

Auctions with Heterogeneous Entry Costs*

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Abstract

It is well known that if bidders have independent private values and homogeneous entry costs a first- or second-price auction with a reserve price equal to the seller's value maximizes social surplus and seller revenue, and leaves bidders with no surplus. Further, in mixed strategy entry equilibria, social surplus and seller revenue decrease with the number of bidders. We show that when entry costs are heterogeneous (and private information) the revenue maximizing reserve price is above the seller's value, a positive inspection fee (and a reserve price equal to the seller's value) generates even more revenue, and in either case bidders capture informational rents. Further, seller revenue and social surplus may either increase or decrease with the number of bidders. Nevertheless, seller revenue is asymptotically the same whether entry costs are homogeneous or heterogeneous. Our results are framed in terms of screening values rather than reserve prices, and apply to any standard auction.

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1 Introduction

A classic result of the auction literature is that in a standard auction with an exogenously fixed number of bidders who have independent private values, maximizing seller revenue requires screening bidders; i.e., the rules of the optimal auction are such that a bidder whose value is below the *screening value* will find it unprofitable to bid. Moreover, the optimal screening value is above the seller's value and is independent of the number of bidders – see Riley and Samuelson (1981) and Myerson (1981). In first- and second-price auctions with a reserve price, the screening value is just the reserve price. Hence the optimal (i.e., revenue maximizing) reserve is above the seller's value and is independent of the number of bidders.

In many instances, however, the number of bidders is endogenously determined as the result of costly entry decisions. As noted by Milgrom (2004), "... auctions for valuable yet highly specialized assets often fail because of insufficient interest by bidders ... [since] buyers are naturally reluctant to begin an expensive, time-consuming evaluation of an asset when they believe that they are unlikely to win at a favorable price." Indeed, McAfee and McMillan (1987) and Levin and Smith (1994) have shown that the endogenous entry of bidders has important implications in first- and second-price auctions. Specifically, when all bidders have the same entry cost, a reserve price equal to the seller's value is optimal both for the seller and society.

In this paper we study standard auctions with endogenous entry, but where bidders have heterogeneous entry costs. In the sale of a firm, for example, prospective buyers may face different regulatory restrictions: some bidders may have to seek approval by regulatory authorities while others may not. Hence different bidders may have substantially different costs of discovering their value for the firm. In Internet auctions a bidder's cost of discovering her value is the opportunity cost of her time, and it varies across bidders. Our theoretical analysis of reserve prices provides a richer framework for empirical studies of Internet auctions using data either from the field or from experiments – e.g., Reiley (2006).

In our setting, like in McAfee and McMillan (1987) and Levin and Smith (1994), bidders simultaneously choose whether to enter the auction. Each bidder who enters the auction observes her value for the object and then bids. Our setting differs in that each bidder's entry cost is an independent draw from a common distribution,

and is privately observed prior to entry.

Heterogeneity in entry costs alters the conclusions obtained for the homogenous entry cost case. We show that while a screening value equal to the seller's value remains socially optimal, the revenue maximizing screening value is above the seller's value. Thus, in first- and second-price auctions the revenue maximizing reserve price is above the seller's value.

When entry costs are homogenous, the seller has no incentive to charge an inspection fee or subsidy (i.e., a fee which a bidder must pay, in addition to her entry cost, in order to learn her value).¹ We show that when entry costs are heterogeneous, if an inspection fee is feasible, then the optimal screening value is, once again, the seller's value, and the optimal inspection fee is positive. In other words, to maximize revenue the seller screens bidders according to entry costs but not according to values (for values exceeding the seller's value). As a consequence, there is less entry than would be socially optimal.

In order to understand the intuition for our results, it is useful to review the results and intuition when entry costs are homogeneous. Let us assume for simplicity that the seller's value for the object is zero. A key result in this setting is that in a standard auction with a screening value of zero the contribution to social surplus of an additional bidder is exactly equal to the bidder's expected utility to entering.² Thus, when entry costs are homogeneous, the interests of an entrant and society are aligned: a bidder enters only if her expected utility to entering is above her entry cost; that is, only if her contribution to social surplus is positive. Since bidders enter the auction so long as their contribution to social surplus is positive, the number of entering bidders maximizes social surplus. If the auction is sufficiently competitive, then in equilibrium each bidder is indifferent between entering or not. Therefore bidder surplus is competed away and the seller captures the entire social surplus. Hence a screening value equal to zero maximizes both seller revenue and social surplus.

When entry costs are heterogeneous a version of the key result described above

¹In the literature, "entry fee" usually refers to a fee paid by the bidder to submit a bid when she already knows her value. Such a fee is captured in our setting through its effect on the screening value. An inspection fee is paid by a bidder before learning her value.

²A version of this result is established in Engelbrech-Wiggans (1993)'s Proposition 1, and is also observed in both McAfee and McMillan (1987) and Levin and Smith (1994).

also holds: in a standard auction with a screening value of zero the contribution to social surplus of a marginal increase of the equilibrium entry threshold is proportional to the bidder's expected utility to entering; that is, the interests of bidders and society are also aligned when entry costs are heterogeneous. Consequently, a standard auction with a zero screening value maximizes social surplus whether entry costs are homogeneous or heterogeneous. With heterogeneous entry costs, however, not all bidder surplus is competed away by entry: whereas the surplus of a bidder with an entry cost equal to the equilibrium threshold is exactly zero, the surplus of bidders with lower entry costs (who also enter) is positive. Therefore bidders capture a positive share of the surplus. Hence, even though setting a positive screening value reduces total surplus, it increases the seller's share of social surplus and, as we show, increases revenue.

In fact, when entry costs are heterogeneous the optimal screening value is above the seller's value, regardless of the distribution of entry costs, and depends on the number of bidders as well as on the distribution of values and entry costs. (As noted above, however, an even greater revenue can be obtained with a positive inspection fee and a screening value of zero.) The optimal screening value, however, is always below the screening value that is optimal when the number of bidders is exogenously fixed.

There is another important difference between homogeneous and heterogeneous entry costs. For homogeneous entry costs, Levin and Smith (1994) show that seller revenue decreases with the number of bidders in an entry equilibrium in mixed strategies. We describe simple examples that show that a direct extension of this result does not hold when entry costs are heterogeneous: even if the number of bidders is such that a bidder enters with probability less than one, an increase in the number of bidders may either increase or decrease seller revenue depending upon the distribution of values and entry costs.

Auctions with homogeneous and heterogeneous entry costs are, however, closely related as the number of bidders grows large. In particular, asymptotic seller revenue is the same when (i) bidders have a homogeneous entry cost $c > 0$, and (ii) when bidders have heterogeneous entry costs and the lower bound of entry costs is c . In other words, heterogeneity of entry costs does not matter asymptotically. We also

show that if bidders' values are uniformly distributed and the lower bound of entry costs is zero, then seller revenue approaches the maximum surplus as the number of bidders becomes large (i.e., bidders' informational rents vanish).

Other models of auctions with endogenous entry have been studied in the literature. Samuelson (1985) studies a procurement sealed-bid auction with entry where bidders have the same homogeneous entry cost, but make entry decisions after observing their procurement costs. He shows that when the reserve is equal to the buyer's value, equilibrium is socially optimal. He also shows by means of examples that an increase in the number of bidders may either increase or decrease procurement costs. Analogous results are obtained by Menezes and Monteiro (2000) who study the equilibria of first- and second-price sealed-bid auctions in this framework – see also Tan and Yilankaya (2007). Kaplan and Sela (2003) study auctions where entry costs are private information, but the bidders' values are commonly known. Green and Laffont (1984) study the existence of equilibrium in a model where, as in our setting, both entry costs and values are private information, but they assume, as in Samuelson (1985), that a bidder makes entry decisions having observed both her entry cost and her value. Lu (2007) provides an interesting characterization of optimal entry fees in second price auctions with heterogeneous entry costs.³ While a seller can generally vary the screening value (e.g., by varying the reserve price), in many settings it is not feasible for the seller to set an entry fee. Pevnitskaya (2004) studies endogenous entry in first-price auctions with heterogeneous risk attitudes.

