# COMPETITIVE ECONOMY AS A RANKING DEVICE OVER NETWORKS

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**Abstract:** We propose a novel approach to generating a ranking of items in a network (e.g., of web pages connected by links or of articles connected by citations). We transform the network into an exchange economy, and use the resulting competitive equilibrium prices of the network nodes as their ranking. The widely used Google's PageRank comes as a special case when the nodes are represented by Cobb-Douglas utility maximizers. We further use the economic metaphor to combine between the Citation Index and PageRank by imposing a redistributive taxing scheme. Finally, we study the outcome of an interaction between a (CES utility) ranking system and agents who bias their link intensities in response to the published ranking. This outcome coincides with that of a related ranking system (and unbiased agents). A modification of the utility function's parameter allows us to cancel out the effect of the bias.

Keywords: Network, exchange economy, competitive prices, ranking, economy-based ranking, PageRank, taxation, citation index, normalized citation index, biased agents.

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## 1. Introduction

In a world with abundant information, ranking systems are of utmost importance. Well known examples include Google's PageRank, which helps internet users to identify web pages that are more likely to interest them, and the Citation Index, which helps academic researchers to assess the quality of academic articles. In both cases, as well as in many other contexts, the ranking of items is based solely on the information embodied in the network such as links between web sites or citations between scientific articles.

Whereas the citation index merely counts the number of citations from other articles and does not discriminate by the source of the citation, more sophisticated ranking systems attempt to give more weight to votes from items which are of a higher rank (as ascribed by the system itself). The approach taken by PageRank, for example, is based on the idea of translating the link structure to a Markov process as follows: each web page is viewed as a state, and after the random walk hits a state it moves on randomly to one of the states that the current state gives them a link. The ranking of a web page is defined as the long-run proportion of time that the process spends in a given state. Since this value depends not only on the number of incoming links but also on the proportion of time the system spends on the states that send these links, the induced ranking indeed grants more weight to links from higher-ranked pages.

We propose a different approach to ranking that consists of constructing an economy based on the network of links and deriving the ranks from the competitive equilibrium prices. Our approach employs a neoclassical pure-exchange economy. Each web page is represented by one consumer, who is initially endowed with one unit of a specific good. The consumer has no utility from consuming his own good, and thus supplies it in an inelastic fashion (i.e., sells the entire unit whatever its price is). The price of one's good then becomes one's budget, which serves to buy other goods. The consumer's utility derives from consuming the specific goods provided by exactly those consumers that he or she sends a link to. For example, if page $i$ has links only to pages $j$ and $k$, then consumer $i$ has utility $u(x^i_j, x^i_k)$ where $x^i_j, x^i_k$ are the quantities that $i$ consumes from goods $j$ and $k$.

The main idea of this paper is to use the competitive prices[1] of this economy as a ranking system. That is, the ranking of a web page coincides with the price of the corresponding good. In this pure-exchange economy, higher-ranked pages correspond to more expensive goods. Moreover, since the budget of a consumer equals the price of the good he initially owns, the initial owners of highly demanded goods are rich. These owners demand larger quantities of the goods they like, thus pushing their prices higher. Hence, those web pages that are pointed to by highly ranked web pages, are highly ranked themselves.

The specific ranking obtained depends on the modeler's choice of utility functions. We assume throughout that all consumers, although consuming different goods, have the same type of utility function. We first consider the Cobb-Douglas utility function which is perhaps

[1]More precisely, we use quasi-equilibrium prices, as in Debreu (1962).

the most widely used in economic modeling. We show that when all consumers have Cobb-Douglas preferences, the resulting vector of competitive equilibrium prices coincides with the PageRank ranking system.

We next show that the Citation Index cannot be derived from any exchange economy that treats the value of a paper as its reviewing power. This impossibility extends also to the 'Normalized' Citation Index, in which the value of a citation from an article is inversely proportional to the number of citations the article makes. One may view this result as an argument against the relevance of the Citation Index.

A ranking system based solely on the network information[2] actually considers each item (e.g., web page, article) both as a reviewer, whose judgement (link, references) determines the rank of others, and as a refereed, whose assessed quality depends on the links it obtained from others. The following economic metaphor can sharpen this distinction: the value of an agent as refereed is the market price of its good. Its power as a reviewer equals its budget. In the formulation we employed so far, these two powers are, by definition, the same, since the budget comes from selling the agent's specific good.

The Citation Index, that does not make any connection between the value of a paper and its refereeing power seems to lack the desirable property of allocating greater weight to citations originating from more important papers. However, there are many cases in which also PageRank, or in fact any ranking system derived from an exchange economy, fails to produce meaningful results. Consider, for example, a sequence of papers published sequentially, each at a distinct time. As citations can only refer to earlier work, all these papers will have zero value. This happens precisely because the value of an item as a reviewer is equated with its value as a refereed entity. The latest article has no incoming links, implying that its price and budget are 0. In turn, it has 0 demand for the articles it has links to. Thus, the penultimate article has 0 value as well, and so forth. In similar cases the Citation Index – where the reviewing power of all items are the same and independent of their power as refereed items – may perform better than the PageRank.

One can combine the advantages of these two ranking systems by developing the economic metaphor a bit further. We add to the exchange economy a taxation scheme, thus allowing to disentangle the value of an item as a reviewer and its quality as determined by others. Each consumer pays a proportion, say $\alpha$, of his income as a tax. The tax revenue is then equally redistributed between all the consumers. When $\alpha = 0$, a consumer's budget equals the price of his specific good, leading us back to the original model. With a 100% tax (i.e., $\alpha = 1$), the budgets of all consumers are equal. As a result, their reviewing power is equal regardless of the prices of their goods. The competitive equilibrium prices in this case could serve as a ranking system that bears the spirit of the Citation Index.[3] By choosing a tax

[2]This is as opposed to information obtained from other sources, such as the quality of the journal in which an article was published, etc.

[3]What is actually obtained is a normalized citation index, in which any article obtains an allotment of one unit to be equally shared between all the articles it cites. The normalized citation index of an article is then the sum of all the (normalized) citations it obtains. A similar idea applied to clusters of papers, based on their field, was proposed by Moed et al (1995) and applied by the University of Leiden.

rate between the two extremes, one can control the extent to which the reviewing power of an item depends on its quality (as assessed by the ranking system). We show examples in which the ranking system derived from an intermediate tax seems to do better than both PageRank and the Citation Index.

Finally, we consider a real-world scenario in which agents are influenced by the Google ranking system when they decide which links to include in their own page. A designer of a web page, for instance, learns about other sites from Google, and is thus more likely to insert links to sites that already have a high ranking. (In fact, some of the citations that appear at the end of this paper were found using Google's search engine. It is therefore more likely that we have cited those papers that have high ranking and neglected others.) Another reason why high ranking papers are more likely to be cited has to do with incentives. An author of an academic article may prefer to cite highly ranked papers so as to attract other authors, who are naturally drawn to look first at more famous papers and follow their citations, to read and cite his own paper.

In a world with such "positive feedback", what we observe is in fact a fixed point of the interaction between Google's ranking, which is based on observed citations, and the agents' citation decisions, which are affected by Google's ranking. We show that if the bias is linear, i.e., each agent sets the intensities of his links to be proportional to their Google ranking, then competitive equilibria in the economy governed by the *minimum* utility function are fixed points of the above interaction. More generally, if the the ranking system is based on a CES utility function with parameter $r$ (see Section 6 for its definition), then the outcome of the interaction between the ranking system and linearly biased agents is the same as that of ranking system is based on a CES utility function with parameter $r+1$ and unbiased agents. This result can be used to get rid of the effect of agents' bias. For example, if Google's objective is to obtain the PageRank ranking (based on the links of "sincere" agents), then it can achieve this objective in a world with linearly-biased agents by employing an economy-based ranking with a CES utility function with parameter $r = -1$.

