B) is that one needs to have a bound in the variance of the shocks, and the smaller the advantage of the dominating strategy, the stricter is the bound. This happens on account of an extra term containing the variances that appears after the application of Itô's rule. This term appears because V_p^i is a nonlinear function of random variables, and the extra term containing the variances is something like a second-order term from a Taylor series expansion. When the variances are small enough, we can be sure that they will not reverse the effect of the domination, but if the variance is large, we cannot be sure. One can conclude from this that in a world with randomness, we can only be certain that a dominated strategy will eventually vanish (or be unimportant most of the time) if it is quite clearly dominated (or the randomness is unimportant).

In the case of the process with mutations, we have to consider two forces. One is the force that tends to make x^i_γ small because it is strictly dominated. This one is present as long as x^i_γ is not too small (where by "too small" I understand an order of magnitude smaller than the mutations). The other is the mutations, which tend to make x^i_γ grow as long as it is small. The proof shows that if $\overline{\lambda}$ is small, the proportion of time when the x^i_γ is large has to be small, because the force that really counts when x^i_γ is large is the one that is related to payoffs.

Something worth noting about Proposition 1B is that x^i_γ will sometimes be large, even in the limit as $t \to \infty$ (although this will happen more and more infrequently as λ becomes smaller). This contrasts with what happens when there are no mutations, and it requires some explanation. The first thing to notice is that the mutations act as a sort of barrier so that x^i_γ cannot be for too long below some value $\lambda_{x^i_\gamma}$. This is important because in a model with shocks to payoffs the way in which the weight of a dominated strategy becomes large is through a series of shocks that make it look good for a while. But the shocks have to be proportionally larger to make x^i_γ large the smaller the departure point is. In the model with mutations, x^i_γ always returns to $\lambda_{x^i_\gamma}$ through deterministic drift, and from that level the shocks that make x^i_γ grow substantially are of a fixed size. In the model without mutations, x^i_γ has no lower bound, and once it hits lower and lower levels, the shocks necessary to make x^i_γ large become less and less likely.

4. RELATIONSHIP WITH OTHER STOCHASTIC DYNAMICS

I will now present an example that shows that the stochastic dynamics I use can have an ergodic distribution whose weight is concentrated, when both mutation rates and the variances of the stochastic shocks are small, on an equilibrium that is not the one with the largest basin of attraction for the deterministic replicator dynamics if the number of players N is large.

Suppose that individuals are randomly matched every period in groups of N players to play a game that has two strategies. Since the game I will present is symmetric, all the players in a group can be assumed to come from the same population. The strategy played by player i is denoted α^i, and α^i can be either 1 or 2. The payoff for player i is

$$u^i(\alpha^1, \ldots, \alpha^N) = a \min_j\{\alpha^j\} - b\alpha^i$$

where $a > b$. Given the random matching structure of the game, if we let x be the proportion of people in the population using strategy 2, the expected payoff for player i using strategy $\alpha^i = 1$ will be equal to

$$E\Big(a \min_j\{\alpha^j\}\Big|x\Big) - b\alpha^i = a - b$$

The expected payoff for player i using strategy $\alpha^i = 2$ will be equal to

$$E\Big(a \min_j\{\alpha^j\}\Big|x\Big) - b\alpha^i = 2ax^{N-1} + a(1 - x^{N-1}) - 2b = ax^{N-1} + a - 2b$$

This game has two strict equilibria in pure strategies that are Pareto ranked. The deterministic replicator dynamics converge to one of them from all initial states except from the unstable mixed-strategy equilibrium. The size of the basin of attraction of the Pareto-superior equilibrium becomes smaller as N gets larger, to the point that it converges to zero as N goes to infinity (to be more precise, the basin of attraction of the high-effort equilibrium becomes smaller than the other when $1/2^{N-1} < b/a$). However, when $a > 2b$, and the number of players is 2, the the basin of attraction of the Pareto-superior equilibrium is larger than the other equilibrium's basin of attraction (in this case, the Pareto-superior equilibrium is also risk-dominant).

Assume that $d = 2$, $\sigma_{11} = \sigma_1$, $\sigma_{12} = 0$, $\sigma_{21} = 0$, and $\sigma_{22} = \sigma_2$. Then the evolution of x can be modeled as (from Equation (9)

$$
\begin{aligned}
(10) \qquad dx(t) =& \Big(x(t)[1 - x(t)]\{\sigma_1^2[1 - x(t)] - \sigma_2^2 x(t) + ax(t)^{N-1} - b\} \\
&+\lambda_2[1 - x(t)] - \lambda_1 x(t)\Big)\, dt \\
&+x(t)[1 - x(t)][\sigma_1 dW_1(t) + \sigma_2\, dW_2(t)]
\end{aligned}
$$

Let $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ and $W(t) = [\sigma_1 W_1(t) + \sigma_2 W_2(t)]/\sigma$. Then

$$
\begin{aligned}
(11) \qquad dx(t) = & \Big\{ x(t)[1 - x(t)][\sigma_1^2 - \sigma^2 x(t) + ax(t)^{N-1} - b] + \lambda_2[1 - x(t)] \\
&-\lambda_1 x(t)\Big\}\, dt \;+\; x(t)[1 - x(t)]\sigma\, dW(t)
\end{aligned}
$$

Since $W(t) = [\sigma_1 W_1(t) + \sigma_2 W_2(t)]/\sigma$ is a standard Wiener process, we will able to use the theory of one-dimensional stochastic differential equations to get closed forms for the ergodic distribution of $x(t)$.

PROPOSITION 2. (a) *The process* $x(t)$ *defined in Equation* (11) *has an ergodic distribution*. (b) *If* $a > 2b$, *the limit of the ergodic distribution puts probability one on the state* $x = 1$ *where all the population is using the high effort strategy, as* λ_1, λ_2, σ_1 *and* σ_2 *go to zero, if* λ_1/λ_2 *is bounded*.

PROOF. See the Appendix.

This equilibrium selection can be explained intuitively in terms of the imitation models underlying the replicator dynamics. The equilibrium that has more weight under the ergodic distribution is the one for which the shocks to payoffs that convince players to switch equilibrium are less likely to arise. In this model, the difficulty in changing from a state where most of the people are playing one strategy to one where mostly the other one is played lies in getting the first few people to defect from the popular strategy. The reason is that it is more difficult to imitate something that almost nobody is doing. The first few defectors have to see that playing the other strategy has been good lately, and this will happen when payoffs suffer a shock that makes the strategy played by the majority less attractive than the alternative. Then it is necessary to compare how likely are the shocks that move the dynamics from the different equilibria to know the ergodic distribution. When $a > 2b$, the shocks necessary to move the dynamics from the Pareto-dominant equilibrium to the other one are much more unlikely than the shocks that produce the opposite transition, if the variance of the shocks is small. Thus the Pareto-dominant equilibrium has more weight under the ergodic distribution.

The ergodic distribution would concentrate its weight on a different equilibrium for the dynamics that Kandori et al. (1993) study when N is large. In this case, their dynamics would select the Pareto-inferior equilibrium, while the ones I use select the Pareto-superior equilibrium.

In the model of Kandori et al. (1993), the factor that determines which equilibrium has more weight under the ergodic distribution is the number of mutations necessary for the rest of the population to start thinking that it is a good idea to change their action. When N is large, fewer mutants are necessary to change from the Pareto-dominant equilibrium to the Pareto-inferior equilibrium than the ones necessary to do the opposite transition. Thus the Pareto-dominated equilibrium has more weight under the ergodic distribution.

When there are only two players in each match, the two criteria, size of the shocks and number of mutants, coincide, which is why the articles of Fudenberg and Harris (1992) and Kandori et al. (1993) give the same conclusions in this respect.

The key difference is that in 2×2 games the payoffs are linear in the proportion of players using every pure strategy, while in the game with more than two players the payoffs are nonlinear. In the example, the nonlinearity makes the basin of attraction, which in this case is related to the number of mutants necessary to switch equilbria, of the Pareto-dominant equilibrium smaller as the number of player grows. We have seen, however, that instead of the numbers of mutants/sizes of the basins of attraction, the determinant of equilibrium selection in the case of aggregate shocks is the relative differences of payoffs between the strategies at the different equilibria. In 2×2 games, both criteria coincide due to linearity in the other player's strategy, but with more players, the difference becomes apparent.