The paper is organized as follows. In Section 2 we layout the basic setting. Section 3 reviews the results for homogenous entry costs. Section 4 presents our results for heterogeneous entry costs. Section 5 concludes. Proofs are in the Appendix.

2 Preliminaries

Consider a market for a single object for which there are N risk-neutral bidders and a risk-neutral seller. In this market the object is allocated using a *standard* auction (i.e., the auction rules are anonymous and allocate the object to the highest bidder).

³Since bidder's pay the entry fee prior to learning their values, in our terminology Lu's entry fee would be an inspection fee.

Each bidder must decide whether to enter the auction, and thereby incur an entry cost. A bidder who enters the auction learns her value, and then bids. The bidders' values V_1, \dots, V_N are independently and identically distributed on $[0, \bar{v}]$ according to an increasing *c.d.f.* F with an increasing hazard rate, and *p.d.f.* f . The seller's value for the object is zero.

Assume that the number of bidders follows a binomial distribution $B(N, p)$, where p is the probability that a single bidder enters, and $p_n^N(p)$ is the probability that exactly $n \in \{0, 1, \dots, N\}$ bidders enter. It will be useful to calculate the seller revenue and the expected utility of a bidder in a standard auction with a *screening value* $v \in [0, \bar{v}]$, where v is independent of the number of actual bidders n . (The *screening value* is the minimum value for which bidding is worthwhile – see Riley and Samuelson (1981).) Assuming that bidding strategies form an increasing symmetric equilibrium of the auction, then by the Revenue Equivalence Theorem (RET) seller revenue is

$$\Pi(v, p) = \sum_{n=1}^N p_n^N(p) \pi(v, n),$$

where

$$\pi(v, n) = n \int_v^{\bar{v}} (yf(y) + F(y) - 1) F^{n-1}(y) dy.$$

The expected utility to a bidder entering the auction is

$$U(v, p) = \sum_{n=0}^{N-1} p_n^{N-1}(p) u(v, n+1),$$

where

$$u(v, n) = \int_v^{\bar{v}} \left(\int_v^y F(x)^{n-1} dx \right) f(y) dy.$$

Here $\pi(v, n)$ and $u(v, n)$ are seller revenue and bidder utility, respectively, in a standard auction with screening value v and n actual bidders – see Riley and Samuelson (1981) for these formulas. We study the symmetric equilibria of the *entry game*. In this game, the payoff to a bidder who enters, when every other bidder enters with the same probability p , is $U(v, p)$ minus her entry cost.

By the RET, in a standard auction the screening value captures everything about the rules of the auction that is payoff-relevant; e.g., the amount of the reserve price, the size of the entry fee, etc. Our assumption that the screening value is independent

of the number of actual bidders, n , is appropriate when either (i) bidders do not observe the number of entering bidders so that their bidding strategies are independent of n , or (ii) the rules of the auction are such that the screening value is the same for every n . The latter holds in first, second, and k^{th} price auctions, for example, where the screening value equals the reserve price regardless of the number of bidders. In this case whether bidders observe the number of entrants is irrelevant (i.e., their payoffs in the entry game are the same). In contrast, in an all-pay auction with a fixed reserve, the screening value depends of the number of bidders. In this case the formulas above describe the payoffs in the entry game only if bidders do not observe the number of entrants.⁴

Note that $\pi(v, n)$ is increasing in n and $u(v, n)$ is decreasing in both v and n . It is easy to see that $U(v, p)$ is decreasing in p : If $p'' > p'$, then $B(N, p'')$ first order stochastically dominates $B(N, p')$, and therefore since $u(v, n)$ is decreasing with respect to n , we have $U(v, p'') < U(v, p')$. Also, since $u(v, n)$ is decreasing in v , then $U(v, p)$ is also decreasing in v .

In an auction with n bidders and a screening value $v \in [0, \bar{v}]$, the gross social surplus, i.e., the social surplus ignoring entry costs, is

$$s(v, n) = \int_v^{\bar{v}} y dF^n(y).$$

Note that $s(v, n)$ is decreasing in v and increasing in n . We adopt the convention $s(0, 0) = 0$. It is easy to show that

$$s(v, n) = \pi(v, n) + nu(v, n),$$

and

$$s(0, n) = E(V_{(n)}),$$

where for $n \in \{1, \dots, N\}$, $V_{(n)}$ is the highest order statistic of $\{V_1, \dots, V_n\}$.

Proposition 1 below establishes that in a standard auction where the screening value is zero the expected utility of each bidder is equal to the contribution to social surplus of the n -th bidder to enter. We provide a simple proof of this result in the

⁴The RET applies even when there is uncertainty about the number of bidders in the auction, provided that bidders have symmetric expectations – see Krishna (2002), Section 3.2.2.

Appendix. A version of this formula is established in Proposition 1 of Engelbrecht-Wiggans (1993).

Proposition 1. For $n \in \{1, \dots, N\}$: $u(0, n) = s(0, n) - s(0, n - 1)$.

As will be seen later, this fact is key to understanding the intuition for our results.

3 Homogenous entry costs

In this section we derive existing results and simple extensions that identify the *optimal* screening value (i.e., the screening value that maximizes seller revenue) for the case of homogenous entry costs. Note that for every value $v \in [0, \bar{v}]$ there is a standard auction with v as the screening value; e.g., a first- or second-price auction with a reserve price $r = v$.

Suppose that all bidders have the same fixed entry cost $c > 0$. We assume that $U(0, 1) < c$; i.e., when the screening value is zero, if bidders enter with probability one, then the expected utility of each bidder is less than c . We further assume that $c < U(0, 0)$. These assumptions rule out uninteresting equilibria in which either every bidder enters or no bidder enters.

In this setting McAfee and McMillan (1987) establish that in a pure strategy entry equilibrium of a first-price sealed-bid auction with a zero reserve price (i) the maximum social surplus is realized (i.e., the socially optimal number of bidders enters the auction and the object is allocated to the bidder with the maximum value), and (ii) the seller captures the entire surplus; hence (iii) the optimal reserve price is zero. Levin and Smith (1994) show that results analogous to (i)-(iii) hold in a symmetric mixed strategy equilibrium of any auction “... for which a bidder wins and pays for the item only if his bid is the highest.” It is straightforward to extend these results to any standard auction. In particular, any standard auction with a screening value of zero maximizes seller revenue. This exercise will help provide intuition for our results for the perhaps more realistic case where entry costs are heterogenous.

The maximum social surplus that can be achieved by *any* mechanism with a fixed number n of bidders is

$$w(n) = E(V_{(n)}) - nc = s(0, n) - nc.$$

A standard auction with a screening value equal to zero attains this maximum. Write $w^* = \max_{n \in \{0, 1, \dots, N\}} w(n)$.

Since $u(0, n) = s(0, n) - s(0, n - 1)$ by Proposition 1, then the social contribution of the n -th bidder is

$$\begin{aligned} w(n) - w(n - 1) &= s(0, n) - s(0, n - 1) - c \\ &= u(0, n) - c. \end{aligned}$$

Since $u(0, n)$ is decreasing in n this contribution is decreasing in n .