1.1. **Related literature.** In the 1960's, Garfield introduced the first citation index for papers published in academic journals: the Science Citation Index (SCI). This index was followed by the Social Sciences Citation Index (SSCI) and later by the Arts and Humanities Citation Index (AHCI). In 1972 Garfield established a ranking of scientific journals, known as Impact Factor. Liebowitz and Palmer (1984) analyzed the impact factors of economic journals by using an iteration (impact adjusted) method, without making the mechanism explicit.

The Markov-chain approach behind PageRank has been originally proposed by Wei (1951) and Kendall (1955). The approach was applied to create PageRank by Brin and Page who describe the algorithm in detail in Brin and Page (1998).

Another major direction that was applied in the literature has been the axiomatic approach. By postulating a number of desired requirements that the ranking system must satisfy, one attempts to identify a specific ranking. This approach was adopted by Palacios-Huerta and Volij (2004) and Altman and Tennenholtz (2008) who axiomatized PageRank.

Demange (2011) also takes an axiomatic approach and derives an index based on treating members in the network both as referees and being refereed at the same time.

Posner (2000) criticizes many ranking methods and states that 'citation analysis is not an inherently economic methodology' due to its lack of theoretical or empirical grounding. Amir (2002) studies properties of various indices. The influence model of Demange (2011) describes a dynamics whereby the ranking of journals affect the intensities of citations.

1.2. **Structure of the paper.** Section 2 introduces the model of an exchange economy induced by a network and governed by a quasi-equilibrium as a ranking scheme. In Section 3 we show that if an exchange economy is governed by Cobb-Douglas utility maximizers, then the resulting ranking coincides with PageRank. Section 4 studies the Citation Index and shows that it cannot be derived from an exchange economy. Section 5 introduces taxation into the economy and deals with the resulting distinction between quality and refereeing power. Section 6 looks at markets governed by CES utility functions and in Section 7 we study the interaction between a ranking system and biased agents. This section demonstrates a way to remove the bias by a appropriately using CED functions. Final remarks appear in Section 8.

## 2. Exchange Economy as a Ranking System

2.1. **The graph of citations.** Our initial data is a directed graph $G$. Each node (vertex) in $V = \{1, ..., n\}$ represents a web page (or an article) and each directed edge $(i, j) \in E \subset V \times V$ represents a link (or citation) from web page $i$ to web page $j$. To the fact that $(i, j) \in E$, we refer as '$i$ has a link to $j$', '$j$ is pointed at by $i$', '$i$ cites $j$', '$j$ is cited by $i$' and alike. Let $O(i) = \{j : \ (i, j) \in E\}$ be the set of outgoing links from $i$, i.e., the nodes that $i$ has a link to. Denote the cardinality of $O(i)$ by $c(i)$. Also denote $I(i) = \{j : \ (j, i) \in E\}$ the set of nodes sending links to $i$ (i.e., incoming links to $i$). For the sake of simplicity we assume that every site sends links, that is, for every $i \in V$, $c(i) > 0$,[4] and that there are no self links, i.e., for any $i$, $(i, i) \notin E$. It is also useful to define the *coincidence matrix* $\Pi = \left(\pi_{ij}\right)_{ij}$ in which $\pi_{ij} = 1$ if $(i, j) \in E$ and $\pi_{ij} = 0$, otherwise. A ranking system is a function from such directed graphs to vectors of valuations, one coordinate for each node.

2.2. **The "governing" utility function.** The second element needed in order to define the exchange economy is a set of utility functions $\{u^k\}_{k=1,...,n}$ where $u^k : \mathbb{R}^k_+ \to \mathbb{R}$ will be the utility function of a consumer who interested in consuming $k$ out of the $n$ goods. We require that for every $k$, $u^k$ be a symmetric function, i.e., $u^k(x) = u^k(\pi(x))$ for any $k$-vector of goods $x$ and any permutation $\pi$ of $k$ elements. Symmetry implies that a consumer's utility depends only on the quantities of the goods consumed and not on their identity; moreover, by construction, all the consumers who are interested in $k$ goods have the same utility function.

[4]This assumption is quite innocuous when applied to papers rather than to web sites, and when $\pi_{ij} = 1$ is interpreted as a reference given in paper $i$ to paper $j$. The reason is that the list of references of any paper (we know) is non-empty.

Thus, any difference between items in their induced ranking will stem only from the structure of links and is not imposed by the utility function.

At this stage we do not require the utility function to satisfy any properties other than symmetry. Specifically, we do not impose any relation between $u^\ell$ and $u^m$. Nevertheless, in all the concrete examples we study (e.g., Cobb-Douglas, minimum, sum, CES) the utility functions employed induce the same consumption patterns across all consumers, regardless of the number of goods they consume. We discuss additional restrictions on the utility function in the concluding comments (Section 8).

2.3. **The induced exchange economy.** We now construct the $u$-exchange-economy induced by the graph $G$. There are $n$ consumers, one for each node. There are also $n$ distinct goods. Consumer $i$ has an initial endowment of one unit of good $i$. The utility of consumer $i$ with set of outgoing links $O(i)$ is $u^{c(i)}\left((x_i^j)_{j\in O(i)}\right)$. Thus, his utility depends on the quantities that he consumes from the $c(i)$ goods that correspond to the vertices he has links to; he is not interested in consuming other goods (including his own).

2.4. **Quasi-equilibrium.** We intend to base our ranking of web pages on the equilibrium prices of the goods in the $u$-exchange-economy. However, there are graphs and utility functions for which a competitive equilibrium fails to exists. Roughly, the reason is that there are consumers whose budget is 0 and some of the goods they consume have 0 price. With positive marginal utility from consuming the 0-priced goods, they will demand large amounts without violating their budget constraint, therefore leading demand to exceed supply. We thus resort to Debreu's weaker notion of quasi-equilibrium (see Debreu, 1962).

As in a competitive equilibrium, also in a quasi equilibrium aggregate demand must not exceed aggregate supply, and consumers must choose, among all the baskets of goods that they can afford, the one that maximizes their utility. However, the second requirement is less strict and hence applies only to consumers with positive budget; consumers whose initial basket is worth zero at the equilibrium prices do not maximize their utility and consume only leftovers. Formally,

**Definition 1.** Quasi-equilibrium *in the $u$-exchange-economy induced by the graph $G$ is a tuple $(x_1, x_2, ..., x_n; p)$, where $x_i = (x_i^j)_{j\in O(i)} \in \mathbb{R}^{O(i)}$ is a basket consumed by consumer $i$ and $p$ is a nonzero vector in $\mathbb{R}_+^n$, such that:*

1. *For every consumer $i$ with $p_i > 0$, if $y_i \in \mathbb{R}^{O(i)}$ and $u^{c(i)}(y_i) > u^{c(i)}(x_i)$, imply $\sum_{j;\ j\in O(i)} p_j y_i^j > p_i$, where $y_i^j$ is the quantity of good $j$ in $y_i$.*
2. *For every good $j$, $\sum_{i;\ j\in O(i)} x_i^j = 1$.*

*The price-vector $p$ is called a* quasi-equilibrium price system *or* vector of quasi-equilibrium prices*. We refer to quasi-equilibrium price system also as* competitive prices*.*

Note that in this definition, the maximization of the utility function subject to the budget constraint (i.e., $\sum_{j;\ j\in O(i)} p_j x_i^j \le p_i$) is restricted only to those consumers $i$ whose budget is positive, that is, $p_i > 0$.