Young and Foster (1991) consider an example in which the set of equilibria with the largest basin of attraction would not be the one to which the ergodic distribution gives the highest weight. In their example, however, the dynamics of Kandori et al. (1993) would have the same limiting ergodic distribution. The reason is that Young and Foster (1991) study an infinitely repeated prisoner's dilemma where the players are restricted to three strategies: always cooperate (C), always defect (D), and tit-for-tat (T). This game has a connected set of equilibria, all of which are mixtures of C and T (call this set CT), and a pure-strategy equilibrium where D is the only strategy used. The set CT has a larger basin of attraction than the state where everybody plays D. But one of the states in CT is just one mutation away from the basin of attraction of the state where everybody plays D, and from any state in CT to any of the other states in CT there is also only one mutation. On the other hand, the state where all players choose D represents a strict equilibrium, and more than one mutation is necessary to exit from it. Since fewer mutations are necessary to go from CT to D than from D to CT, the ergodic distribution puts all the weight in D when mutation rates are very small.

Binmore and Samuelson (1994), Robson and Vega-Redondo (1996), and Vega-Redondo (1993) also obtain equilibrium selection properties that need not coincide with those of Kandori et al. (1993).

5. APPENDIX

I need some notation before I can proceed with the proof of the propositions. Let

$$m_\alpha(x,\lambda) = u^i(\alpha, x^{-i}) - u^i(x^i, x^{-i}) - \sum_\beta \lambda^i_{\beta\alpha}.$$

The function $m_\alpha(x,\lambda)$ gathers all the terms in Equation (9) that are multiplied by x^i_α (and are not multiplied by the variances).

$$M = \max_{x,\lambda,\alpha}\{|m_\alpha(x,\lambda)|\}$$

$$\sigma = \max_{\alpha,i,l}\{\sigma^i_{\alpha l}\}$$

Let $\delta_{\beta\alpha} = 0$ if $\beta \neq \alpha$ and $\delta_{\alpha\alpha} = 1$.

$$A^t_{\alpha s} = \sum_{\beta=1}^{n_i}\sum_{l=1}^{d}\int_s^t [\delta_{\beta\alpha} - x^i_\alpha(u)]\sigma^i_{\alpha l} dW_l(u)$$

$A^t_{\alpha s}$ gathers all the stochastic terms in dx^i_α and integrates them from s to t.

Let p be any $n_i \times 1$ vector, and

$$V^i_p(t) = \Pi^{n_i}_{\alpha=1}[x^i_\alpha(t)]^{p_\alpha}$$

By Itô's rule,

$$\begin{aligned} dV^i_p(t) =& \sum_{\alpha=1}^{n_i} p_\alpha m_\alpha[x^i(t)]V^i_p(t)\,dt \\ & - \sum_{\alpha=1}^{n_i} p_\alpha \left[\sum_{\beta=1}^{n_i}\sum_{l=1}^{d} x^i_\beta(t)\sigma^i_{\alpha l}\sigma^i_{\beta l} - \sum_{\beta=1}^{n_i}\sum_{h=1}^{n_i}\sum_{l=1}^{d} x^i_h(t)x^i_\beta(t)\sigma^i_{\beta l}\sigma^i_{hl}\right] V^i_p(t)\,dt \\ & + \frac{1}{2}\sum_{l=1}^{d}\sum_{\alpha=1}^{n_i}\sum_{\beta=1}^{n_i}\sum_{\gamma=1}^{n_i}\sum_{\zeta=1}^{n_i}[\delta_{\alpha\gamma} - x^i_\gamma(t)][\delta_{\beta\zeta} - x^i_\zeta(t)]\sigma^i_{\gamma l}\sigma^i_{\zeta l}(p_\beta - \delta_{\alpha\beta})p_\alpha V^i_p(t)\,dt \\ & + \sum_{\alpha=1}^{n_i}\frac{p_\alpha}{x^i_\alpha(t)}\left[\sum_{\beta=1}^{n_i}\lambda_{\alpha\beta}x^i_\beta(t)\right]V^i_p(t)dt + \sum_{\alpha=1}^{n_i}\sum_{l=1}^{d}\sum_{\gamma=1}^{n_i}[\delta_{\alpha\gamma} - x^i_\gamma(t)]\sigma^i_{\gamma l}p_\alpha V^i_p(t)dW^l(t) \end{aligned}$$

I am going to collect now some terms and give them a name to save space. Let

$$A(p)^t_s = \sum_{\alpha=1}^{n_i}\sum_{l=1}^{d}\sum_{\gamma=1}^{n_i}\int_s^t [\delta_{\alpha\gamma} - x^i_\gamma(u)]\sigma^i_{\gamma l}p_\alpha dW^l(u)$$

$A(p)_s^t$ collects the stochastic terms in dV_p^i and integrates from s to t. Let

$$a_p^s(t) = \sum_{l=1}^{d} \int_s^t \left\{ \sum_{\alpha=1}^{n_i} \sum_{\gamma=1}^{n_i} [\delta_{\alpha\gamma} - x_\gamma^i(u)] \sigma_{\gamma l}^i p_\alpha \right\}^2 du$$

$a_p^s(t)$ is the quadratic variation of $A(p)_s^t$. Let

$$\sigma_p(t) = -\sum_{\alpha=1}^{n_i} p_\alpha \left[\sum_{\beta=1}^{n_i} \sum_{l=1}^{d} x_\beta^i(t) \sigma_{\alpha l}^i \sigma_{\beta l}^i - \sum_{\beta=1}^{n_i} \sum_{h=1}^{n_i} \sum_{l=1}^{d} x_h^i(t) x_\beta^i(t) \sigma_{\beta l}^i \sigma_{hl}^i \right]$$
$$+ \frac{1}{2} \sum_{l=1}^{d} \sum_{\alpha=1}^{n_i} \sum_{\beta=1}^{n_i} \sum_{\gamma=1}^{n_i} \sum_{\zeta=1}^{n_i} [\delta_{\alpha\gamma} - x_\gamma^i(t)][\delta_{\beta\zeta} - x_\zeta^i(t)] \sigma_{\gamma l}^i \sigma_{\zeta l}^i (p_\beta - \delta_{\alpha\beta}) p_\alpha$$

The function $\sigma_p(t)$ collects the deterministic terms that are multiplied by the variances of the stochastic shocks. These are the terms that appear in stochastic calculus but would not appear in deterministic calculus when using the chain rule.

In the remainder of this appendix, I will suppress the superindex when it is clear that we are referring to strategies for player i, as well as the summation indices (α, β from 1 to n_i, l from 1 to d).

LEMMA 1.

$$\text{(A.1)} \quad \text{(a)} \quad x_\alpha(t) = \exp\left(\int_0^t \{ m_\alpha[x(s), \lambda] + \sigma_\alpha(s) \} \, ds - \frac{1}{2} a_\alpha^0(t) + A_{\alpha 0}^t \right) x_\alpha(0)$$
$$+ \int_0^t \exp\left(\int_s^t \{ m_\alpha[x(u), \lambda] + \sigma_\alpha(u) \} \, du - \frac{1}{2} a_\alpha^s(t) + A_{\alpha s}^t \right)$$
$$\times \left[\sum_\beta \lambda_{\alpha\beta} x_\beta(t) \right] ds$$

$$\text{(A.2)} \quad \text{(b)} \quad V_p(t) = \exp\left(\int_0^t \left\{ \sum_\alpha p_\alpha m_\alpha[x(s), \lambda] + \sigma_p(s) \right\} ds \right.$$
$$\left. - \frac{1}{2} a_p^0(t) + A(p)_0^t \right) V_p(0)$$
$$+ \int_0^t \exp\left(\int_s^t \left\{ \sum_\alpha p_\alpha m_\alpha[x(u), \lambda] + \sigma_p(u) \right\} du - \frac{1}{2} a_p^s(t) + A(p)_s^t \right)$$
$$\times \sum_\alpha \frac{p_\alpha}{x_\alpha(s)} \left[\sum_\beta \lambda_{\alpha\beta} x_\beta(t) \right] V_p(s) \, ds$$

(A.3) (b′)
$$V_p(t) = \exp\left(\int_0^t \left\{\sum_\alpha p_\alpha m_\alpha[x(s), \lambda] - f[x(s)] + \sigma_p(s)\right\} ds - \frac{1}{2}a_p^0(t) + A(p)_0^t\right) V_p(0)$$
$$+ \int_0^t \exp\left(\int_s^t \left\{\sum_\alpha p_\alpha m_\alpha[x(u), \lambda] - f[x(u)] + \sigma_p(u)\right\} du - \frac{1}{2}a_p^s(t) + A(p)_s^t\right) \sum_\alpha \frac{p_\alpha}{x_\alpha(s)} \left[\sum_\beta \lambda_{\alpha\beta} x_\beta(t)\right] V_p(s)\, ds$$
$$+ \int_0^t \exp\left(\int_s^t \left\{\sum_\alpha p_\alpha m_\alpha[x(u), \lambda] - f[x(u)] + \sigma_p(u)\right\} - \frac{1}{2}a_p^s(t) + A(p)_s^t\right) f[x(s)] V_p(s)\, ds$$

PROOF. See Karatzas and Shreve (1991, problem 5.6.15 on p. 361, solved on p. 393). □

To understand why this lemma is true, notice that the solution to the ordinary differential equation

$$\dot{y}(t) = a(t)y(t) + g(t) \tag{A.4}$$

is given by

$$y(t) = \exp\left[\int_0^t a(s)\, ds\right] y(0) + \int_0^t \exp\left[\int_s^t a(u)\, du\right] g(s)\, ds \tag{A.5}$$

[You can check this by differentiating Equation (A.5).]