Consider the incentives of bidders when they sequentially decide whether to enter a standard auction with a zero screening value. The n -th bidder enters if her payoff to entering is at least her cost, i.e., if

$$u(0, n) - c \geq 0. \tag{1}$$

As shown above, the left hand side of this expression is just the social contribution of the n -th bidder. Hence, when the screening value is zero a bidder enters if and only if her entry raises social surplus. Therefore in a pure strategy entry equilibrium the number of entering bidders n^* maximizes social surplus; i.e., $w(n^*) = w^*$. If we ignore that n^* must be an integer, then n^* satisfies (1) with equality, and bidders capture none of the surplus.⁵

This argument establishes that a standard auction with a screening value equal to zero maximizes social surplus and, moreover, the seller captures the entire social surplus. A positive screening value reduces the social surplus and, because seller revenue is at most the social surplus, also reduces seller revenue. Hence the optimal screening value is zero.

The key insight above was that the private and social benefit of the entry of a bidder coincide in a standard auction with a screening value equal to zero. The same logic applies to symmetric entry equilibria in mixed strategies. If each bidder enters with probability p , then the number actual of bidders in the auction follows

⁵However, our argument ignores that in the pure strategy equilibria studied by McAfee and McMillan (1987) the number of entrants is an integer, and therefore bidder surplus typically will be positive, and may be non-negligible. When this is the case, an inspection fee (see Section 4) equal to the surplus of a bidder allows the seller to capture the entire social surplus. If an inspection fee is not feasible, then the optimal screening value is positive.

a binomial distribution $B(N, p)$, and the maximum (constrained) social surplus that can be achieved by any mechanism is

$$W(p) = \sum_{n=1}^N p_n^N(p) s(0, n) - Npc. \quad (2)$$

A standard auction with a screening value equal to zero attains this maximum. Write $W^* = \max_{p \in [0,1]} W(p)$. Note that W^* is a “constrained” maximum surplus; i.e., it is the maximum surplus when all bidders enter with the same probability.⁶

Since $u(0, n) = s(0, n) - s(0, n - 1)$, then we have⁷

$$\begin{aligned} W'(p) &= N \left(\sum_{n=1}^N p_{n-1}^{N-1}(p) s(0, n) - \sum_{n=1}^{N-1} p_n^{N-1}(p) s(0, n) - c \right) \\ &= N \left(\sum_{n=0}^{N-1} p_n^{N-1}(p) u(0, n+1) - c \right) \\ &= N(U(0, p) - c), \end{aligned}$$

i.e., the marginal social contribution of an increase in the probability of entry is proportional to the payoff of an entering bidder. In a symmetric mixed strategy entry equilibrium bidders are indifferent between entering and not entering;⁸ i.e., bidders enter with a probability p^* satisfying

$$U(0, p^*) - c = 0.$$

Hence $W'(p^*) = 0$. Since U is decreasing in p , then $W''(p) < 0$, i.e., W is a concave function. Therefore $W^* = W^*(p^*)$; i.e., a symmetric mixed strategy entry equilibrium maximizes social surplus. Since the seller captures the entire social surplus, the optimal screening value is zero.

These results are summarized in the Proposition below.

Proposition MM-LS (*Homogeneous entry costs – McAfee and McMillan (1987), Levin and Smith (1994).*) *In a standard auction with a screening value equal to zero, if*

⁶It is easy to show that our assumption $U(0, 1) < c < U(0, 0)$ implies the number of bidders n^* that maximizes social surplus $w(n)$ satisfies $1 < n^* < N$. This in turn implies that if bidders use a symmetric entry rule, then social surplus is below w^* . Hence $w^* > W^*$.

⁷A version of this formula can be found in Milgrom (2004)’s proof of Theorem 6.5.

⁸It is easy to see that a symmetric mixed-strategy equilibrium p^* exists, is unique, and satisfies $p^* > 0$.

bidders follow a (symmetric mixed) pure strategy entry equilibrium, the (constrained) maximum social surplus is realized and is captured by the seller. Hence either a first- or a second-price auction with a reserve price equal to zero and no entry fee maximizes seller revenue.

4 Heterogenous entry costs

In this section we study the general case where bidders have heterogenous entry costs. Specifically, each bidder i has a privately known entry cost Z_i . Bidders' entry costs Z_1, \dots, Z_N are independently and identically distributed on \mathbb{R}_+ according to a *c.d.f.* H with support $[\underline{c}, \bar{c}]$, where $0 \leq \underline{c} < \bar{c} \leq \infty$. As in the homogenous entry cost case (i.e., the case where H is degenerate), we assume that $U(0, 1) < \bar{c}$ and $\underline{c} < U(0, 0)$ in order to rule out uninteresting equilibria. For simplicity, we assume also that H is increasing, satisfies $H(\underline{c}) = 0$, and has a *p.d.f.* h .

Under these assumptions, an entry strategy for a bidder can be described by a number $t \in [\underline{c}, \bar{c}]$ indicating the threshold (the maximum entry cost) for which the bidder enters the auction; that is, the bidder enters if her entry cost is less than t , and does not enter if it is greater than t – whether the bidder enters when her entry cost is exactly t is inconsequential.⁹ If all bidders employ the same threshold t , then the number of bidders in the auction is distributed according to a binomial distribution $B(N, p)$ where $p = H(t)$.

Consider any standard auction with a screening value $v \in [0, \bar{v}]$. A *symmetric entry equilibrium* is a threshold $t \in [\underline{c}, \bar{c}]$ such that for all $z \in [\underline{c}, \bar{c}]$: $U(v, H(t)) > z$ implies $t > z$, and $U(v, H(t)) < z$ implies $t < z$; i.e., in a symmetric entry equilibrium t a bidder enters if her expected utility to entering is above her entry cost, and does not enter if it is below.

We now define a mapping that will describe the symmetric entry equilibrium threshold of a standard auction for every screening value $v \in [0, \bar{v}]$. This mapping, $t^* : [0, \bar{v}] \rightarrow [\underline{c}, \bar{c}]$, is given by $t^*(v) = \underline{c}$ if $U(v, 0) \leq \underline{c}$, and by the unique solution to

⁹In general, entry decisions are described by a mapping from $[\underline{c}, \bar{c}]$ into $[0, 1]$ indicating for each entry cost the probability with which the buyer enters the auction. When H is atomless, however, it is without loss of generality to restrict attention to entry strategies described by a threshold.

the equation

$$U(v, H(t)) = t \tag{3}$$

if $U(v, 0) > \underline{c}$. The mapping $t^*(\cdot)$ is a continuous function on $[0, \bar{v}]$ – see Lemma 1 in the Appendix.

Proposition 2 establishes that a standard auction has a unique symmetric entry equilibrium.¹⁰

Proposition 2. *For each screening value $v \in [0, \bar{v}]$, $t^*(v)$ is the unique symmetric entry equilibrium.*

Assume that each bidder enters when her entry cost is less than $t \in [\underline{c}, \bar{c}]$. Then the social surplus generated in a standard auction with a screening value $v \in [0, \bar{v}]$ is

$$\hat{W}(v, t) = \sum_{n=1}^N p_n^N(H(t))s(v, n) - Nc(t), \tag{4}$$

where

$$c(t) = \int_{\underline{c}}^t z dH(z)$$

is the expected entry cost incurred by each bidder. It is easy to show that social surplus can be calculated as

$$\hat{W}(v, t) = \Pi(v, H(t)) + N[H(t)U(v, H(t)) - c(t)]. \tag{5}$$

In this expression, the term

$$N[H(t)U(v, H(t)) - c(t)],$$

is the bidder surplus.