2.5. **Economy-based ranking.** Under mild conditions[5], a quasi equilibrium exists. Our ranking system is based on quasi-equilibrium prices. Thus, if $p = (p_1, ..., p_n) \in \mathbb{R}^n_+$ is a vector of quasi-equilibrium prices, we define the rank, or quality, of node $i$ as the price of the corresponding good $p_i$. Since a quasi-equilibrium exists, the definition is non empty. However, since a quasi equilibrium is in general not unique, our approach might generate multiple ranking. In the sequel we look for utility functions and network structures that generate a unique equilibrium and thereby a uniquely defined ranking.

Different utility functions typically produce different rankings. Thus, the exchange economy could serve as a mechanism that generates different ranking systems. By simply 'plugging in' different utility functions the exchange economy can produces different rankings. It is important to note though that whatever the utility function, the budget of consumer $i$ coincides with the price of good $i$. Moreover, this price depends on the individual purchasing power of its consumers, which coincides with the prices of their goods. The latter, on their part, depend on the purchasing power of their own consumers, and so forth. Hence, the price of a good depends on the entire network's structure. In particular, a good consumed by a rich consumer (i.e., one who initially owns an expensive good) is likely to be expensive itself.

## 3. Cobb-Douglas utility yields Google's ranking

In this section we illustrate the main idea using the Cobb-Douglas utility function and show the connection with Google's PageRank.

3.1. **PageRank in detail.** Google's ranking, known as PageRank, attempts to capture not only the number of links a site receives from others, but also the significance of each link. The rank of a site depends on the values of the links it receives, where the value of a link is determined by the rank of the site that gave the link (divided by the number of links coming out from that site). The approach taken by Brin and Page (1998) to tackle the circular definition, is to transform the link data into a Markov chain; PageRank then takes the ranks from the resulting invariant distribution . We now describe their methodology in detail.

Let the set of states of the Markov chain be $V$. The probability of transition from state $i$ to state $j$, denoted $m_{ij}$, is defined by:

$$m_{ij} = \frac{\pi_{ij}}{c(i)}. \tag{1}$$

In words, the total probability to exit state $i$ is equally divided between all the vertices $j$ to which $i$ refers. Note that $m_{ij}$ is well defined. Denote by $M$ the transition matrix $(m_{ij})_{(i,j)\in V\times V}$.

[5]Two conditions are required in order to guarantee the existence of quasi-equilibrium. First, that utility functions be continuous. Second, that for every $i$ and basket $x$, the set of baskets preferred by $i$ over $x$ be convex. The second condition is referred to as *quasi concavity.* The word 'quasi' is coincidentally used in the two separate contexts of concavity and equilibrium. The reader is referred to the proof in Jehle and Reny (2011), for instance.

PageRank is based on an invariant distribution of $M$. Suppose that $p$ is a distribution over $V$, that is, $p \in \Delta(V)$. We say that $p$ is an *invariant distribution* of $M$, if

$$pM = p. \tag{2}$$

The interpretation of $p$, which makes it a good candidate for ranking, is that the probability $p_i$ assigned to $i$ by the distribution $p$ is the frequency in which state $i$ is visited by the system, when run for a long period of time.

A technical comment is due here. Typically, since $M$ might not be ergodic, there could be a few invariant distributions. In practice, $M$ is slightly perturbed in order to make all its transition probabilities strictly positive. Brin and Page [4] make use of the "damping factor" technique to make the matrix ergodic, which guarantees uniqueness of its invariant distribution. Later on (in section 5.2) we propose an alternative way of guaranteeing uniqueness other than perturbing the matrix.

**Example 1.** *Consider the following graph with* 3 *nodes.*

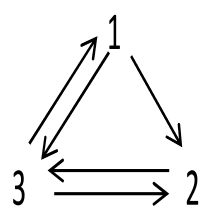


FIGURE 1. *Example 1.*

*The coincidence and transition matrices are:*

$$\Pi = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

*There is only one vector whose coordinates sum up to* 1 *that satisfies Equation (2), which is* $p = (2/9, 3/9, 4/9)$. *Indeed,*

$$(2/9, 3/9, 4/9) \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (2/9, 3/9, 4/9).$$

3.2. **The Cobb-Douglas Economy.** Let $u$ be the symmetric Cobb-Douglas utility function, and consider the $u$-exchange-economy defined above. That is, $i$'s utility is the product of the quantities he consumes from all the goods in $O(i)$:

$$u_i^{c(i)}(x) = \Pi_{j \in O(i)} x_j, \tag{3}$$

where $x \in \mathbb{R}_+^{O(i)}$, and $x_j$ is the quantity of good $j$ in basket $x$. We denote this economy by $E(\Pi, \mathrm{CD})$.

The following proposition states that there is a unique normalized vector of competitive equilibrium prices and this vector coincides with PageRank.

**Proposition 1.** *There is always a quasi equilibrium price system of in $E(\Pi, \mathrm{CD})$, denoted $p$. Moreover, $pM = p$. In case the directed graph of the network is connected,[6] $p$ is necessarily unique. Furthermore, it coincides with PageRank or the Invariant Method.*

*Proof.* Since the Cobb-Douglas utility function is continuous and quasi-concave, a quasi equilibrium price system exists.

The budget of consumer $i$, who is the original owner of one unit of good $i$, is $p_i$. As the utility function is Cobb-Douglas, when his budget is positive, he divides it equally between the goods in $c(i)$, i.e., he purchases quantity $\frac{p_i \pi_{ij}}{c(i) p_j}$ of commodity $j$ (note that for $j \notin O(i)$, $\pi_{ij} = 0$). His expenditure on good $j$ is then $\frac{p_i \pi_{ij}}{c(i) p_j} p_j = \frac{p_i \pi_{ij}}{c(i)} = p_i m_{ij}$.

The total budget of consumer $j$ is the sum of all the expenditures, made by all other agents, on good $j$. That is, for every $j$ with $p_j > 0$,

$$p_j = \sum_i p_i m_{ij}. \tag{4}$$

However, Eq. (4) holds also when $p_j = 0$. Indeed, in case $p_j = 0$, $p_i = 0$ for every $i$ such that $\pi_{ij} = 1$ and therefore, $p_i m_{ij} = 0$ for every $i$.

Note that since $p$ is normalized, it becomes a distribution over $V$. Finally, notice that Eq. (4) precisely means that $p$ satisfies Eq. (2), and therefore coincides with PageRank.

Suppose now that the network's graph is connected. Let $V' \subseteq V$ be the largest subset such that for any $i \in V$ and $j \in V'$, there is a directed path from $i$ to $j$. Since the graph is finite and connected, $V'$ is non-empty. By definition, any two nodes $i, j \in V'$ are connected by two paths, one in each direction (that is, from $i$ to $j$ and from $j$ to $i$).

Suppose that $p = (p_1, ..., p_n)$ is a quasi-equilibrium price system. We show that for any $i \in V'$, $p_i \neq 0$. If, to the contrary, there is $i' \in V'$ such that $p_{i'} = 0$, then since $p \neq 0$ and the graph is connected, there are $j$ and $i$, such that $p_j > 0$ and $p_i = 0$ and moreover, $j \in I(i)$. In this case, consumer $j$ could purchase a positive amount of each of his desirable goods and an unbounded amount of good $i$. That would increase his utility to infinity, and generate an excesses demand for good $i$. Such a case violates the equilibrium condition. We conclude that for any quasi-equilibrium price system all the goods in $V'$ have positive prices.