The $x_\alpha(t)$ process is the solution of Equation (9), which is the stochastic differential version of Equation (A.4). To go from Equation (9) to (A.1), which is the stochastic analogue of Equation (A.5), since you cannot use differentiation, it is necessary to use Itô's rule, which introduces the extra term $-1/2a_\alpha^0(t)$.

PROOF OF PROPOSITION 1A. I will do the proof by induction on the rounds of deletion of strictly dominated strategies. Let $p \notin H_i^{(1)}$. Then there is a $p' \in H_i^{(1)}$ such that

$$u^i(p, x) - u^i(p', x) < 0 \qquad \text{for all } x \in H_{-i}^{(0)} \tag{A.6}$$

so that p' strictly dominates p relative to the whole strategy space.

Let $m = \max_x \{\sum_\alpha (p_\alpha - p'_\alpha) m_\alpha(x, 0)\}$. Notice that $m < 0$ by Equation (A.6). Since $V'_p(t) \leq 1$, $V_p(t) \leq V_{p-p'}(t)$. By Lemma 1(b),

$$V_{(p-p')}(t) = \exp\left(\int_0^t \left\{\sum_\alpha (p_\alpha - p'_\alpha) m_\alpha[x(s), 0] + \sigma_{(p-p')}(s)\right\} ds \right.$$
$$\left. - \frac{1}{2} a^s_{(p-p')}(t) + A[(p-p')]_0^t \right) V_{(p-p')}(0)$$

By the definition of m, and given that $\sigma_{(p-p')}(s)$ is bounded by $4\sigma^2 d$ and $-1/2a^s_{(p-p')}(t)$ is negative, we have that

$$V_{(p-p')}(t) \leq \exp\left[(m + 4\sigma^2 d)t + A(p-p')_0^t\right] V_{(p-p')}(0)$$

By Theorem 3.4.6 in Karatzas and Shreve (1991, p. 174, eq. 4.17), we know that there is a Wiener process $W(t)$ such that almost surely $A(p-p')_0^t = W[a^0_{p-p'}(t)]$, so we have almost surely that

$$V_p(t) \leq V_{(p-p')}(t) \leq \exp\left[(m + 4\sigma^2 d)t + W(a^0_{p-p'}(t))\right] V_{(p-p')}(0)$$

Let σ be sufficiently small for $m + 4\sigma^2 d < 0$. The assumption that $x^i_\alpha(0) > 0$, for all i, implies that $V_{(p-p')}(0) < \infty$. Now we have to distinguish two cases. Suppose first that $\lim_{t\to\infty} a^0_{p-p'}(t) = \infty$. Then we have that $\lim_{t\to\infty} W[a^0_{p-p'}(t)]/a^0_{p-p'}(t) \to 0$ almost surely, by the strong law of large numbers (Karatzas and Shreve, 1991, problem 2.9.3, on p. 104, solved on p. 124). Thus we can write $W[a^0_{p-p'}(t)] = a^0_{p-p'}(t) w(t)$, where $\lim_{t\to\infty} w(t) = 0$ (a.s.). Thus

$$V_p(t) \leq \exp\left\{\left[m + 4\sigma^2 d + \frac{a^0_{p-p'}(t)}{t} w(t)\right] t\right\} V_{(p-p')}(0) \text{ (a.s.)}$$

but since $a^0_{p-p'}(t)/t$ is a bounded function, $\lim_{t\to\infty} w(t) = 0$ and $m + 4\sigma^2 d < 0$,

$$\lim_{t\to\infty} \left[m + 4\sigma^2 d + \frac{a^0_{p-p'}(t)}{t} w(t)\right] t = -\infty$$

and we have then that $\lim_{t\to\infty} V_p(t) = 0$ almost surely.

When $\lim_{t\to\infty} a^0_{p-p'}(t) < \infty$, we have that $W[a^0_{p-p'}(t)]$ converges to a normal random variable with mean zero and variance $\lim_{t\to\infty} a^0_{p-p'}(\infty)$; thus we have that $W[a^0_{p-p'}(\infty)] < \infty$ almost surely.

Since $\lim_{t\to\infty} (m + 4\sigma^2 d)t = -\infty$,

$$\lim_{t\to\infty} \exp\left\{(m + 4\sigma^2 d)t + W[a^0_{p-p'}(t)]\right\} V_{(p-p')}(0) = 0$$

almost surely. Since this exhausts all cases, the result follows for $H_i^{(1)}$.

Now let $r > 1$ and assume that for all $p \notin H_i^{(r-1)}$, $\lim_{t\to\infty} V_p(t) = 0$ almost surely. Let $p \notin H_i^{(r)}$ and $p \in H_i^{(r-1)}$. Then there is a $p' \in H_i^{(r)}$ such that

$$u^i(p,x) - u^i(p',x) < 0 \qquad \text{for all } x \in H_{-i}^{(r-1)} \tag{A.7}$$

so that p' strictly dominates p relative to $H_{-i}^{(r-1)}$. For all $x \in S^{-i}$,

$$u(p,x) = \sum_{h\in P^{-i}} u(p,h)\Pi_{j\neq i} x^j_{h_j}$$

where h_j is the pure strategy of agent j in the pure strategy profile h.

Let the set $C_j^{(r)} = \left\{x \in P^j \bigcap H_j^{(r-1)}\right\}$. $C_j^{(r)}$ is the set of pure strategies for player j that are in $H_j^{(r-1)}$, and therefore, if $\alpha \notin C_j^{(r)}$, $x^j_\alpha(t)$ converges to 0 as time tends to infinity almost surely by the induction assumption. Let $C_{-i}^{(r)} = \Pi_{j\neq i} C_j^{(r)}$ and $(C_{-i}^{(r)})^c = \{h \in P^{-i} | h \notin C_{-i}^{(r)}\}$. $C_{-i}^{(r)}$ is the set of pure-strategy profiles (for agents other than i) such that all strategies in the profile have survived r rounds of deletions of strictly dominated strategies. $(C_{-i}^{(r)})^c$ is the set of pure-strategy profiles (for agents other than i) such that at least one strategy in the profile has not survived r rounds of deletions of strictly dominated strategies. The payoff function for player i facing a mixed-strategy profile x can be divided into the payoff against pure profiles in $C_{-i}^{(r)}$ and pure profiles in $(C_{-i}^{(r)})^c$. The set $C_{-i}^{(r)}$ is never empty (it is impossible for all pure strategies to be strictly dominated), but $(C_{-i}^{(r)})^c$ may be empty. In this case, the sum over pure strategies in $(C_{-i}^{(r)})^c$ is zero.

$$u(p,x) = \sum_{h\in C_{-i}^{(r)}} u(p,h)\Pi_{j\neq i} x^j_{h_j} + \sum_{h\in \left(C_{-i}^{(r)}\right)^c} u(p,h)\Pi_{j\neq i} x^j_{h_j}$$

Thus, for all $x \in S^{-i}$,

$$\begin{aligned} u(p,x) - u(p',x) &= \sum_{h\in C_{-i}^{(r)}} \left[u(p,h) - u(p',h)\right] \Pi_{j\neq i} x^j_{h_j} \\ &\quad + \sum_{h\in \left(C_{-i}^{(r)}\right)^c} \left[u(p,h) - u(p',h)\right] \Pi_{j\neq i} x^j_{h_j} \end{aligned}$$

which by Equation (A.7) gives

$$u(p,x) - u(p',x) < \sum_{h\in \left(C_{-i}^{(r)}\right)^c} \left[u(p,h) - u(p',h)\right] \Pi_{j\neq i} x^j_{h_j} \tag{A.8}$$

If we denote $M_{pp'} = |\max_{x\in S^{-i}} [u(p,x) - u(p',x)]| + 1$ and $x_c = \sum_{h\in\left(C_{-i}^{(r)}\right)^c} \Pi_{j\neq i} x^j_{h_j}$, Equation (A.8) implies

$$u(p,x) - u(p',x) - M_{pp'} x_c < 0 \qquad \text{for all } x \in S^{-i} \tag{A.9}$$

By definition of $C_{-i}^{(r)}$, for all $h \in \left(C_{-i}^{(r)}\right)^c$ there is at least one agent $k \neq i$ such that $h_k \notin H_k^{(r)}$ and thus by the induction assumption $\lim_{t\to\infty} x_{h_k}^k(t) = 0$, almost surely. Therefore, $x_c(t)$ goes to zero almost surely. Note also that $x_c = 0$ if $(C_{-i}^{(r)})^c$ is empty and $x_c \leq 1$.