Given a common entry threshold $t \in [\underline{c}, \bar{c}]$, the maximum social surplus that can be achieved by any mechanism is $\hat{W}(0, t)$. Write $\hat{W}^* = \max_{t \in [\underline{c}, \bar{c}]} \hat{W}(0, t)$ for the “constrained” maximum social surplus; i.e., \hat{W}^* is the maximum surplus if we restrict attention to symmetric entry rules.

Recall that a standard auction with a screening value equal to zero maximizes social surplus when entry costs are homogeneous. Proposition 3 establishes that a

¹⁰Tan and Yilankaya (2006) obtain an analogous result in Samuelson’s model.

standard auction with a screening value equal to zero also maximizes social surplus when entry costs are heterogeneous. In particular, the symmetric entry equilibrium threshold $t^*(0)$ induces socially optimal entry; that is, $\hat{W}(0, t^*(0)) = \hat{W}^*$.

Proposition 3. *A screening value equal to zero maximizes social surplus, i.e., $\hat{W}(0, t^*(0)) = \hat{W}^*$.*

For screening values v for which there is entry (i.e., $t^*(v) > \underline{c}$), the surplus of a bidder is

$$H(t^*(v))U(v, H(t^*(v))) - c(t^*(v)) = \int_{\underline{c}}^{t^*(v)} (t^*(v) - z)dH(z) > 0,$$

where the equality follows from the equilibrium condition $U(v, H(t^*(v))) = t^*(v)$. Hence the seller does not capture the entire social surplus. (It is easy to see that if $t^*(v) > \underline{c}$, then $t^*(v)$ is decreasing in v – see Lemma 1 – and therefore bidder surplus decreases with v .) This result is stated in Proposition 4 below.

Proposition 4. *If there is entry (i.e., $t^*(v) > \underline{c}$), then bidders capture a positive surplus; i.e., seller revenue is less than the social surplus.*

Recall that if entry costs are homogenous, then in an entry equilibrium bidder surplus is zero for any screening value. This difference between the homogeneous and heterogeneous entry cost cases has important implications for determining the optimal screening value when entry costs are heterogeneous, as we see shall shortly.

When bidders have heterogeneous entry costs, seller revenue in a standard auction with a screening value $v \in [0, \bar{v}]$ is $\Pi(v, H(t^*(v)))$. An *optimal screening value* v^* satisfies $v^* \in \arg \max_{v \in [0, \bar{v}]} \Pi(v, H(t^*(v)))$. It is well known that when the number of bidders is exogenously given, then the optimal screening value v^F is positive, and is the solution to the equation

$$v = \frac{1 - F(v)}{F'(v)}, \tag{6}$$

independently of the number of bidders present in the auction – see Riley and Samuelson (1981) and Myerson (1981).

Proposition 5 establishes bounds on optimal screening values when entry costs are heterogeneous. Recall that the optimal screening value is zero when entry costs are homogeneous.

Proposition 5. *If v^* is an optimal screening value, then $0 < v^* < v^F$.*

The intuition for this result is as follows: When the screening value is zero, an increase in the screening value has a negative impact on both social surplus and bidder surplus. Since social surplus is maximized when the screening value is zero, the impact on social surplus is negligible. However, its impact on bidder surplus is non-negligible. Hence seller revenue, which is social surplus less bidder surplus, increases. Lemma 2 in the Appendix formalizes this argument, showing that the total derivative of seller revenue with respect to the screening value is positive at zero.

A similar intuition explains why the optimal screening value is below v^F . When the screening value is v^F , a decrease in the screening value has a negative impact on revenue holding the entry threshold $t^*(v^F)$ fixed, and a positive impact on revenue through increased entry. Since for a fixed entry threshold seller revenue is maximized at v^F , the first effect is negligible. However, the effect on revenue of increasing the entry threshold is non-negligible. (Lemma 4 shows that the total derivative of revenue with respect to the screening value is negative at v^F .)

INSPECTION FEES

Assume that the seller may set an anonymous inspection fee (or subsidy) $e \in \mathbb{R}$ which a bidder must pay, in addition to her entry cost, in order to learn her value. While a bidder's entry cost represents her own idiosyncratic cost of discovering her value, the inspection fee is an extra cost that the seller imposes on a bidder who chooses to enter the auction. A bidder might, for example, need to view the item for auction in order to discover her value, in which case the seller may charge the bidder for making the item available.

Proposition 6 establishes that an inspection fee enables the seller to obtain more revenue than he obtains by choosing a screening value alone. In fact, when the seller may set an inspection fee, then the optimal screening value is zero. Thus, when bidders have heterogeneous entry costs, an inspection fee is a more effective instrument to increase seller revenue than screening bidders by value (e.g., by setting a reserve

price or charging an entry fee). In contrast, when the number of bidders is endogenous but entry costs are homogeneous, the optimal screening value is zero, and it is easy to show that the optimal inspection fee is also zero.¹¹

Proposition 6. *If an inspection fee is feasible, then the optimal screening value is zero, the optimal inspection fee is positive, and seller revenue is greater than with no inspection fee.*

A sketch of the proof of this result is as follows: If the screening value is positive, then the seller can reduce the screening value to zero and at the same time raise the inspection fee so that the expected utility to a bidder to entering the auction is unchanged. This inspection fee (combined with a zero screening value) induces the same entry by bidders without incurring the ex-post inefficiencies of a positive screening value. Seller revenue increases since social surplus increases while bidder surplus is unchanged.

It's easy to see that a result analogous to Proposition 3 holds for a standard auction with an inspection fee; namely, that social surplus is maximized when the screening value is zero and there is no inspection fee. Thus, the optimal inspection fee can not be negative since raising the fee to zero increases social surplus and decreases bidder surplus (since the entry threshold decreases), thereby increasing seller revenue. Although the outcome with a positive inspection fee and a screening value equal to zero is ex-post efficient, a positive inspection fee induces less entry than would be socially optimal.

MARKET THICKNESS

In this section we study the impact on seller revenue and social surplus of an increase in the number of bidders. When entry costs are homogeneous, Levin and Smith (1994) show that in a symmetric mixed strategy entry equilibrium, seller revenue and social surplus (which in this case coincide) decrease as the number of bidders increases. Simple examples show that a direct extension of the result of Levin and Smith (1994) to the case of heterogeneous entry costs does not hold: whether seller revenue and social surplus increase or decrease with the number of bidders depends

¹¹Of course, often it is not possible for the seller to charge either inspection or entry fees – none of the Internet auction websites allow sellers to charge either fee.

on the distribution of entry costs and values. For example, if values are uniformly distributed on $[0, 1]$, then as the number of bidders increases from $N = 1$ to $N = 2$ both the social surplus and seller revenue increase when the distribution of entry costs is uniform on $[0, 1]$, but decrease when it is uniform on $[\underline{c}, .5]$.

We show that as the number of bidders N grows large, the asymptotic properties of equilibrium in Levin and Smith (1994) are closely related to the asymptotic properties of equilibrium with heterogeneous entry costs. In particular, when bidders have a homogeneous entry cost, $c > 0$, asymptotic seller revenue and asymptotic social surplus are the same as when bidders have heterogeneous entry costs and the lower bound of the support of entry costs is $\underline{c} = c$. Consequently, when entry costs are heterogeneous, asymptotic seller revenue is invariant to changes in the distribution of entry costs that preserve the lower bound of its support. Further, asymptotic seller revenue equals asymptotic social surplus, and hence asymptotic bidder surplus is zero. In addition, a screening value equal to zero is *asymptotically optimal*; that is, seller revenue with a screening value of zero is asymptotically the same as with an optimal screening value. These results are established in Proposition 7 below.