Next we show that for any $i \notin V'$, $p_i = 0$. If to the contrary, there is $i \notin V'$ with $p_i > 0$. By the definition of $V'$, there is a path from $i$ to any $j \in V'$. Fix a path of this kind. In an economy governed by Cobb-Douglas utility functions all the budgets (i.e., prices) of all consumers on this path must be positive. In particular, there are consumers, say $i' \notin V'$ and $j \in V'$ on this path, such that $i' \in I(j)$ and $p_{i'} > 0$. In a Cobb-Douglas economy consumer $i'$ spends money on good $j$. In other words, there is money going from $V \setminus V'$ to $V'$. But no money goes backward – from $V'$ to $V \setminus V'$, simply because there is no path from $V'$ out. Recall that in quasi-equilibrium each individual has a balanced budget: his spending equals his budget. This therefore applies also to groups of consumers. Here, however, the spending

[6]That is, there is a directed path between any two nodes.

of $V \setminus V'$ is greater than its budget, contradicting the fact that $p$ is a quasi-equilibrium price system.

Assume now that $p = (p_1, ..., p_n)$ and $p' = (p'_1, ..., p'_n)$ are two competitive equilibrium price systems. Without loss of generality we may assume that $p$ is coordinate-wise greater than $p'$ and that for one $j$, $0 < p_j = p'_j$. Due to the connectedness of the graph, we can assume that there is $\ell$ such that $\ell \in I(j)$ and $p_\ell > p'_\ell$.

We now reduce all $p_i$, $i \in V' \setminus \{j\}$ to the level $p'_i$ without touching $p_k$, $k \notin V' \setminus \{j\}$. Two effects can be perceived from the perspective of each consumer $i$ $(i \neq j)$. First, the budget has been reduced from $p_i$, to $p'_i$ and second, all prices are not higher compared to what they were before the reduction.

Consider $i \in I(j)$. In the case of Cobb-Douglas utility functions, when the prices of the goods $i$ is interested in have been reduced and when his budget has been cut down, his demand for good $j$ cannot grow up. Moreover, since the budget of $\ell$ is strictly lower than what it was prior to the reduction, his demand for good $j$ is strictly smaller than what is was prior to the change. Consequently, following the change (i.e., after prices turned into $p'$), total demand for good $j$ becomes smaller than 1 – the amount supplied. In other words, there is an excess supply of good $j$, which contradicts the assumption that $p'$ is a competitive equilibrium price system. We conclude that there is a unique quasi equilibrium price. □

**Remark 1.** *When a Web user visits a Web page, it is likely that he would randomly follow one of the page's hyperlinks to surf the Web. Thus, the Markov chain is believed to a good model for the Web surfing behavior. Furthermore, the stationary distribution of a Markov chain is used to measure the importance of a page. This coincides with the intuition that a link in the Web can be treated as a vote; meanwhile the more votes a page gets, the more important it is. Proposition 1 provides another perspective on PageRank. When surfing the Web, a Web user's goal is to satisfy his information needs. He can do so by following outgoing links of a page to another. This can be modeled by a Cobb-Douglas economy graph. Each Web page is considered as an agent, the information of the page is corresponding to the agent's initial endowments, and a link from page p to page q means that the agent on page p has a demand for the information of page q. Intuitively, the more "demand" a page gets, the more important it is. Therefore, we can use an equilibrium price of the exchange economy to measure the importance of a page.*

*Proposition 1 provides further insights to PageRank. That is the substitution and complementarity effects of a page's outgoing links. For instance, suppose we have a directory page of a university, which has outgoing links pointing to the home pages of the university's departments. If a Web user wants to visit the home page of a particular department (e.g. the economics department), he is likely to click one of the outgoing links, but not any other one. Thus, for this Web user, the outgoing links are substitutes for each other instead of complements. On the other hand, another Web user could be a potential applicant of the university. He may be interested in a few departments, for instance, the economics department, the finance department, as well as the public policy department. Then, he may click more than one of the outgoing links. Thus, for this Web user, the outgoing links are more* *likely to be complements for each other than substitutes. By Proposition 1 and the properties of the Cobb-Douglas utility function, the PageRank algorithm treats the set of pages that a Web page points to as a mix of the substitution and complementarity effects with elasticity of substitution* 1.

**Remark 2.** *Eaves* [8] *showed that the computation of an equilibrium for a Cobb-Douglas market can be reduced to solving a linear equation system. Although it was not explicitly claimed in* [8]*, Eaves's result implies that an equilibrium of a Cobb-Douglas market actually corresponds to a principle eigenvector of a stochastic matrix. In Proposition 1, we show the reverse direction of Eaves' reduction. That is a principle eigenvector of a stochastic matrix corresponds to an equilibrium of a special Cobb-Douglas economy. Therefore, our result is essentially different from Eaves' finding in terms of motivations.*

## 4. The Citation Index

The ranking that is most widely used by the scientific community is the Science Citation Index (SCI). Under this index, the ranking of a scientific paper is simply proportional to the number of citations it received from other papers. This index does not take into consideration the ranking of the citing paper nor its venue. One can also define a modified version of this index, in which a citation from an article that makes $n$ citations is counted as $1/n$ rather than 1. Thus, each article has the same reviewing power independently of the number of citation it makes. We call this the Normalized Citation index, henceforth NCI.[7]

**Proposition 2.** *There is no u-exchange-economy such that for every citations graph the competitive equilibrium generated coincides with SCI or with NCI.*

*Proof.* In any exchange economy, $p_j \leq \Sigma_{i \in I(j)} p_i$. Thus, if all links to $j$ come from nodes which themselves receive no links (and therefore have price 0), then $p_j$ itself must be also 0. However, the citation index of a node that receives links from others is never 0. □

In fact, the reason why the Science Citation Index cannot be captured by an exchange economy is that it makes a complete separation between an item's rank and its refereeing power. That is, the value of a link from item $i$ is independent of $i$'s rank. In our exchange economy demand from a 'richer' consumer, ceteris paribus, has an higher impact on a good's price. One may view this result as an argument against the use of the citation index (whether normalized or not).

**Remark 3.** *The SCI and NCI index can be interpreted as an equilibrium of a Fisher's economy* [16] *with Cobb-Douglas utilities. In that setting, an item is splitted into two parts: one is a consumer that has an initial endowment of cash (for SCI index, the initial endowment is $k$ dollars if the item has $k$ references, while for NCI index, the initial endowment is* 1 *dollar), while the other part is one unit of a unique good. As we know, Fisher's economy can be treated as a special case of the an exchange economy (with an extra artificial consumer*

[7]The NCI can be helpful in comparing articles coming from different fields, as these often differ in the average length of their citations' lists.

*who is only interested in money and has all the goods as his initial endowments). The induced graph of this exchange economy is strongly connected. Thus, this Fisher's economy is guaranteed to have an equilibrium. By following a similar argument as Proposition 1, it is straightforward to see that the equilibrium price of goods is corresponding to the SCI or NCI index (by using different initial endowment values). The key difference between a ranking based on Fisher's economy and a ranking based on u-exchange-economy is that in a Fisher's economy the referencing power of a consumer is fixed, while in a u-exchange-economy the referencing power of a consumer is based on its authority rank, i.e., the price of his good. Therefore, we would argue that the SCI and NCI index is more suitable for the situation that the referencing power of an item is known, but the authority power or the rank is not.*

## 5. Quality and refereeing power

Any ranking systems induced by an exchange economy equates, by definition, the refereeing power of a paper with its quality (its rank). The reason is that the price of good $i$, which is identified as $i$'s rank, becomes the budget of consumer $i$. This budget is then split between the papers that $i$ cites, so it is exactly $i$'s refereeing power. By contrast, the Science Citation index disregards completely the rank of an article when it determines its refereeing power. In the SCI a citation from any article has the same importance. In the NSI, the weight of a citation from an article is also independent of its rank, but decreases with the number of citations an article makes. In this section we introduce a new element – taxation – that allows to control the extent to which the price and the budget of $i$ are tied to each other.