Let $g(x) = \sum_\alpha (p_\alpha - p'_\alpha) m_\alpha(x, 0) - M_{pp'} x_c$ and $m = \max_x g(x)$. Notice that $m < 0$ by Equation (A.9).

Since $V'_p(t) \leq 1$, $V_p(t) \leq V_{p-p'}(t)$. By Lemma 1(b),

$$V_{(p-p')}(t) = \exp\left(\int_0^t \left\{ g[x(s)] + \sigma_{(p-p')}(s) + 2M_{pp'} x_c(s) \right\} ds - \frac{1}{2} a^s_{(p-p')}(t) + A[(p-p')]_0^t \right) V_{(p-p')}(0)$$

By the definition of m, and given that $\sigma_{(p-p')}(s)$ is bounded by $4\sigma^2 d$ and $-1/2a^s_{(p-p')}(t)$ is negative, we have that

$$V_{(p-p')}(t) \leq \exp\left[(m + 4\sigma^2 d)t + 2\int_o^t M_{pp'} x_c(s)\, ds + A(p-p')_0^t\right] V_{(p-p')}(0)$$

By Theorem 3.4.6 in Karatzas and Shreve (1991, p. 174, eq. 4.17), we know that there is a Wiener process $W(t)$ such that almost surely $A(p-p')_0^t = W[a^0_{p-p'}(t)]$, so we have almost surely that

$$V_p(t) \leq V_{(p-p')}(t) \leq \exp\left\{(m + 4\sigma^2 d)t + 2\int_0^t M_{pp'} x_c(s)\, ds + W[a^0_{p-p'}(t)]\right\} V_{(p-p')}(0)$$

Let ϵ and σ be sufficiently small for $m + 4\sigma^2 d + \epsilon < 0$. Since $x_c(t)$ converges to zero as time goes to infinity, for all sample paths ω, there is a time $b(\omega) < \infty$ such that $2M_{pp'} x_c(t) < \epsilon$ for $t > b(\omega)$. Therefore, since $x_c \leq 1$, we have that $2\int_0^t M_{pp'} x_c(s) ds < \epsilon t + 2M_{pp'} b(\omega)$. The assumption that $x^i_\alpha(0) > 0$, for all i, implies that $V_{(p-p')}(0) < \infty$. Now we have to distinguish two cases. Suppose first that $\lim_{t\to\infty} a^0_{p-p'}(t) = \infty$. Then we have that $\lim_{t\to\infty} W[a^0_{p-p'}(t)]/a^0_{p-p'}(t) \to 0$ almost surely, by the strong law of large numbers (Karatzas and Shreve, 1991, Theorem 2.9.3, on p. 104, solved on p. 124). Thus we can write $W[a^0_{p-p'}(t)] = a^0_{p-p'}(t) w(t)$, where $\lim_{t\to\infty} w(t) = 0$ (a.s.). Thus

$$V_p(t) < \exp\left\{(m + 4\sigma^2 d + \epsilon)t + 2M_{pp'} b(\omega) + W[a^0_{p-p'}(t)]\right\} V_{(p-p')}(0)$$

$$= \exp 2M_{pp'} b(\omega) \exp\left\{\left[m + 4\sigma^2 d + \epsilon + \frac{a^0_{p-p'}(t)}{t} w(t)\right] t\right\} V_{(p-p')}(0)$$

but since $a^0_{p-p'}(t)/t$ is a bounded function $\lim_{t\to\infty} w(t) = 0$ and $m + 4\sigma^2 d + \epsilon < 0$,

$$\lim_{t\to\infty} \left[m + 4\sigma^2 d + \epsilon + \frac{a^0_{p-p'}(t)}{t} w(t)\right] t = -\infty \tag{A.10}$$

Since $b(\omega) < \infty$ almost surely and $V_{(p-p')}(0) > 0$, Equation (A.10) implies that $\lim_{t\to\infty} V_p(t) = 0$ almost surely.

When $\lim_{t\to\infty} a^0_{p-p'}(t) < \infty$, we have that $W[a^0_{p-p'}(t)]$ converges to a normal random variable with mean zero and variance $\lim_{t\to\infty} a^0_{p-p'}(\infty)$; thus we have that $W[a^0_{p-p'}(\infty)] < \infty$ almost surely. We also know that $b(\omega) < \infty$ almost surely.

Since $\lim_{t\to\infty}(m + 4\sigma^2 d + \epsilon)t = -\infty$,

$$\lim_{t\to\infty} \exp\left\{(m + 4\sigma^2 d + \epsilon)t + M_{pp'}b(\omega) + W[a^0_{p-p'}(t)]\right\} V_{(p-p')}(0) = 0$$

almost surely. Since this exhausts all cases, the result follows by induction. □

For the proof of Proposition 1B, I will need a few more lemmas.

Lemma 2 proves a generalized version of the fact that

$$E\left\{\exp\left[\int_b^t x_\alpha(s)\sigma_\alpha dW_\alpha(s)\right]\right\} < \exp\left[\frac{1}{2}\sigma_\alpha^2(t-b)\right]$$

To understand this, think of the simple case where we did not have $x_\alpha(s)$ inside the integral. Then we could rewrite the expectation as $E(\exp\{\sigma_\alpha[W_\alpha(t) - W_\alpha(b)]\})$. Notice that $\sigma_\alpha[W_\alpha(t) - W_\alpha(b)]$ is a normal random variable with mean 0 and variance $\sigma_\alpha^2(t-b)$, so its exponential is log normal. It is well known (see Greene, 1991, p. 60) that in this case the log-normal distribution has mean equal to $\exp\left[1/2\sigma_\alpha^2(t-b)\right]$. Lemma 2 is used to take expectations wherever $dW_\alpha(t)$ appears.

LEMMA 2. *Let some constant $c > 0$.*

(a) $$E(\exp cA^t_{\beta s}) \leq \exp\left[dc^2\sigma^2(t-s)\right]$$

(b) $$E(\exp cA(p)^t_s) \leq \exp\left[dc^2\sigma^2(t-s)\right]$$

PROOF. (a) Let

$$Z^t_s(x) = \exp\left(\sum_l \sum_\alpha \int_s^t 2c[\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l} dW_l(u) - \frac{1}{2}\int_s^t \sum_l \left\{\sum_\alpha 2c[\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l}\right\}^2 du\right)$$

By applying Itô's rule to the exponential function (Karatzas and Shreve, 1991, example 3.3.9, on p. 153), we have

$$Z^t_s(x) = 1 + \sum_l \sum_\alpha \int_s^t Z^u_s(x) 2c[\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l} dW_l(u)$$

By Novikov's (1972) sufficient condition to Girsanov's theorem (Karatzas and Shreve, 1991, Corollary 3.5.13, on p. 199), $Z^t_s(x)$ is a martingale if

$$E\left[\exp\left(\frac{1}{2}\int_s^t \sum_l \left\{\sum_\alpha 2c[\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l}\right\}^2 du\right)\right] < \infty \qquad \text{for } s \leq t < \infty$$

which in this case is true because $0 \le x_\alpha(t) \le 1$. If $Z_s^t(x)$ is a martingale, $E[Z_s^t(x)] = 1$. Note that $\sum_\alpha [\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l} = \sigma_{\beta l} - \sum_\alpha x_\alpha(u)\sigma_{\alpha l} \le \sigma$. Using that and Hölder's inequality,

$$E(\exp cA^t_{\alpha s}) \le \left\{E[Z_s^t(x)]\right\}^{1/2} \left\{E\left[\exp\left(\frac{1}{2}\int_s^t \sum_l \left\{\sum_\alpha 2c[\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l}\right\}^2 du\right)\right]\right\}^{1/2}$$

$$\le \exp\left[dc^2\sigma^2(t-s)\right]$$

The same argument applies for (b). $\square$

Lemma 3 shows that

$$\lim_{t\to\infty} E\left\{\frac{1}{[x_\alpha(t)]^c}\right\} < \frac{C}{\underline{\lambda}^c}$$

where C is a constant that depends on the payoffs c and σ. The proof exploits the fact that when there are mutations, x_α cannot be an order of magnitude smaller than $\underline{\lambda}$ for very long because the deterministic part of the flow is positive in that region.