For each positive integer N , denote by W_N^* and Π_N^* the maximum constrained social surplus and seller revenue in Levin and Smith (1994)'s setting. Recall that $W_N^* = \Pi_N^*$. Likewise, for each N denote by \hat{W}_N^* and $\hat{\Pi}_N^*$ the maximum constrained social surplus and the seller revenue with an optimal screening value in our setting. By Proposition 4, $\hat{W}_N^* > \hat{\Pi}_N^*$.

Proposition 7. *If $c = \underline{c} > 0$, then*

$$\lim_{N \rightarrow \infty} W_N^* = \lim_{N \rightarrow \infty} \Pi_N^* = \lim_{N \rightarrow \infty} \hat{W}_N^* = \lim_{N \rightarrow \infty} \hat{\Pi}_N^* > 0.$$

Further, a screening value equal to zero is asymptotically optimal.

An interesting case not covered by Proposition 7 is when the lower bound of the support of entry costs is zero, i.e., $\underline{c} = 0$. Proposition 8 below establishes that if values are uniformly distributed, then both asymptotic seller revenue and asymptotic social surplus equal \bar{v} (the upper bound of the support of values). An immediate implication of this result is that the total entry costs incurred by bidders, as well as total bidder surplus, are asymptotically zero. As when $\underline{c} > 0$, a screening value equal to zero is asymptotically optimal.

Proposition 8. *If the lower bound of the support of entry costs is zero, i.e., $\underline{c} = 0$, and values are distributed uniformly on $[0, \bar{v}]$, then asymptotic seller revenue (and asymptotic social surplus) is \bar{v} , i.e., $\lim_{N \rightarrow \infty} \hat{\Pi}_N^* = \lim_{N \rightarrow \infty} \hat{W}_N^* = \bar{v}$. Further, a screening value equal to zero is asymptotically optimal.*

As illustrated in Figure 1 below, when entry costs are heterogeneous seller revenue may increase with the number of bidders. Assume that values are distributed uniformly on $[0, 1]$. The top curve shows seller revenue as a function of the number of bidders when all bidders have an entry cost of $\frac{1}{8}$. Consistent with Proposition 9 of Levin and Smith (1994), seller revenue decreases with the number of bidders. The bottom curve is the graph of seller revenue as a function of the number of bidders when entry costs are distributed uniformly on $[\frac{1}{8}, \frac{1}{2}]$. It shows that seller revenue increases with the number of bidders. Consistent with Proposition 7, the two curves approach each other as the number of bidders becomes large.

Figure 1 goes here.

5 Conclusions

The conclusions obtained when entry costs are homogeneous, namely that (i) the optimal screening value is the seller's value, (ii) social surplus is maximized at the optimal screening value, and (iii) the seller captures the entire social surplus, are not robust to the introduction of heterogeneity in entry costs. In the generic case of heterogeneous entry costs, we rather find that (I) the optimal screening value is above the seller's value; (II) the social surplus is less than the (constrained) maximum surplus – because a screening above the seller's value both induces less entry than would be socially optimal and generates ex-post inefficient outcomes with positive probability; and (III) seller revenue is less than the social surplus – heterogeneity of entry costs generates informational rents, allowing bidders to capture a positive share of the social surplus. While auctions are of greatest interest for small numbers of bidders, as the number of bidders grows large, asymptotic seller revenue depends only on the lower bound of entry costs \underline{c} and is the same as when entry costs are homogeneous and equal to \underline{c} .

6 Appendix

Proof of Proposition 1: For $n > 1$, by interchanging the order of integration we obtain

$$\begin{aligned} u(0, n) &= \int_0^{\bar{v}} \left(\int_0^y F(x)^{n-1} dx \right) f(y) dy \\ &= \int_0^{\bar{v}} \left(\int_x^{\bar{v}} f(y) dy \right) F(x)^{n-1} dx \\ &= \int_0^{\bar{v}} (1 - F(x)) F(x)^{n-1} dx. \end{aligned}$$

Integrating by parts we get

$$\begin{aligned} \int_0^{\bar{v}} F(x)^n dx &= xF^n(x)|_0^{\bar{v}} - \int_0^{\bar{v}} nxF(x)^{n-1}f(x)dx \\ &= \bar{v} - E\left(Y_1^{(n)}\right). \end{aligned}$$

Hence

$$\begin{aligned} u(0, n) &= \int_0^{\bar{v}} F(x)^{n-1} dx - \int_0^{\bar{v}} F(x)^n dx \\ &= \left(\bar{v} - E\left(Y_1^{(n-1)}\right) \right) - \left(\bar{v} - E\left(Y_1^{(n)}\right) \right) \\ &= s(0, n) - s(0, n-1). \end{aligned}$$

For $n = 1$ we have

$$u(0, 1) = \int_0^{\bar{v}} yf(y)dy = E(Y^{(1)}) = s(0, 1) = s(0, 1) - s(0, 0). \quad \square$$

Before proving Proposition 2 we establish some properties of the mapping t^* .

Lemma 1: *The mapping t^* is a continuous function on $[0, \bar{v}]$. Further, t^* is decreasing and satisfies $\underline{c} < t^*(v) < \bar{c}$ on $[0, \hat{v})$, where $\hat{v} \in (0, \bar{v}]$ is the unique solution to the equation $U(v, 0) = \underline{c}$.*

Proof: Let $v \in [0, \bar{v}]$; we have

$$U(v, 0) = \sum_{n=0}^{N-1} p_n^{N-1}(0)u(v, n+1) = u(v, 1).$$

(Note that $p_0^{N-1}(0) = 1$, and $p_n^{N-1}(0) = 0$ for $n \in \{1, \dots, N-1\}$.) Since $U(0, 0) > \underline{c}$ by assumption, $U(\bar{v}, 0) = u(\bar{v}, 1) = 0 \leq \underline{c}$, and $U(\cdot, 0) \equiv u(\cdot, 1)$ is continuous and decreasing on $[0, \bar{v}]$, the equation $U(v, 0) = \underline{c}$ has a unique solution, $\hat{v} \in (0, \bar{v}]$.

For $v \in [0, \hat{v})$ we have $U(v, 0) > \underline{c}$, and

$$U(v, 1) = \sum_{n=0}^{N-1} p_n^{N-1}(1)u(v, n+1) = u(v, N) \leq u(0, N) < \bar{c}.$$

(Note that $p_n^{N-1}(1) = 0$ for $n \in \{0, 1, \dots, N-2\}$ and $p_{N-1}^{N-1}(1) = 1$.) Hence, since $U(v, H(\cdot))$ is continuous the equation

$$t = U(v, H(t))$$

has a solution on $[\underline{c}, \bar{c}]$; and since $U(v, H(\cdot))$ is decreasing on $[\underline{c}, \bar{c}]$ (because $U(v, p)$ is decreasing in p and H is increasing), there is a unique solution. Therefore the function $t^*(\cdot)$ is well defined, and since $U(v, H(\cdot))$ is continuous (because each $u(\cdot, n)$ for $n \in \{1, \dots, n\}$ is continuous), then $t^*(\cdot)$ is also continuous.