5.1. **Ranking induced by an exchange economy with taxes.** We add the following taxation scheme: every consumer is required to pay a fraction $\alpha$ of his income as a tax, and receives back $1/n$ of the total tax revenue as a lump sum transfer.

The purpose of the tax is to allow a separation between the rank of an item as evaluated by others (i.e., the *price* of its specific good) and its reviewing power (the *budget* of its representing consumer). Without the tax, the two coincide. With with a 100% tax rate, the reviewing power of all articles coincide, regardless of their ranking, because all have the same budget ($1/n$ of the tax revenue). In this case the induced ranking system is the Normalized Citation Index, namely, every article has the same budget, and this budget is equally split between the articles it cites. Hence, the money each article (consumer) gets amounts to the sum of all its normalized citations. By fixing a tax rate in-between the two extremes, one could attain control over the extent of the entanglement or disentanglement between the reviewing power and the ranking of an item.

In order to receive a clearer picture of the effect of taxation, consider the following example that deals with articles of two generations: two old papers, denoted $2a, 2b$, and two young ones, denoted $1a, 1b$. The two articles in each generation cite each other, and one of the younger generation makes an intergenerational citation and cites an article from the old generation.

**Example 2.** *Four items, $V = \{1a, 1b, 2a, 2b\}$ have citations given by*

$$E = \{(1a, 1b), (1b, 1a), (2a, 2b), (2b, 2a), (2a, 1a)\}.$$

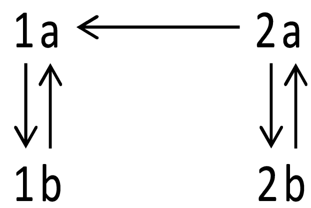


FIGURE 2. *Two generations.*

*The rankings provided by the various systems are shown in the following table:*

| *Ranking System* | *Economy* | $p_{1a}, p_{1b}, p_{2a}, p_{2b}$ |
|---|---|---|
| *PageRank* | *Any exchange economy without tax ($\alpha$=0) (Except for perfect substitutes)* | $\frac{1}{2}, \frac{1}{2}, 0, 0$ |
| | *CD with tax $\alpha = 0.5$* | $\frac{11}{28}, \frac{9}{28}, \frac{5}{28}, \frac{3}{28}$ |
| *NCI* | *CD with tax $\alpha = 1$* | $\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8}$ |
| *SCI* | *NON* | $\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$ |

*Indeed, the articles of the old generation are more important, since they receive a citation from the younger generation and not vice versa. However, giving the articles of the younger generation a rank of* $0$ *makes PageRank too extreme in differentiating between the two generations. Moreover, since the younger generation has* $0$ *rank, its refereeing power is also* $0$*, and thus article* $1a$ *has the same rank as* $1b$ *(*$1/2$*), even though the former enjoys an additional citation.*

*The SCI does grant positive ranking to the articles of the younger generation, and it also differentiates between the two articles of the old generation. However, it does not differentiate between* $2a$ *and* $2b$*, even though – while each of them cites the other –* $2b$ *enjoys only part of the citations of* $2a$ *and* $2a$ *enjoys all the citations of* $2b$*.*

*The NCI does take this last argument into consideration, and gives* $2a$ *a rank higher than that of* $2b$*. But like the SCI, article* $1b$*, which receives a unique link from the most important article* $1a$*, has the same rank as* $2a$*, which has a unique link from the least important article* $2b$*.*

The economy with an intermediate tax rate does succeed to grant appropriate ranking to all the articles: the old generation has a higher ranking than that of the young one, and the articles are properly ranked within each generation as well.

In the Cobb-Douglas case, when the tax rate is $\alpha$, the resulting competitive equilibrium system $p$ satisfies (compare with Eq. (4))

$$(\alpha \cdot (1, ..., 1) + (1 - \alpha)p)M = p. \tag{5}$$

While in the above example the tax rate affects only the cardinal ranking, this is not the general case. The following example shows that different tax rates may induce different ordinal rankings:

**Example 3.** *Consider the following variation of Example 1, with Cobb-Douglas preferences and tax $\alpha$: With no tax ($\alpha = 0$), we have $p_4 = 0$, and thus the prices of the other node*

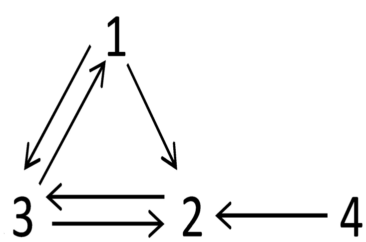


FIGURE 3. *Node no. 4 is added to Example 1.*

*remain the same as before: $p_1 = \frac{2}{9}, p_2 = \frac{3}{9}, p_3 = \frac{4}{9}$. In particular, the price of node 3 is strictly higher than that of node 2. With tax $\alpha = 1$ we obtain the NCI, and the price of node 2 (which receives $1 + \frac{1}{2} + \frac{1}{2}$ normalized citations) is strictly higher than that of node 3 (which receives only $1 + \frac{1}{2}$ normalized citations). Thus, by changing the tax rate, the ordinal ranking of nodes 2 and 3 is reversed.*

5.2. **Ranking by budget rather than by prices: an algorithmic advantage.** So far we defined the economy-based ranking by the equilibrium prices. An alternative approach could be to rank nodes based on their respective budgets, rather than on their specific-goods prices. While in the case of an exchange economy with no taxes there is no distinction between the two approaches, with taxes they differ. However, it turns out that the budget vector is just a constant translation of the price vector. More precisely, when the tax rate is $\alpha$ and the price vector is normalized to a probability distribution, the budget of consumer $i$ is $I_i = (1 - \alpha)p_i + 1/n$. Thus, as far as ordinal ranking is concerned, the price vector and the budget vector induce the same ranking. Moreover, one can be retrieved from the other.

In the case of Cobb-Douglas utilities, the budget vector can be calculated using the eigenvector idea. Consider the original $n$ states and the transition probability matrix $M$. Now we add another state that represents, say, the government. We introduce a new Markov matrix, denoted $T$, applied now to $n + 1$ states: $n$ original ones plus the newly added state.

The probability $T_{ij}$ of moving from $i$ to $j$, where $i, j \leq n$, is $(1 - \alpha)m_{ij}$, while $T_{ij} = alpha$ when $i < j = n + 1$. In other words, the probability of moving to $n + 1$ is always $\alpha$ and the remaining probability is divided proportionally to the probabilities in $M$. Finally, the probability of moving from $n+1$ to $j$ $(j \leq n)$ is always $1/n$. To summarize it,

$$T = T(\alpha) = \begin{pmatrix} & & & & \alpha \\ & & & & . \\ & & (1-\alpha)M & & . \\ & & & & . \\ & & & & \alpha \\ 1/n & . & . & . \quad 1/n & 0 \end{pmatrix} \tag{6}$$

Provided that $M$ is a Markov matrix, this (ergodic) matrix, is a Markov one and it has a unique invariant distribution.