LEMMA 3. *Let $c > 0$ and $C(M, c, \sigma) = \exp[c(M + 2d\sigma^2)]C_1^{1/2}\exp[6c^2 d\sigma^2]$, where C_1 is a constant independent of both the time index and the particular stochastic process we consider.*

$$E[x_\beta(t)^{-c}] \le C(M, c, \sigma)\underline{\lambda}^{-c}$$

PROOF.

$$x_\beta(t) = \exp\left(\int_0^t \left\{m_\beta[x(s), \lambda] + \sigma_\beta(s)\right\} ds - \frac{1}{2}a^0_\beta(t) + A^t_{\beta 0}\right)x_\beta(0)$$

$$+ \int_0^t \exp\left(\int_s^t \left\{m_\beta[x(u), \lambda] + \sigma_\beta(u)\right\} du - \frac{1}{2}a^s_\beta(t) + A^t_{\beta s}\right)$$

$$\times \left[\sum_\beta \lambda_{\alpha\beta} x_\beta(t)\right] ds$$

By the positivity of the exponential function, λ, and x,

$$E[x_\beta(t)^{-c}] \le E\left\{\left[\int_{t-1}^t \exp\left(\int_s^t \left\{m_\beta[x(u), \lambda] + \sigma_\beta(u)\right\} du\right.\right.\right.$$

$$\left.\left.\left. - \frac{1}{2}a^s_\beta(t) + A^t_{\beta s}\right) \sum_\beta \lambda_{\alpha\beta} x_\alpha(s)\, ds\right]^{-c}\right\}$$

Since $|m_\beta[x(u), \lambda]| \le M$, $|\sigma_\beta(u)| \le \sigma^2 d$ and $|a^s_\beta(t)| \le \sigma^2 d$,

$$E[x_\beta(t)^{-c}] \le \exp[c(M + 2\sigma^2 d)](\underline{\lambda})^{-c}$$

$$E\left[\left(\int_{t-1}^t \exp\left\{\sum_l \sum_\alpha \int_s^t [\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l} dW_l(u)\right\} ds\right)^{-c}\right]$$

Letting $k = \exp[c(M + 2\sigma^2 d)]\underline{\lambda}^{-c}$ we have then by Hölder's inequality

$$E[x_\beta(t)^{-c}] \leq k\left\{E\left[\left(\exp\left\{\sum_l\sum_\alpha\int_{t-1}^t[\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l}dW_l(u)\right\}\right)^{-2c}\right]\right\}^{1/2}$$

$$\times \left\{E\left[\left(\int_{t-1}^t \exp\left\{-\sum_l\sum_\alpha\int_{t-1}^s[\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l}dW_l(u)\right\}ds\right)^{-2c}\right]\right\}^{1/2}$$

$$\leq k\left[E\left(\exp\left\{\sum_l\sum_\alpha\int_{t-1}^t[\delta_{\beta\alpha} - x_\alpha(u)](-2c)\sigma_{\alpha l}dW_l(u)\right\}\right)\right]^{1/2}$$

$$\times \left[E\left(\sup_{t-1\leq s\leq t}\exp\left\{\sum_l\sum_\alpha\int_{t-1}^s 2c[\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l}dW_l(u)\right\}\right)\right]^{1/2}$$

which by Lemma 2 and Hölder's inequality gives

$$\leq k\left[\exp\left(4c^2d\sigma^2\right)\right]^{1/2}\left(E\left\{\left[\sup_{t-1\leq s\leq t} Z_{t-1}^s(x)\right]^2\right\}\right)^{1/4}$$

$$\times \left(E\left\{\left[\sup_{t-1\leq s\leq t}\exp\left(\sum_l\int_{t-1}^s\left\{\sum_\alpha 2c[\delta_{\beta\alpha} - x_\alpha(u)]\sigma_{\alpha l}\right\}^2 ds\right)\right]^2\right\}\right)^{1/2}$$

By the proof of Lemma 2, we know that $Z_{t-1}^s(x)$ is a martingale, so we can use Novikov's (1971) martingale moment inequalities (Karatzas and Shreve, 1991, Proposition 3.3.26, on p. 163) to bound the expectation of the square of its supremum:

$$E[x_\beta(t)^{-c}] \leq k\left[\exp\left(4c^2d\sigma^2\right)\right]^{1/2} C_1^{1/2}\exp(4c^2d\sigma^2)$$

Since C_1 is a constant independent of both the time index and the particular martingale, the result follows. □

Lemma 4 is an auxiliary lemma that allows me to prove Lemma 5.

LEMMA 4. *Let*

$$C_1(M, 2, \sigma) = C(M, 2, \sigma)^{\frac{1}{2}}\exp(M + 2\sigma^2)\exp(2d^2\sigma^2)$$

and

$$C_2(M, 2, \sigma) = \exp[M + 2\sigma^2]C(M, 2, \sigma)$$

For any pure strategy β,

$$E\left[\frac{x_\beta^j(t)}{V_p(t)}\right] \leq C_1(M, 2, \sigma)\frac{E[x_\beta^j(t-1)]}{\underline{\lambda}} + C_2(M, 2, \sigma)\frac{\overline{\lambda}}{\underline{\lambda}}$$

PROOF. By Lemma 1 and the definition of σ and M, we have,

$$\frac{x^j_\beta(t)}{V_p(t)} \le \frac{1}{V_p(t)}\left\{\exp(M+2\sigma^2+A^t_{\beta t-1})x^j_\beta(t-1) + \int_{t-1}^t \exp[(M+2\sigma^2)(t-s)+A^t_{\beta s}]\sum_{\gamma=1}^{n_j}\lambda^j_{\beta\gamma}x^j_\gamma(s)\,ds\right\} \tag{A.11}$$

By Lemma 1, the positivity of the exponential function, λ, and x, and using Hölder's inequality, we have that

$$E_{t-1}\left\{\frac{1}{[V_p(t)]^2}\right\} \le \prod_\alpha\left(E_{t-1}\left\{\left[\int_{t-1}^t \exp\left(\int_s^t\{m_\alpha[x(u),\lambda]+\sigma_\alpha(u)\}\,du - \frac{1}{2}a^s_\alpha(t)+A^t_{\alpha s}\right)\sum_{\gamma=1}^{n_j}\lambda_{\alpha\gamma}x_\gamma(s)\,ds\right]^{-2}\right\}\right)^{p_\alpha}$$

We can then show by Lemma 3 that

$$E_{t-1}\left\{\frac{1}{[V_p(t)]^2}\right\} \le \prod_\alpha\left[C(M,2,\sigma)\frac{1}{\underline{\lambda}^2}\right]^{p_\alpha} \tag{A.12}$$

As in the proof of Lemma 3, we can show that

$$E_{t-1}\left\{\left[\int_{t-1}^t \exp(A^t_{\beta s})\sum_{\gamma=1}^{n_j}\lambda^j_{\beta\gamma}x^j_\gamma(s)\,ds\right]^2\right\}$$
$$\le \exp[2(M+2\sigma^2 d)]\overline{\lambda}^2 E\left[\left(\int_{t-1}^t \exp\left\{\sum_l\sum_{\gamma=1}^{n_j}\int_s^t[\delta_{\beta\gamma}-x^j_\gamma(u)]\sigma^j_{\gamma l}dW_l(u)\right\}ds\right)^2\right]$$

which implies as in Lemma 3,

$$E_{t-1}\left\{\left[\int_{t-1}^t \exp(A^t_{\beta s})\sum_{\gamma=1}^{n_j}\lambda^j_{\beta\gamma}x^j_\gamma(s)\,ds\right]^2\right\} \le C(M,2,\sigma)\overline{\lambda}^2 \tag{A.13}$$