We show that $t^*(\cdot)$ is decreasing on $[0, \hat{v})$. Differentiating implicitly the equation $t = U(v, H(t))$, and noticing that $U(v, p)$ is decreasing in both v and p yields

$$\frac{dt^*(v)}{dv} = \frac{\partial U}{\partial v} \left(1 - \frac{\partial U}{\partial p} h(t) \right)^{-1} < 0. \square$$

Proof of Proposition 2: Consider a standard auction with a screening value $v \in [0, \bar{v}]$. We show that $t^*(v)$ is the unique symmetric entry equilibrium. If $v \in [\hat{v}, \bar{v}]$, clearly $t^*(v) = \underline{c}$ is the unique symmetric entry equilibrium. If $v \in [0, \hat{v})$, then $t^*(v)$ is a symmetric entry equilibrium. We show that no other symmetric entry equilibrium exists. By Lemma 1 $\underline{c} < t^*(v) < \bar{c}$. Let $\bar{t} \in [\underline{c}, t^*(v))$. We show that \bar{t} is not a symmetric entry equilibrium. Since $U(v, H(\cdot))$ is decreasing we have

$$U(v, H(\bar{t})) > U(v, H(t^*(v))) = t^*(v).$$

Therefore for $\bar{t} < z < t^*(v)$ we have $z < U(v, H(\bar{t}))$. Hence \bar{t} is not a symmetric entry equilibrium. An analogous argument establishes that no $\bar{t} \in (t^*(v), \bar{c}]$ is a symmetric entry equilibrium either. \square

Proof of Proposition 3: Differentiating $\hat{W}(0, t)$ yields

$$\hat{W}'(0, t) = \sum_{n=1}^N \frac{dp_n^N(H(t))}{dt} s(0, n) - Nth(t).$$

For $n \leq N - 1$ we have

$$\frac{dp_n^N(H(t))}{dt} = N(p_{n-1}^{N-1} - p_n^{N-1})h(t),$$

and

$$\frac{dp_N^N(H(t))}{dt} = Np_{N-1}^{N-1}h(t).$$

(All binomial probabilities are calculated for $p = H(t)$.) Substituting these expressions and using Proposition 1, we have

$$\begin{aligned} \hat{W}'(0, t) &= Nh(t) \left(p_{N-1}^{N-1} s(0, N) + \sum_{n=1}^{N-1} (p_{n-1}^{N-1} - p_n^{N-1}) s(0, n) - t \right) \\ &= Nh(t) \left(\sum_{n=0}^{N-1} p_n^N u(0, n+1) - t \right) \\ &= Nh(t) (U(0, H(t)) - t). \end{aligned}$$

Since $U(0, H(t^*(0))) = t^*(0)$ by Lemma 1, we have

$$\hat{W}'(0, t^*(0)) = 0.$$

Moreover, since $h(t) > 0$ and $U(0, H(\cdot))$ is decreasing on $[\underline{c}, \bar{c}]$, then $\hat{W}'(0, t) > 0$ for $t < t^*(0)$, and $\hat{W}'(0, t) < 0$ for $t > t^*(0)$. Hence $t = t^*(0)$ uniquely maximizes $\hat{W}(0, t)$ on $[\underline{c}, \bar{c}]$. Clearly $\hat{W}(0, t) > \hat{W}(v, t)$ for $v > 0$. Hence $\hat{W}(0, t^*(0)) \geq \hat{W}(0, t) \geq \hat{W}(v, t)$ for all (v, t) , where the first inequality is strict if $t \neq t^*(0)$ and the second inequality is strict if $v > 0$. \square

The following lemmas are useful in the proof of Proposition 5.

Lemma 2. $\left. \frac{d\Pi(v, H(t^*(v)))}{dv} \right|_{v=0} > 0$.

Proof: Using equation (5), for $v \in [0, \hat{v}]$ we can calculate seller revenue as

$$\begin{aligned} \Pi(v, H(t^*(v))) &= \hat{W}(v, t^*(v)) - N(H(t^*(v))U(v, H(t^*(v))) - c(t^*(v))) \\ &= \hat{W}(v, t^*(v)) - N \left(\int_{\underline{c}}^{t^*(v)} (t^*(v) - z) dH(z) \right). \end{aligned}$$

Differentiating with respect to v ,

$$\left. \frac{d\Pi(v, H(t^*(v)))}{dv} \right|_{v=0} = \left. \frac{d\hat{W}(v, t^*(v))}{dv} \right|_{v=0} - N \frac{dt^*(0)}{dv} H(t^*(0)) > 0.$$

Since $\hat{W}(v, t^*(v))$ is maximized at $v = 0$ by Proposition 3, we have

$$\left. \frac{d\hat{W}(v, t^*(v))}{dv} \right|_{v=0} = 0.$$

Therefore since t^* is decreasing on $[0, \hat{v})$ we have

$$\left. \frac{d\Pi(v, H(t^*(v)))}{dv} \right|_{v=0} = -N \frac{dt^*(0)}{dv} H(t^*(0)) > 0. \quad \square$$

Recall that v^F , the solution to the equation $v = (1 - F(v))/f(v)$, maximizes $\pi(\cdot, n) \in [0, \bar{v}]$ for all $n \in \{1, \dots, N\}$ – see Proposition 3 in Riley and Samuelson (1981).

Lemma 3. *If $v^F < \hat{v}$, then $\Pi(v, H(t^*(v))) > \Pi(v^F, H(t^*(v^F)))$ for $v \in (v^F, \bar{v}]$.*

Proof: Assume that $v^F < \hat{v}$, and let $v \in (v^F, \bar{v}]$. Since $t^*(v^F) > t^*(v)$ by Lemma 1, the *c.d.f.* of the binomial $B(N, p(H(t^*(v))))$ first order stochastically dominates the *c.d.f.* of the binomial $B(N, p(H(t^*(v^F))))$. Thus, because π is strictly increasing with respect to n , and $\pi(v^F, n) > \pi(v, n)$ for all $n \in \{1, \dots, N\}$ we have

$$\begin{aligned} \Pi(v^F, H(t^*(v^F))) &= \sum_{n=1}^N p_n^N(H(t^*(v^F))) \pi(v^F, n) \\ &> \sum_{n=1}^N p_n^N(H(t^*(v))) \pi(v^F, n) \\ &> \sum_{n=1}^N p_n^N(H(t^*(v))) \pi(v, n) \\ &= \Pi(v, H(t^*(v))). \quad \square \end{aligned}$$

Lemma 4. *If $v^F < \hat{v}$, then $\left. \frac{d\Pi(v, H(t^*(v)))}{dv} \right|_{v=v^F} < 0$.*

Proof: Assume that $v^F < \hat{v}$. Since H is differentiable, then both $t^*(\cdot)$ and $\Pi(\cdot, H(t^*(\cdot)))$ are differentiable on $(0, \hat{v})$. We have

$$\left. \frac{d\Pi(v, H(t^*(v)))}{dv} \right|_{v=v^F} = \sum_{n=1}^N \left(\left. \frac{dp_n^N(H(t^*(v)))}{dv} \right|_{v=v^F} \pi(v^F, n) + p_n^N(H(t^*(v^F))) \left. \frac{d\pi(v, n)}{dv} \right|_{v=v^F} \right).$$

Since v^F maximizes $\pi(\cdot, n) \in [0, \bar{v}]$ for all $n \in \{1, \dots, N\}$, we have

$$\left. \frac{d\pi(v, n)}{dv} \right|_{v=v^F} = 0$$

for all $n \in \{1, \dots, N\}$. Denote by $p^* = H(t^*(v^F))$ the binomial probability at $t^*(v^F)$.