**Proposition 3.** *Suppose that $0 \leq \alpha < 1$ and $q$ is an invariant distribution of $T$. Then, the first $n$ coordinates of $q$ are a translation of quasi equilibrium prices of the exchange economy rulled by Cobb-Douglas utilities and tax rate of $\alpha$.*

*Proof.* It is easy to show that $q_{n+1} = 1/(\alpha+1)$. Thus, $q' = (\alpha+1)(q_1, ..., q_n)$ is a probability distribution that satisfies,

$$(1-\alpha)q'M + \alpha \cdot e = q', \tag{7}$$

where $e$ is an $n$-dimensional vector whose all coordinates are equal to $1/n$. Set $p = \frac{q'-\alpha\cdot c}{1-\alpha}$. Then, $p$ is a probability distribution, and $q' = (1-\alpha)p + \alpha \cdot c$. From Eq. (7) we obtain, $(1-\alpha)((1-\alpha)p+\alpha\cdot c)M+\alpha\cdot e) = (1-\alpha)p+\alpha\cdot c$, which is equivalent to $(1-\alpha)p+\alpha\cdot c)M = p$. The last equality means that $p$ is a quasi equilibrium price system of the exchange economy governed by Cobb-Douglas utilities and tax rate of $\alpha$. □

The idea of adding a state has two advantages. First, for every Markov matrix $M$, the matrix $T$ – defined in Eq. (6) – is ergodic and therefore there is no need to perturb the probability in $M$ in order to obtain ergodicity. Thus, a uniqueness of an invariant distribution of $T$ is automatically guaranteed. The second advantage is that the algorithm that finds the invariant distribution in the same way as in the PageRank, could find the equilibrium prices with tax, if applied to $T$, rather than to $M$.

**Remark 4.** *The Web graph is almost certainly reducible, which makes its PageRank vector not unique. Brin and Page* [4] *solved the problem by introducing a "damping factor". Langville et. al.* [13] *showed that the technique of damping factor is mathematically equivalent to the idea of introducing the "government" as a new state (Langville et. al. called it "teleportation state") and taxation. However, we provide a totally new perspective here. By introducing the taxation system, the ranking of an item is based on its budget instead of its price.*

## 6. Symmetric CES utility functions

We now extend our domain of utility functions to the class of symmetric CES utility functions (Arrow, Chenery, Minhas, and Solow, 1961):[8]

$$u_i(x,\beta) = \left(\sum_{j\in O(i)} x_j^\beta\right)^{\frac{1}{\beta}}, \qquad \beta \in [-\infty, 1] \;\; (\beta = \infty \text{ and } \beta = 0 \text{ are understood as limits.})$$

This one-parameter class of functions spans many types of preferences. The case of perfect substitutes, $u_i(x) = \sum_{j\in O(i)} x_j$, is obtained when $\beta = 1$; the case of Cobb-Douglas preferences, $u_i(x) = \sum_{j\in O(i)} x_j$, is obtained (as a limit) when $\beta \to 0$; and the case of perfect complements, $u_i(x) = \min\{x_j\}_{j\in O(i)}$, is obtained (again, as a limit) when $\beta \to -\infty$. The goods are substitutes if $\beta > 0$ and complements if $\beta < 0$. We now study the effect of the parameter $\beta$ on the ranking, for $\beta$ ranging from $-\infty$ (perfect complements) to $+1$ (perfect substitutes).

Denoting $r = \frac{\beta}{\beta-1}$, we obtain the demands when $p_j > 0$ (recall $x_i^j$ is the quantity of good $j$ that $i$ consumes):

$$x_i^j(p, I_i) = \frac{p_j^{r-1}}{\sum_{k\in O(i)} p_k^r} I_i = \frac{p_j^{r-1}\pi_{ij}}{\sum_{k\in O(i)} p_k^r} p_i \tag{8}$$

and thus the amount that $i$ spends on good $j$ when $p_j > 0$ is:

$$e_i^j(p, I_i) = x_i^j(p, I_i)\, p_j = \frac{p_j^r \pi_{ij}}{\sum_{k\in O(i)} p_k^r} p_i.$$

When the price of good $j$ is positive, the sum of expenditures on good $j$ coincides with this price (since its supply consists of one unit). That is, when $p_j > 0$,

$$p_j = \sum_{i\in I(j)} e_i^j(p, I_i) = \sum_{i\in I(j)} \frac{p_j^r \pi_{ij}}{\sum_{k\in O(i)} p_k^r} p_i \tag{9}$$

(Note that in the Cobb-Douglas case, $\beta = 0$ and thus $r = 0$, therefore this expression reduces to $p_j = \sum_{i\in I(j)} p_i \frac{1}{c(i)}$.)

Let $M(\Pi, r, p) = (m_{ij})_{(i,j)\in V\times V}$, where

$$m_{ij} = \frac{\pi_{ij} p_j^r}{\sum_{k\in O(i)} p_k^r},$$

and recall the definition of $V'$ in the proof of Proposition 1: $V'$ is the largest set of nodes $j$ such that every node $i$ has a directed path connecting $i$ to $j$.

**Proposition 4.** *In case the directed graph of the network is connected and $r < 1$, $p$ is a quasi-equilibrium if and only if (a)*

$$p_i > 0 \text{ iff } i \in V'$$

[8] In order to keep things simple, when ambiguity may not arise, $u_i$ will stands for $u_i^{c(i)}$ and $x$ will be understood to be in $\mathbb{R}^{O(i)}$.

*and (b)*

$$p = p \cdot M(\Pi, r, p).^{9} \tag{10}$$

Note that in the Cobb-Douglas case ($r = 0$), Eq. (10) reduces to $p = p \cdot M(\Pi, 0, p) = pM$.

*Proof.* Suppose that $p$ is quasi-equilibrium. As in the Cobb-Douglas case, if $p_i = 0$ for every $i \in I(j)$ then $p_j = 0$. It implies, like in the proof of Proposition 1, that if $i \notin V'$, then $p_i = 0$. Furthermore, if $i \in I(j)$ and $p_i > 0$, then $p_j > 0$, and hence, (a). (b) is the translation of Eq. (9) to matrix terms. We have seen that Eq. (9) holds true in case $p_j > 0$. If $p_j = 0$ and $\pi_{ij} = 1$, then $p_i = 0$. Thus, $\frac{p_j^r \pi_{ij}}{\sum_{k \in O(i)} p_k^r} p_i$, that appears in the right hand side of Eq. (9), equals 0. Therefore Eq. (9) holds also when $p_j = 0$ and therefore, (b).

Suppose now that (a) and (b) are satisfied. Then, (b) implies that when all consumers whose budgets are positive maximize their CES utility, the total demand of any good in $V'$ is equal to its total supply. As for the other goods, their demand comes only from consumers whose budget is 0. These consumers may consume any quantity as long as there is no violation of market clearing. We thus obtain market clearing of all goods and thereby a quasi-equilibrium. $\square$

Now consider again Example 1 above, with the coincidence matrix $\Pi = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

With general CES preferences one obtains,

$$\begin{aligned} p_1^{1-r} &= \frac{p_3}{p_1^r + p_2^r} \\ p_2^{1-r} &= \frac{p_1}{p_2^r + p_3^r} + \frac{p_3}{p_1^r + p_2^r} \\ 1 &= p_1 + p_2 + p_3 \end{aligned}$$

The following figure graphs the three prices as $\beta$ ranges from $-\infty$ ($r = 1$, case of perfect complements), through 0 (Cobb-Douglas) to 1 ($r = -\infty$, case of perfect substitutes):

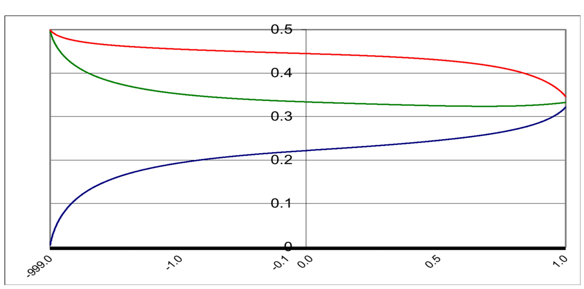


FIGURE 4. *$p_1$ (bottom-red), $p_2$(middle-green), $p_3$(top-blue) as a function of beta.*

[9] $\frac{0}{0}$ is understood as 0.