By Equation (A.11), we have that

$$E_{t-1}\left[\frac{x^j_\beta(t)}{V_p(t)}\right] \le \left\{E_{t-1}\left[\frac{1}{V_p^2(t)}\right]\right\}^{\frac{1}{2}}$$
$$\times\left[\exp(M+2\sigma^2)\{E[\exp(2A^t_{\beta t-1})]\}^{\frac{1}{2}}x^j_\beta(t-1) + \exp(M+2\sigma^2)\left(E\left\{\left[\int_{t-1}^t \exp(A^t_{\beta s})\sum_{\gamma=1}^{n_j}\lambda^j_{\beta\gamma}x^j_\gamma(s)\,ds\right]^2\right\}\right)^{\frac{1}{2}}\right]$$

and by Lemma 2 and Equations (A.12) and (A.13),

$$E_{t-1}\left[\frac{x^j_\beta(t)}{V_p(t)}\right] \le C(M,2,\sigma)^{\frac{1}{2}}\frac{1}{\underline{\lambda}}\Big[\exp(M+2\sigma^2)\exp(2d^2\sigma^2)x^j_\beta(t-1)$$

$$+\exp(M+2\sigma^2)C(M,2,\sigma)^{\frac{1}{2}}\overline{\lambda}\Big]$$

From this we get

$$E\left[\frac{x^j_\beta(t)}{V_p(t)}\right] \le C_1(M,2,\sigma)\frac{E[x^j_\beta(t-1)]}{\underline{\lambda}} + C_2(M,2,\sigma)\frac{\overline{\lambda}}{\underline{\lambda}} \qquad \square$$

PROOF OF PROPOSITION 1B. I will do the proof by induction. Let the sets $C_j^{(r)}$, $C_{-i}^{(r)}$, and $\left(C_{-i}^{(r)}\right)^c$ and the constants $M_{pp'}$ and x_c be defined as in the proof of Proposition 1A. Let $r > 1$, and assume that there is a constant $K^{(r-1)}$ such that for all $\beta \notin C_i^{(r-1)}$,

$$\lim_{\lambda\to 0}\limsup_{t\to\infty}\left\{E[x^i_\beta(t)]\right\} < K^{(r-1)}\overline{\lambda} \tag{A.14}$$

for $\max_{\alpha,i,l}\{\sigma^i_{\alpha l}\}$ small enough. Let $\gamma \notin C_i^{(r)}$ and $\gamma \in C_i^{(r-1)}$. Then there is a $p' \in H_i^{(r)}$ such that

$$u^i(\gamma,x) - u^i(p',x) < 0 \qquad \text{for all } x \in H_{-i}^{(r-1)} \tag{A.15}$$

Equation (A.14) and the fact that $x_c(t) \le \sum_{j\ne i}\sum_{\beta\notin C_j^{r-1}} x^j_\beta$ imply that

$$\lim_{\lambda\to 0}\limsup_{t\to\infty} E[x_c(t)] \le \sum_{j\ne i} n^j K^{(r-1)}\overline{\lambda}$$

Equation (A.15) and the definition of x_c imply that

$$u(\gamma,x) - u(p',x) - M_{\gamma p'}x_c < 0 \qquad \text{for all } \ x \in S^{-i} \tag{A.16}$$

Assume that λ is small enough that

$$\max_{x\in S^{-i}}\Big\{u(\gamma,x) - u(p',x) - M_{\gamma p'}x_c\Big\} < \sum_\beta(\lambda_{\gamma\beta} - \sum_\alpha p'_\alpha\lambda_{\alpha\beta}) \tag{A.17}$$

Since

$$m_\gamma(x,\lambda) - \sum_\alpha p'_\alpha m_\alpha(x,\lambda) = u(\gamma,x) - u(p',x) - \sum_\beta(\lambda_{\gamma\beta} - \sum_\alpha p'_\alpha\lambda_{\alpha\beta})$$

Equations (A.16) and (A.17) imply that

$$m_\gamma(x,\lambda) - \sum_\alpha p'_\alpha m_\alpha(x,\lambda) - M_{\gamma p'}x_c < 0 \qquad \text{for all } \ x \in S^{-i} \tag{A.18}$$

Let $g(x) = m_\gamma(x, \lambda) - \sum_\alpha p'_\alpha m_\alpha(x, \lambda) - M_{\gamma p'} x_c$, and let $m = \max_x\{g(x)\}$. Notice that $m < 0$ by Equation (A.18).

Before proceeding with the proof of Proposition 1B, I will show in Lemma 5 that the expectation of $x_\gamma / V_{p'}$ is bounded. Samuelson and Zhang (1992) show that strictly dominated strategies disappear by showing that for γ dominated by p', $x_\gamma / V_{p'}$ goes to zero when there are no shocks. I cannot do this because mutations prevent the weights of strategies from becoming arbitrarily small. But Lemma 5 shows that for any γ and p, $E(x_\gamma / V_{p'})$ has a bound that is independent of the mutation rates if these are not orders of magnitude apart. This will be used to show that for γ dominated, $E(x_\gamma)$ is asymptotically small when the mutation rates are small. $E(x_\gamma / V_{p'})$ has a bound because far from the boundaries the dynamics tend to make it small, so the first term in Equation (A.19) is small, but near the boundaries the movement depends on mutation rates to a greater extent, and the second term in Equation (A.19) reflects this.

LEMMA 5. *For all t large enough,*

$$E\left[\frac{x_\gamma(t)}{V_{p'}(t)}\right] \leq \exp[(m + 5\sigma^2 d)t] \frac{x_\gamma(0)}{V_{p'}(0)} + \left[-(m + 5\sigma^2 d)\right]^{-1} \times \left\{ M_{\gamma p'}\left[C_1(M, 2, \sigma) \sum_{j \neq i} n_j \frac{K^{(r-1)}\overline{\lambda}}{\underline{\lambda}} + C_2(M, 2, \sigma)\frac{\overline{\lambda}}{\underline{\lambda}} \right] + \frac{\overline{\lambda}}{\underline{\lambda}} C(m, 2, \sigma) \right\} \tag{A.19}$$

PROOF. By Lemma 1(b'), we know that

$$\begin{aligned}
\frac{x_\gamma(t)}{V_{p'}(t)} = &\exp\left(\int_0^t \left\{ (g[x(s)] + \sigma_{(\gamma - p')}(s) \right\} ds - \frac{1}{2} a^0_{(\gamma - p')}(t) + A[(\gamma - p')]_0^t \right) \frac{x_\gamma(0)}{V_{p'}(0)} \\
&+ \int_0^t \exp\left(\int_s^t \left\{ g[x(u)] + \sigma_{(\gamma - p')}(u) \right\} du - \frac{1}{2} a^s_{(\gamma - p')}(t) + A[(\gamma - p')]_s^t \right) \\
&\times M_{\gamma p'} x_c(s) \frac{x_\gamma(s)}{V_{p'}(s)} ds \\
&+ \int_0^t \exp\left(\int_s^t \left\{ g[x(u)] + \sigma_{(\gamma - p')}(u) \right\} du - \frac{1}{2} a^s_{(\gamma - p')}(t) + A[(\gamma - p')]_s^t \right) \\
&\times \left\{ \frac{1}{x_\gamma(s)} \left[\sum_\beta \lambda_{\gamma\beta} x_\beta(s) \right] - \sum_\alpha \frac{p'_\alpha}{x_\alpha(s)} \left[\sum_\beta \lambda_{\alpha\beta} x_\beta(s) \right] \right\} \frac{x_\gamma(s)}{V_{p'}(s)} ds
\end{aligned}$$

By the definition of m, and since $\sigma_{(\gamma-p')}(t) \leq 4d\sigma^2$, $-1/2a^0_{(\gamma-p')}(t)$ is negative, and $x_\gamma \leq 1$,

$$\begin{aligned}\frac{x_\gamma(t)}{V_{p'}(t)} \leq &\exp\Big\{(m+4\sigma^2 d)t + A\,[(\gamma-p')]_0^t\Big\}\frac{x_\gamma(0)}{V_{p'}(0)}\\ &+\int_0^t \exp\Big\{(m+4\sigma^2 d)(t-s) + A\,[(\gamma-p')]_s^t\Big\}\\ &\times\left[M_{\gamma p'}\frac{x_c(s)}{V_{p'}(s)} + \frac{\overline{\lambda}}{x_\gamma(s)}\frac{x_\gamma(s)}{V_{p'}(s)}\right]ds\end{aligned}$$