Hence

$$\begin{aligned} \left. \frac{d\Pi(v, H(t^*(v)))}{dv} \right|_{v=v^F} &= \sum_{n=1}^N \left. \frac{dp_n^N(H(t^*(v)))}{dv} \right|_{v=v^F} \pi(v^F, n) \\ &= \sum_{n=1}^N \left. \frac{dp_n^N(p)}{dp} \right|_{p=p^*} \left. \frac{dH(t)}{dt} \right|_{t=t^*(v^F)} \left. \frac{dt^*(v)}{dv} \right|_{v=v^F} \pi(v^F, n) \\ &= h(t^*(v^F)) \frac{dt^*(v^F)}{dv} \left(\sum_{n=1}^N \left. \frac{dp_n^N(p)}{dp} \right|_{p=p^*} \pi(v^F, n) \right). \end{aligned}$$

In this expression $h(t^*(v^F)) > 0$ and $\frac{dt^*(v^F)}{dv} < 0$ (by Proposition 1). The term

$$\sum_{n=1}^N \left. \frac{dp_n^N(p)}{dp} \right|_{p=p^*} \pi(v^F, n),$$

is positive: an increase in the binomial probability induces a new binomial distribution whose *c.d.f.* first order stochastically dominates the *c.d.f.* of $B(N, p^*)$ which, because π is increasing with respect to n , increases seller revenue. Therefore

$$\left. \frac{d\Pi(v, H(t^*(v)))}{dv} \right|_{v=v^F} < 0. \quad \square$$

Proof of Proposition 5: Lemma 2 implies $v^* > 0$. We show that $v^* \leq v$. If $v^F \geq \hat{v}$, then $v \geq v^F$ implies $t^*(v) = \underline{c}$; hence $H(t^*(v)) = 0$, and therefore

$$\Pi(v, H(t^*(v))) = 0.$$

Since $\Pi(v, H(t^*(v))) > 0$ for $0 < v < \hat{v}$, we have $v^* < v^F$. If $v^F < \hat{v}$, then $\Pi(v, H(t^*(v))) < \Pi(v, H(t^*(v)))$ for all $v > v^F$ by Lemma 3, and therefore $v^* \leq v$. Hence Lemma 4 implies $v^* < v^F$. \square

Proof of Proposition 6: Consider an auction with a screening value and an inspection fee $(v, e) \in [0, \bar{v}] \times \mathbb{R}$. The equilibrium entry threshold t is the solution to the equation

$$t = U(v, H(t)) - e.$$

If $\hat{v} > 0$, then the equilibrium entry threshold \hat{t} is the same for $(\hat{v}, 0)$ and $(0, \hat{e})$, where $\hat{e} = U(0, H(\hat{t})) - \hat{t}$ and $\hat{t} = U(\hat{v}, H(\hat{t}))$; hence bidder surplus is the same, while social surplus is greater for $(0, \hat{e})$. Therefore seller revenue is greater for $(0, \hat{e})$ than for $(\hat{v}, 0)$. Hence the optimal screening value and entry fee is $(0, e^*)$.

We show that $e^* > 0$. Assume that $e < 0$. Clearly an analog of Proposition 3 holds for an auction with an inspection fee; namely, that social surplus is uniquely maximized by setting $(e, v) = (0, 0)$. Hence reducing the inspection fee to zero increases social surplus, and does not increase bidder surplus (because the entry threshold is weakly decreasing in e). Hence seller revenue increases; i.e., $e < 0$ is suboptimal. Hence $e^* \geq 0$. Further, by Proposition 5, when no inspection fee is possible the optimal screening value v^* is positive; i.e., seller revenue is larger for $(v^*, 0)$ than for $(0, 0)$. And by the above argument seller revenue is larger setting an inspection fee $\tilde{e} = U(0, H(t^*(v^*))) - t^*(v^*)$ and a screening value of zero; i.e., seller revenue is greater for $(0, \tilde{e})$ than for $(v^*, 0)$. Hence $e^* \neq 0$, and therefore $e^* > 0$. \square

Proof of Proposition 7. Assume $c = \underline{c} > 0$. Using (2) we can calculate the social surplus for each p and N , $W_N(p)$. As established in Section 2, a standard auction with a zero screening value generates the maximum “constrained” social surplus that can be achieved by any mechanism – see also Levin and Smith (1994), Proposition 6; i.e.,

$$W_N^* \equiv W_N(p_N^*),$$

where p_N^* is the equilibrium probability of entry. Further, the sequence $\{W_N^*(p_N^*)\} \subset [0, \bar{v}]$ is decreasing by Proposition 9 in Levin and Smith (1994), and hence has a limit, which we denote by W .

For each N , denote by $\hat{W}_N(v, t)$ the social surplus generated in a standard auction with a reserve $v \in [0, \bar{v}]$ when bidders have heterogeneous entry costs and use the entry threshold $t \in [\underline{c}, \bar{c}]$ – this surplus can be calculated using (4). Also for each N and $v \in [0, \bar{v}]$ denote by $t_N^*(v)$ the equilibrium entry threshold – the mapping $t_N^*(\cdot)$ is well defined by Lemma 1. By Proposition 3, a standard auction with a zero screening value generates the maximum “constrained” social surplus that can be achieved by any mechanism; i.e.,

$$\hat{W}_N^* \equiv \hat{W}_N(0, t_N^*(0)).$$

We first show that

$$W_N^* \geq \hat{W}_N(v, t_N^*(v))$$

for each N and $v \in [0, \bar{v}]$; i.e., equilibrium social surplus is greater when entry costs are homogeneous than when they are heterogeneous. When entry costs are heterogeneous and the screening value is v , then the expected cost of each entrant is

$$E[z | \underline{c} \leq z \leq t_N^*(v)] > \underline{c},$$

whereas it is only $c = \underline{c}$ with homogeneous costs. Hence writing $\hat{p}_N = H(t_N^*(v))$ we have

$$\begin{aligned} W_N^* &\geq W_N(\hat{p}_N) \\ &= \sum_{n=1}^N p_n^N(\hat{p}_N) s(0, n) - N\hat{p}_N c \\ &\geq \sum_{n=1}^N p_n^N(\hat{p}_N) s(0, n) - N\hat{p}_N E(z | z \leq t_N^*(v)) \\ &\geq \sum_{n=1}^N p_n^N(\hat{p}_N) s(v, n) - N\hat{p}_N E(z | z \leq t_N^*(v)) \\ &= \hat{W}_N(v, t_N^*(v)). \end{aligned}$$

The above inequalities imply

$$W_N^* \geq \hat{W}_N(0, t_N^*(0)) \geq \hat{W}_N(v_N^*, t_N^*(v_N^*)) \geq \Pi_N(v_N^*, t_N^*(v_N^*)) \geq \Pi_N(0, t_N^*(0)).$$

We show $\lim_{N \rightarrow \infty} \hat{W}_N(0, t_N^*(0)) = W$. For each N let $\hat{t}_N \in [\underline{c}, \bar{c}]$ be such that $H(\hat{t}_N) = p_N^*$. Then

$$\hat{W}_N(0, \hat{t}_N) = \sum_{n=1}^N p_n^N(p_N^*) s(0, n) - Np_N^* E(z | z \leq \hat{t}_N).$$

Since $W_N^* \geq 0$, then $0 \leq Np_N^* \leq \frac{\bar{v}}{\underline{c}}$ for each N , and hence $\lim_{N \rightarrow \infty} p_N^* = \lim_{N \rightarrow \infty} H(\hat{t}_N) = 0$. Therefore $\lim_{N \rightarrow \infty} \hat{t}_N = \underline{c} = \lim_{N \rightarrow \infty} E(z | z \leq \hat{t}_N)$. Since

$$0 \leq W_N^* - \hat{W}_N(0, \hat{t}_N) = Np_N^*(\underline{c} - E(z | z \leq \hat{t}_N)),$$

and $\{Np_N^*\}$ is a bounded sequence, then $\lim_{N \rightarrow \infty} (W_N^* - \hat{W}_N(0, \hat{t}_N)) = 0$, and therefore

$$W = \lim_{N \rightarrow \infty} W_N^* - \lim_{N \rightarrow \infty} (W_N^* - \hat{W}_N(0, \hat{t}_N)) = \lim_{N \rightarrow \infty} \hat{W}_N(0, \hat{t}_N).$$