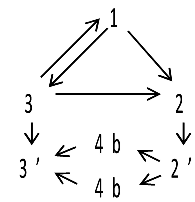


FIGURE 5. *Example 1 modified: an edge is replaced by a sub-graph*

Note that in the simple example given above different utility functions induce different cardinal rankings, while the ordinal ranking were left unchanged. In general, ordinal ranking may change, as can be witnessed when considering the following slight modification of this example:

The old direct link from 2 to 3 now goes through nodes 2', 4, 4', and 2". Since all the budget of 2 goes exclusively to node 2' and then (through 4 and 4') to node 2", we have $p_{2''} = p_{2'} = p_2$. Thus, the equations that determine the expenditures on good 3, and therefore $p_1$, $p_2$ and $p_3$ are unchanged (whatever the utility function). By symmetry, $p_4 = p_{4'}$, and these are equal to $\frac{1}{2}p_2$. It can be easily demonstrated that in the economy induced by the network in Figure 5 $\frac{1}{2}p_2$ (half the level of the middle, green line) may be higher or lower than $p_1$ (the lower, blue line).

**Remark 5.** *While the economic approach to ranking can span different ranking systems depending on the governing utility function, the Markov approach can lead only to PageRank. Using the Markov approach to obtain a ranking that is equivalent to that generated by a non Cobb-Douglas economy, one would need to define transition probabilities from a node to its outgoing links that depend on their final rank. In other words, the definition of the process will have to involve its outcome. The only Markov model that can be well defined without resorting to the outcome is the one of equal transition probabilities, i.e., PageRank. In contrast, in the economic approach it is possible to span different ranking systems using models with a closed definition, i.e., whose definition can be made without reference to their outcome. The way in which consumers split their budget depending on the equilibrium prices is a derivative of the equilibrium itself, rather than a part of the (modeler's) definition.*

Analogously to Proposition 1 in the case of CES, we have,

**Proposition 5.** *For any $0 \leq \beta < 1$ , when the directed graph of the network is connected, and all consumers are CES consumers with parameter $\beta$, there exists a unique quasi-equilibrium.*

*Proof.* The proof idea is similar to the one used in the proof of Proposition 1. Let $V'$ be the set of nodes that has been defined there. As in the Cobb-Douglas case, when $-\infty < \beta \leq 1$, $p_i > 0$ if and only if $i \in V'$. Suppose that $p = (p_1, ..., p_n) \geq p' = (p'_1, ..., p'_n)$ are two distinct quasi-competitive equilibrium price systems. Without loss of generality we may assume that there is $j \in V'$ such that $p_j = p'_i$ and furthermore, there is $\ell$ such that $\ell \in I(j)$ and $p_\ell > p'_\ell$. We now reduce all $p_i$, $i \in V' \setminus \{j\}$ down to $p'_i$ without touching $p_j$. Since $0 \leq \beta < 1$ means

$-\infty < r \leq 0$, when the $p_k$'s increase, the numerator, $p_j^{r-1} p_i$, in Eq. (8), goes down while the denominator, $\sum_{k \in O(i)} p_k^r$, goes up. Thus, the quantity consumer $i$ demands from good $j$ reduces. When applied to consumer $\ell$, his demand for good $j$ becomes strictly smaller compared to his demand prior to the change in prices. We thus obtain that under $p'$ there is an excess supply for $j$, which contradicts quasi-equilibrium conditions. □

It turns out that when $\beta < -1$ there are examples with multiple equilibria. The reader is referred to Lehrer and Pauzner (2012) for an explicit example of a network that induces an exchange economy with multiple equilibrium. This example is based on Gjerstad (1996). In the case of minimum utility (i.e., when goods are perfect complements) there might even exist a continuum of quasi-equilibrium (i.e., in any neighborhood of any competitive price system there is another competitive price system.) This is illustrated in the next example:

**Example 4.** *Consider the following network, the same as in Example 1 only without the link from 1 to 2.*

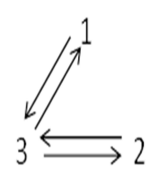


FIGURE 6. *An economy with multiple equilibria.*

*In the economy induced by this network with minimum utility functions, every vector* $(\alpha, 1/2 - \alpha, 1/2)$ *where* $0 \leq \alpha \leq 1/2$ *is a competitive equilibrium price system.*

## 7. Biased agents

In reality the decisions of agents regarding which items to cite are often biased by the ranking system itself. The fact that an item is highly ranked makes other agents more likely to cite it. This further increases the number of citations the item receives, pushing its rank even higher. In this section we model such a scenario and look for an exchange economy – with unbiased agents – which can replicate the outcome. We also show how a different exchange economy can be used to "undo" the agents' bias.

7.1. **The interaction between Google and biased agents.** Consider then the following simultaneous interaction between Google and agents. Google looks at the links between web pages and their intensities in order to determine the ranking, while agents observe the ranking generated by Google and bias the link intensities of their web pages accordingly. We are interested in the rest point of this interaction and in constructing a simple exchange economy whose competitive equilibrium coincides with this rest point.

In order to construct a model with biased agents, we extend our definition of a citation network to allow for link intensities. Let $\Pi = (\pi_{ij})$ be the coincidence matrix that represents the original network of links. Given the ranking vector $p$ generated by Google's PageRank, the agents bias their link intensities proportionally to $p$. The modified coincidence matrix then becomes $B(\Pi, p) = (b_{ij})_{ij \in V \times V}$ where

$$b_{ij} = \frac{p_j}{\sum_k p_k \pi_{ik}} \pi_{ij}. \tag{11}$$

Note that, subject to the assumption that $\sum_k p_k \pi_{ik} = 0$ implies $p_i = 0$,[10] $B(\Pi, p)$ is a Markov matrix (perhaps with less than $n$ states).

Given the agents modified link intensity matrix $B = B(\Pi, p)$, Google updates its ranking and computes the PageRank as the solution $p$ to $pB = p$. Thus, a rest point of the interaction between Google's publicly announced ranking and the agents is an invariant distribution $p = (p_1, ..., p_n)$ of the matrix $B(\Pi, p)$, i.e., a vector $p$ that satisfies:

$$pB(\Pi, p) = p \quad \text{(i.e., for every } j,\ p_j = \sum_i p_i \frac{p_j \pi_{ij}}{\sum_k p_k \pi_{ik}}). \tag{12}$$

7.2. **An exchange economy with perfect complements.** From the original coincidence matrix $\Pi$ we now construct a simple exchange economy whose competitive equilibria are rest points of the above interaction. That is, we consider agents who are not biased by the ranking and retain their original link intensities provided by $\Pi$. We change the utility function and obtain the same outcome.

Consider an exchange economy in which consumer $i$'s utility function is $u^i(x_1, ..., x_n) = \min\{x_j : \ \pi_{ij} = 1\}$. Denote the corresponding induced economy by $E(\Pi, Min)$.

**Proposition 6.** *Any competitive price system $p$ of $E(\Pi, Min)$ satisfies $pB(\Pi, p) = p$ and is thus a rest point of the interaction between PageRank and linearly-biased agents.*

*Proof.* In equilibrium, consumer $i$'s consumption of good $j$ is $\frac{p_i \pi_{ij}}{\sum_k p_k \pi_{ik}}$. The revenue of $j$, which is identical to $p_j$, is the sum of expenditures on good $j$, and thus equals $\sum_i p_j \frac{p_i \pi_{ij}}{\sum_k p_k \pi_{ik}}$. This is precisely the condition indicated in Eq. (12). □

The following example shows that not every solution of Eq. (12) is a vector of competitive prices of $E(\Pi, Min)$.