Taking expectations and using the fact that $x_c \leq \sum_{j\neq i}\sum_{\beta\notin C_j^{(r-1)}} x_\beta^j$,

$$\begin{aligned}E\left[\frac{x_\gamma(t)}{V_{p'}(t)}\right] \leq &\exp[(m+4\sigma^2 d)t][E(\exp\{A[(\gamma-p')]_0^t\})]\frac{x_\gamma(0)}{V_{p'}(0)}\\ &+\int_0^t \exp[(m+4\sigma^2 d)(t-s)]\\ &\times E\left(E_s\{\exp[A(\gamma-p')_s^t]\}\left[M_{\gamma p'}\sum_{j\neq i}\sum_{\beta\notin C_j^{(r-1)}}\frac{x_\beta^j(s)}{V_{p'}(s)} + \frac{\overline{\lambda}}{V_p'(s)}\right]\right)ds\end{aligned}$$

which by Lemma 2 gives

$$\begin{aligned}E\left[\frac{x_\gamma(t)}{V_{p'}(t)}\right] \leq &\exp[(m+4\sigma^2 d)t]\exp(d\sigma^2 t)\frac{x_\gamma(0)}{V_{p'}(0)}\\ &+\int_0^t \exp[(m+4\sigma^2 d)(t-s)]\exp[d\sigma^2(t-s)]\\ &\times\left\{M_{\gamma p'}\sum_{j\neq i}\sum_{\beta\notin C_j^{(r-1)}} E\left[\frac{x_\beta^j(s)}{V_{p'}(s)}\right] + E\left[\frac{\overline{\lambda}}{V_p'(s)}\right]\right\}ds\end{aligned}$$

and by Lemmas 3 and 4 gives

$$\begin{aligned}E\left[\frac{x_\gamma(t)}{V_{p'}(t)}\right] \leq &\exp[(m+4\sigma^2 d)t]\exp\left(d\sigma^2 t\right)\frac{x_\gamma(0)}{V_{p'}(0)}\\ &+\int_0^t \exp\left[(m+4\sigma^2 d)(t-s) + d\sigma^2(t-s)\right]\\ &\times\left(M_{\gamma p'}\left\{C_1(M,2,\sigma)\sum_{j\neq i}\sum_{\beta\notin C_j^{(r-1)}}\frac{E[x_\beta^j(t-1)]}{\underline{\lambda}} + C_2(M,2,\sigma)\frac{\overline{\lambda}}{\underline{\lambda}}\right\}\right.\\ &\left.+\frac{\overline{\lambda}}{\underline{\lambda}}C(m,2,\sigma)\right)ds\end{aligned}$$

Notice that by the induction assumption for t large enough, $E[x^j_\beta(t-1)] \le K^{(r-1)}\overline{\lambda}$, and the lemma then follows by integration. $\square$

Now I continue with the proof of Proposition 1B.

Let $b < t$. By Lemma 1(b),

$$V_{p'}(t) = \exp\left(\int_b^t \left\{(\sum_\alpha p'_\alpha m_\alpha[x(s), \lambda] + \sigma_{p'}(s)\right\} ds - \frac{1}{2} a^b_{p'}(t) + A(p')^t_b\right) V_{p'}(b)$$

$$+ \int_b^t \exp\left(\int_s^t \left\{(\sum_\alpha p'_\alpha m_\alpha[x(u), \lambda] + \sigma_{p'}(u)\right\} du - \frac{1}{2} a^s_{p'}(t) + A(p')^t_s\right)$$

$$\times \sum_\alpha \frac{p'_\alpha}{x_\alpha(s)} \left[\sum_\beta \lambda_{\alpha\beta} x_\beta(s)\right] V_{p'}(s)\, ds$$

Then, by the positivity of λ, p', and x and the exponential function,

$$1 \ge V_{p'}(t) \ge \exp\left(\int_b^t \left\{\sum_\alpha p'_\alpha m_\alpha[x(s), \lambda] + \sigma_{p'}(s)\right\} ds - \frac{1}{2} a^b_{p'}(t) + A(p')^t_b\right) V_{p'}(b)$$

By Lemma 1(b'), letting $g_\gamma(x) = m_\gamma(x, \lambda) - M_{\gamma p'} x_c$,

$$x_\gamma(t) = \exp\left(\int_b^t \{g_\gamma[x(s)] + \sigma_\gamma(s)\}\, ds - \frac{1}{2} a^b_\gamma(t) + A^t_{\gamma b}\right) x_\gamma(b)$$

$$+ \int_b^t \exp\left(\int_s^t \{g_\gamma[x(u)] + \sigma_\gamma(u)\}\, du - \frac{1}{2} a^s_\gamma(t) + A^t_{\gamma s}\right) \left[\sum_\beta \lambda_{\gamma\beta} x_\beta(s)\right] ds$$

$$+ \int_b^t \exp\left(\int_s^t \{g_\gamma[x(u)] + \sigma_\gamma(u)\}\, du - \frac{1}{2} a^s_\gamma(t) + A^t_{\gamma s}\right) M_{\gamma p'} x_c(s) x_\gamma(s)\, ds$$

Now I divide the first line in the preceding equation by

$$\exp\left(\int_b^t \left\{\sum_\alpha p'_\alpha m_\alpha[x(s), \lambda] + \sigma_{p'}(s)\right\} ds - \frac{1}{2} a^b_{p'}(t) + A(p')^t_b\right) V_{p'}(b)$$

and since I showed that the last expression is less than one, letting $g_{(\gamma-p')}(x) = m_\gamma[x(s), \lambda] - \sum_\alpha p'_\alpha m_\alpha[x(s), \lambda] - M_{pp'} x_c(s)$ gives

$$x_\gamma(t) \le \exp\left(\int_b^t \{g_{\gamma-p'}[x(s)] + \sigma_{(\gamma-p')}(s)\}\, ds - \frac{1}{2} a^b_{(\gamma-p')}(t) + A[(\gamma-p')]^t_b\right) \frac{x_\gamma(b)}{V_{p'}(b)}$$

$$+ \int_b^t \exp\left(\int_s^t \{g_\gamma[x(u)] + \sigma_\gamma(u)\}\, du - \frac{1}{2} a^s_\gamma(t) + A^t_{\gamma s}\right) \left[\sum_\beta \lambda_{\gamma\beta} x_\beta(s)\right] ds$$

$$+ \int_b^t \exp\left(\int_s^t \{g_\gamma[x(u)] + \sigma_\gamma(u)\}\, du - \frac{1}{2} a^s_\gamma(t) + A^t_{\gamma s}\right) M_{\gamma p'} x_c(s)\, ds$$

Taking expectations and applying Lemmas 2 and 3, by definition of M,

$$\text{(A.20)} \qquad \begin{aligned} E[x_\gamma(t)] \leq & \exp[(m+2\sigma^2 d)(t-b)]E\left[\frac{x_\gamma(b)}{V_{p'}(b)}\right] \\ & + \left(M+2\sigma^2 d\right)^{-1}\left\{-1+\exp\left[(M+2\sigma^2 d)(t-b)\right]\right\}\overline{\lambda} \\ & + M_{\gamma p'} \max_{s\in(b,t)}\{E[x_c(s)]\}\left(M+2\sigma^2 d\right)^{-1} \\ & \times\left\{-1+\exp\left[(M+2\sigma^2 d)(t-b)\right]\right\} \end{aligned}$$

I have to show that there is some constant $K^{(r)}$ such that for all t larger than some t_r, $E[x_\gamma(t)]$ is smaller than $K^{(r)}\overline{\lambda}$.