By Proposition 3 and the inequality above we have

$$\hat{W}_N(0, \hat{t}_N) \leq \hat{W}_N(0, t_N^*(0)) \leq W_N^*$$

for all N . Hence

$$\lim_{N \rightarrow \infty} \hat{W}_N(0, \hat{t}_N) = \lim_{N \rightarrow \infty} W_N^* = W,$$

implies

$$\lim_{N \rightarrow \infty} \hat{W}_N(0, t_N^*(0)) = W.$$

Next we show that $\lim_{N \rightarrow \infty} \Pi_N(0, t_N^*(0)) = W$. For each N , write

$$U_N(0, p) = \sum_{n=0}^{N-1} p_n^{N-1}(p) u(0, n+1).$$

Since $u(0, n)$ is decreasing in n , then $U_N(0, p)$ is decreasing in p . Hence the equilibrium entry threshold when entry costs are heterogeneous, $t_N^*(0) \in [\underline{c}, \bar{c}]$, and the equilibrium entry probability when the bidders have homogeneous entry costs, p_N^* , satisfy

$$U_N(0, H(t_N^*(0))) = t_N^*(0) \geq \underline{c} = U_N(0, p_N^*).$$

Hence $0 \leq H(t_N^*(0)) \leq p_N^*$ for all N . Therefore $\lim_{N \rightarrow \infty} p_N^* = 0$ implies $\lim_{N \rightarrow \infty} H(t_N^*(0)) = 0$, $\lim_{N \rightarrow \infty} t_N^*(0) = \underline{c}$, and $\lim_{N \rightarrow \infty} E(z \mid z \leq t_N^*(0)) = \underline{c}$. Hence if the seller sets $v = 0$, the asymptotic total bidder surplus is

$$\lim_{N \rightarrow \infty} NH(t_N^*(0)) [t_N^*(0) - E(z \mid z \leq t_N^*(0))] = 0,$$

and thus the asymptotic seller revenue is

$$\lim_{N \rightarrow \infty} \Pi_N(0, t_N^*(0)) = \lim_{N \rightarrow \infty} \hat{W}_N(0, t_N^*(0)) = W.$$

Since

$$\hat{W}_N(0, t_N^*(0)) \geq \hat{W}_N(v_N^*, t_N^*(v_N^*)) \geq \Pi_N(v_N^*, t_N^*(v_N^*)) \geq \Pi_N(0, t_N^*(0))$$

for all N , and $\lim_{N \rightarrow \infty} \hat{W}_N(0, t_N^*(0)) = \lim_{N \rightarrow \infty} \Pi_N(0, t_N^*(0)) = W$, we have

$$\lim_{N \rightarrow \infty} \hat{W}_N(v_N^*, t_N^*(v_N^*)) = \lim_{N \rightarrow \infty} \Pi_N(v_N^*, t_N^*(v_N^*)) = W. \quad \square$$

Proof of Proposition 8. Without loss of generality, assume that $\bar{v} = 1$. We first establish that $\lim_{N \rightarrow \infty} \hat{W}_N^* = 1$ by showing that for every $\varepsilon > 0$ there is \bar{N} sufficiently large that $\hat{W}_N^* > 1 - \varepsilon$ for all $N \geq \bar{N}$.

Let λ be such that $1 - \frac{1}{\lambda}(1 - e^{-\lambda}) > 1 - \varepsilon$, i.e., $\frac{1}{\lambda}(1 - e^{-\lambda}) < \varepsilon$. Such a λ exists since $\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda}(1 - e^{-\lambda}) = 0$. For each $N > \lambda$, let $t_N \in [0, \bar{c}]$ be such that $H(t_N) = \frac{\lambda}{N}$. Note t_N exists since H is increasing. Also note that for $\varepsilon > 0$, we have $H(\varepsilon) > 0$; hence there is \tilde{N} sufficiently large that for all $N > \tilde{N}$, $NH(\varepsilon) > \lambda$; hence $t_N < \varepsilon$ for $N > \tilde{N}$, and therefore $\lim_{N \rightarrow \infty} t_N = 0$.

We have

$$\hat{W}_N(0, t_N) = \sum_{n=0}^N p_n^N(H(t_N)) \frac{n}{n+1} - NH(t_N) \int_0^{t_N} z dH(z).$$

Since $NH(t_N) = \lambda$ for all N and $\lim_{N \rightarrow \infty} t_N = 0$, we have

$$\lim_{N \rightarrow \infty} NH(t_N) \int_0^{t_N} z dH(z) = \lambda \lim_{N \rightarrow \infty} \int_0^{t_N} z dH(z) = 0.$$

Since the limit of a binomial distribution as N goes to infinity, holding $NH(t_N) = \lambda$ fixed, is the Poisson distribution, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{n=0}^N p_n^N(H(t_N)) \frac{n}{n+1} &= \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \frac{n}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \left(1 - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^{n+1}}{n!} \frac{1}{n+1}. \end{aligned}$$

Letting $k = n + 1$, i.e., $n = k - 1$ we have

$$1 - \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^{n+1}}{n!} \frac{1}{n+1} = 1 - \frac{1}{\lambda} \left(-e^{-\lambda} + \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!}\right) = 1 - \frac{1}{\lambda}(-e^{-\lambda} + 1).$$

Let \bar{N} sufficiently large that for all $N > \bar{N}$

$$\left| \hat{W}_N(0, t_N) - \left(1 - \frac{1}{\lambda}(1 - e^{-\lambda})\right) \right| < \delta,$$

where $0 < \delta < \varepsilon - \frac{1}{\lambda}(1 - e^{-\lambda})$. Then for each $N > \bar{N}$ we have

$$\hat{W}_N^* \geq \hat{W}_N(0, t_N) \geq 1 - \frac{1}{\lambda}(1 - e^{-\lambda}) - \delta > 1 - \varepsilon.$$

Now, since $\hat{W}_N^* = \hat{W}_N(0, t_N^*(0))$ for all N by Proposition 3, we have

$$\hat{W}_N^* = \sum_{n=0}^N p_n^N(H(t_N^*(0))) \frac{n}{n+1} - \lim_{N \rightarrow \infty} NH(t_N^*(0)) \int_0^{t_N^*(0)} z dH(z).$$

Since $\left\{ \sum_{n=0}^N p_n^N(H(t_N^*(0))) \frac{n}{n+1} \right\} \subset [0, 1]$ and $NH(t_N^*(0)) \int_0^{t_N^*(0)} z dH(z) > 0$ for all N , then $\lim_{N \rightarrow \infty} \hat{W}_N^* = 1$ implies

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N p_n^N(H(t_N^*(0))) \frac{n}{n+1} = 1,$$

and therefore

$$\lim_{N \rightarrow \infty} NH(t_N^*(0)) \int_0^{t_N^*(0)} z dH(z) = \lim_{N \rightarrow \infty} \sum_{n=0}^N p_n^N(H(t_N^*(0))) \frac{n}{n+1} - \lim_{N \rightarrow \infty} \hat{W}_N^* = 0.$$

It is easy to see that this implies

$$\lim_{N \rightarrow \infty} t_N^*(0) = 0,$$

which in turn implies that the asymptotic total surplus captured by bidders is

$$\lim_{N \rightarrow \infty} N \int_0^{t_N^*(0)} (t_N^*(0) - z) dH(z) = 0.$$

And since in the limit bidders capture no surplus, all the surplus is captured by the seller; i.e.,

$$\lim_{N \rightarrow \infty} \hat{\Pi}_N^* = \lim_{N \rightarrow \infty} \hat{W}_N^* = 1. \quad \square$$

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Seller Revenue and the Number of Potential Entrants

Bidder Values are Distributed $U[0,1]$