If $\overline{\lambda}/\underline{\lambda}$ is bounded, Lemma 5 shows that $E\left[x_\gamma(b)/V_{p'}(b)\right]$ is bounded by a constant F that depends only on m, M, σ, and $x_\gamma(0)/V_{p'}(0)$ when b is above some b_r. Choose t' such that $t'-b_r>0$ and

$$\exp[2(m+4\sigma^2 d)(t'-b_r)]F<\overline{\lambda}$$

Then for all $t>t'$, choose b such that $t-b=t'-b_r$. This guarantees that the first line in Equation (A.20) is strictly smaller than $\overline{\lambda}$. Since $t-b$ is a constant by definition of b, the second line of Equation (A.20) is also a constant times $\overline{\lambda}$. Since $x_c \leq \sum_{j\neq i}\sum_{\beta\notin C_i^{(r-1)}} x_\beta^j$ and by the induction assumption x_β^j is smaller than $K^{(r-1)}\overline{\lambda}$, the third line in Equation (A.20) also can be made smaller than a constant times $\overline{\lambda}$ for t larger than some t''. Let t' be larger than t'', and the result follows for $r>1$ by making

$$\begin{aligned} K^{(r)} = & 1+\left(M+2\sigma^2 d\right)^{-1}\left\{-1+\exp\left[(M+2\sigma^2 d)(t'-b_r)\right]\right\} \\ & +M_{pp'}\Big(\sum_{j\neq i} n_j K^{(r-1)}\Big)\left(M+2\sigma^2 d\right)^{-1}\left\{-1+\exp\left[(M+2\sigma^2 d)(t'-b_r)\right]\right\} \end{aligned}$$

The proof for $r=1$ is analogous. Just notice that $\left(C_j^{(0)}\right)^c=\varnothing$ and $x_c=0$, and the steps of the proof are identical. The result follows by induction. $\square$

PROOF OF PROPOSITION 2. This proof borrows heavily from the proof of Propositions 3 and 4 in Fudenberg and Harris (1992), so readers familiar with their work can follow my proofs more easily.

(a) Let $\delta(x)=x(1-x)(\sigma_1^2-\sigma^2 x+ax^{N-1}-b)-\lambda_1 x+\lambda_2(1-x)$. Let an arbitrary $z\in(0,1)$, and

$$I_1=\int_0^{x(0)}\exp\left[-2\int_z^x\frac{\delta(y)}{y^2(1-y)^2\sigma^2}dy\right]dx$$

$$I_2 = \int_{x(0)}^{1} \exp\left[-2\int_z^x \frac{\delta(y)}{y^2(1-y)^2\sigma^2}dy\right] dx$$

$$D(x) = \frac{2}{x^2(1-x)^2\sigma^2} \exp 2\int_z^x \frac{\delta(y)}{y^2(1-y)^2\sigma^2}dy$$

The process $x(t)$ is ergodic (see Theorem 1.17 of Skorohod, 1989, p. 48) if I_1 and I_2 are infinite and $\int_0^1 D(x)dx$ is finite.

But $\delta(y)/[y^2(1-y)^2\sigma^2]$ is of order λ_2/y^2 around $y=0$ and of order $-\lambda_1/(1-y)^2$ around $y = 1$. Thus I_1 and I_2 are infinite. $D(x)$ is of order $\exp(-\lambda_1/x)/x^2$ in a neighborhood of $x = 0$ and of order $\exp[-\lambda_2/(1-x)]/(1-x)^2$ in the vicinity of $x = 1$, so $\int_0^1 D(x)dx$ is finite.

(b) By Theorem 1.17 of Skorohod (1989, p. 48), the density of the ergodic distribution is proportional to

$$D(x) = \frac{2}{x^2(1-x)^2\sigma^2} \exp 2\int_z^x \frac{\delta(y)}{y^2(1-y)^2\sigma^2}\, dy$$

But since

$$\frac{1}{x^2(1-x)^2} = \exp\ 2[-\ln\ x - \ln(1-x)] = \exp\ 2\left(\int_z^x \frac{1}{1-y} - \frac{1}{y}\, dy\right) \frac{1}{z^2(1-z)^2}$$

We have then that

$$D(x) = \frac{2}{z^2(1-z)^2\sigma^2}\ \exp\ 2\int_z^x \frac{\delta(y) + (2y-1)y(1-y)\sigma^2}{y^2(1-y)^2\sigma^2}\, dy$$

Let

$$\gamma(y) = \delta(y) + (2y-1)y(1-y)\sigma^2 = y(1-y)$$
$$\times(\sigma^2 y - \sigma_2^2 + ay^{N-1} - b) - \lambda_1 y + \lambda_2(1-y)$$

and
$$F(x) = \exp\ 2\int_z^x \frac{\gamma(y)}{y^2(1-y)^2\sigma^2}\, dy$$

Let y_1 be the smallest $y \in [0, 1]$ such that $\gamma(y) = 0$. Since

$$\gamma(y) > -by - \lambda_1 y + \lambda_2(1-y) - \sigma^2 y$$

then

$$y_1 > \frac{\lambda_2}{b + \lambda_1 + \lambda_2 + \sigma^2}$$

Since $\gamma(y) > 0$ for $y < y_1$, $F(y) < F(y_1)$.

Choose σ_2 so that $a - 2b - \sigma_2^2 > 0$. Choose y_2 so that $ay_2^{N-1} - b - \sigma_2^2 > b + k$ for some $a - 2b - \sigma_2^2 > k > 0$. Let $y_3 = 1 - \lambda_1/(b+k)$. Since $\gamma(y) > 0$ in $[y_2, y_3]$, then $F(y)$ is strictly increasing in that interval.

Now let $x \in [y_1, y_2)$ and $x' \in (y_2, y_3]$:

$$\frac{F(x')}{F(x)} = \exp 2\left[\int_{y_2}^{x'} \frac{\gamma(y)}{y^2(1-y)^2\sigma^2}\,dy + \int_{x}^{y_2} \frac{\gamma(y)}{y^2(1-y)^2\sigma^2}\,dy\right]$$

$$\geq \exp \frac{2}{\sigma^2}\left\{\int_{y_2}^{x'}\left[\frac{b+k}{y(1-y)} - \frac{\lambda_1}{y(1-y)^2}\right]dy - \int_{x}^{y_2}\left[\frac{b+\sigma_2^2}{y(1-y)} + \frac{\lambda_1}{y(1-y)^2}\right]dy\right\}$$

$$\geq \exp \frac{2}{\sigma^2}\left\{-(b+k)[\ln(1-x') - \ln(1-y_2)] + (b+\sigma_2^2+\lambda_1)\ln x\right.$$

$$\left.+\ (b+\sigma_2^2)\ln(1-y_2) + \lambda_1 \ln(1-x') - \frac{\lambda_1}{(1-x')}\right\}$$

If $x' \geq 1 - \lambda_1^{(b+k/4)/(b+k)}$, then given the definition of x' and y_1,

$$\frac{F(x')}{F(x)} \geq \exp \frac{2}{\sigma^2}\left[-(b+k/4)\ln\lambda_1 + (b+\lambda_1+\sigma_2^2)\ln\frac{\lambda_2}{b+\lambda_1+\lambda_2+\sigma_2^2}\right.$$

$$\left.+ (2b+k+\sigma_2^2)\ln(1-y_2) + \lambda_1\ln(1-x') - \frac{\lambda_1}{(1-x')}\right]$$

Since λ_1/λ_2 is bounded, if $x' \geq 1 - \lambda_1^{(b+k/4)/(b+k)}$ and λ_1 and σ_2 are small enough, $F(x') \geq F(x)$. Given that for $y < y_1$, $F(y) < F(y_1)$, and $F(y)$ is increasing in the interval $[y_2, y_3]$, this implies that $F(x') \geq F(y)$ for all $y < x'$.

Let $x_1 = 1 - \lambda_1^{(b+k/3)/b+k}$, $x_2 = 1 - \lambda_1^{(b+k/4)/(b+k)}$, and $x_3 = 1 - \lambda_1^{(b+k/2)/(b+k)}$. Now let the ratio of probabilities under the ergodic distribution

$$\frac{P(x > x_1)}{P(x < x_2)} \geq \frac{(x_3 - x_1)\min_{x\in[x_1,x_3]} D(x)}{\max_{x\in[0,x_2]} D(x)} \geq \frac{(x_3-x_1)F(x_1)}{F(x_2)}$$

$$\geq (x_3 - x_1)\exp\frac{2}{\sigma^2}\left\{-(b+k)[\ln(1-x_1) - \ln(1-x_2)]\right.$$

$$\left.+ \lambda_1\ln x_2 + \lambda_1\ln(1-x_1) - \frac{\lambda_1}{(1-x_1)}\right\}$$

Since the preceding expression tends to infinity as λ_1, λ_2, σ_1^2, and σ_2^2 tend to zero, all the probability mass tends to be concentrated in the interval $[1 - \lambda_1^{(b+k/3)/(b+k)}, 1]$. Since λ_1 goes to zero, the result follows. □

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