**Example 5.** *Consider the network discussed in Example 1. For $p = (0, 1/2, 1/2)$,*

$$M(p) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

*and $pB(\Pi, p) = p$. Thus, $p$ satisfies Eq. (12). Moreover, $p$ is a vector of competitive prices of $E(\Pi, Min)$, with the consumptions of the consumers being $x^1 = (0, 0, 0)$, $x^2 = (0, 0, 1)$ and $x^3 = (1, 1, 0)$.*

[10] As will be shown later, this assumption is automatically satisfied when $p$ is a competitive price system.

*Eq. (12) has a second fixed point: vector* $q = (1/2, 0, 1/2)$ *also satisfies*

$$q = qM(q) = q \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

*However,* $q$ *is not a vector of competitive equilibrium prices of* $E(\Pi, Min)$*. The reason is that under the price vector* $q$*, the optimal consumption of* $1$ *is* $(0,1,1)$ *while the optimal basket of* $3$ *is* $(1,1,0)$*. The total demand for good* $2$ *is therefore* $2$*, while only one unit of this good is available. This is a violation of Definition 1.*

For any coincidence matrix $\Pi$ and $n$-vector $p$ let $D(p) = (d_{ij}(p))_{ij}$ be the matrix defined by

$$d_{ij}(p) = \frac{\pi_{ij}}{\sum_k p_k \pi_{ik}}.$$

Note that $d_{ij}(p)$ might take the value of $+\infty$. As will become clear in the following proof, $p_i d_{ij}$ is the demand of a consumer $i$ with budget $p_i > 0$ from good $j$ in the economy $E(\Pi, Min)$.

**Proposition 7.** *A vector* $p$ *that satisfies* $pB(\Pi, p) = p$ *is a competitive price system of* $E(\Pi, Min)$ *if and only if* $pD(p) \leq (1, ..., 1)$.[11]

*Proof.* Assume that $p$ is a vector of competitive prices of $E(\Pi, Min)$. By Proposition 6 we know that $pB(\Pi, p) = p$, i.e., $\sum_i p_i \frac{p_j \pi_{ij}}{\sum_k p_k \pi_{ik}} = p_j$ for all $j$. In case $p_j \neq 0$, this implies that $\sum_i p_i \frac{\pi_{ij}}{\sum_k p_k \pi_{ik}} = 1$. Consider now the case of $p_j = 0$. The total demand of those $i$'s whose prices are positive from good $j$ does not exceed 1, i.e., $\sum_{i:p_i>0} p_i \frac{\pi_{ij}}{\sum_k p_k \pi_{ik}} \leq 1$. For those $i$'s whose prices are 0 we obviously have $p_i \frac{\pi_{ij}}{\sum_k p_k \pi_{ik}} = 0$. Thus, $\sum_i p_i \frac{\pi_{ij}}{\sum_k p_k \pi_{ik}} \leq 1$.

Suppose now that $pB(\Pi, p) = p$ and $pD(p) \leq (1, ..., 1)$. The equality $pB(\Pi, p) = p$ implies that (i) consumers $i$ with $p_i > 0$ maximize their utility subject to their budget constraint and (ii) the market is clearing for those goods $j$ whose prices are strictly positive. The inequality $pD(p) \leq (1, ..., 1)$ guaranties that the total demand does not exceed the total supply (which is 1) also for goods $j$ whose price is 0. Thus, $p$ is a quasi-equilibrium price system. □

We conclude that every competitive equilibrium of the economy with the minimum utility function is a rest point of the interaction between Google and the biased agents. Rest points of the interaction are competitive equilibria of $E(\Pi, Min)$ provided that they satisfy the additional requirement that $pD(p) \leq (1, ..., 1)$.

7.3. **CES utility and biased agents.** The above results for the case of PageRank (or equivalently, a Cobb-Douglas economy) can be generalized to ranking system based on CES utility functions. Consider the interaction between an economy based ranking system with CES utility function with parameter $r$. Given the agents' modified link intensity matrix

[11]By $pD(p) \leq (1, ..., 1)$ we mean that each coordinate of the vector $pD(p)$ is less than, or equal to, 1. Also, we refer to the product $0 \cdot \infty$ as 0.

$B = B(\Pi, p)$, the ranking system solves the equation $pM(B, r, p) = p$. Thus, a rest point of the interaction is a vector $p = (p_1, ..., p_n)$ that satisfies:

$$pM(B(\Pi, p), r, p) = p.$$

The next proposition states that for $r < 1$, the outcome of the above interaction coincides with that of an economy-based ranking system with CES utility function with parameter $r + 1$, applied to the original coincidence matrix $\Pi$ (i.e., when agents have no bias):

**Proposition 8.** *The vector $p$ is a fixed point of the interaction between linearly-biased agents and the ranking system based on the CES economy with parameter $r < 0$, if and only if it satisfies $pM(\Pi, r+1, p) = p$, i.e., $p$ is a quasi-equilibrium price vector of the CES economy with parameter $r + 1$.*

*Proof.* $M(B(\Pi, p), r, p)$ is defined by

$$m_{ij} = \frac{b_{ij} p_j^r}{\sum_k b_{ik} p_k^r} = \frac{\frac{p_j}{\sum_k p_k \pi_{ik}} \pi_{ij} p_j^r}{\sum_k \frac{p_k}{\sum_{\hat{k}} p_{\hat{k}} \pi_{i\hat{k}}} \pi_{ik} p_k^r} = \frac{\pi_{ij} p_j^{r+1}}{\sum_k \pi_{ik} p_k^{r+1}},$$

which defines $M(\Pi, r+1, p)$. Note that in the case where $r = 0$, there might be a rest point that violates $pD(p) \leq (1, ..., 1)$, which might occur when a consumer with a positive budget buys a good whose price is zero. This cannot happen when $r < 0$, because prices are positive in $V'$ and zero otherwise, and no consumer in $V'$ buys anything from goods out of $V'$. Thus, the additional (i.e., the inequality) requirement is not needed here. □

**Remark 6.** *Note that $B(\Pi, p) = M(\Pi, 1, p)$. Thus, the above calculation shows that $M(M(\Pi, 1, p), r, p) = M(\Pi, r+1, p)$. One could also consider agents whose bias is not linear but rather defined by $B(\Pi, b, p) = M(\Pi, b, p)$; in this case we have $M(M(\Pi, b, p), r, p) = M(\Pi, r+b, p)$, implying that a rest point of the interaction is a quasi-equilibrium price vector of the CES economy with parameter $r + b$.*

**Remark 7.** *The above result hints on a relation between the concept of "strategic complementarities" in interactions and "complement goods" in exchange economies. In the interaction between the agents and the ranking system, the latter gives a higher rank to the items to which the agents send more links. Due to the bias, agents send stronger links to the items that the ranking system ranks higher. Thus, the bias reinforces the strategic complementarity in the interaction between the two parties. Proposition 8 shows that this is equivalent to increasing the parameter $r$ of the utility function by 1, i.e., moving towards preferences in which the goods are stronger complements than in the original economy.*

7.4. **Removing the bias.** The results of the last section can be used in the inverse way. Consider a designer who wishes to implement a ranking scheme based on the economy with CES parameter $r$, based on agents' "sincere" opinions (pertaining to an unmodified incidence matrix $\Pi$). The agents, however, are not sincere: they bias their citations' intensities (say in a linear fashion). That is, the designer has no access to their sincere opinions; rather, the link intensities that he observes are biased toward the ranking he publishes. In order to obtain a ranking scheme that would reflect agents' "sincere" opinions, as if they were unbiased, the designer could simply employ a ranking scheme based on CES with parameter $r-1$. The outcome of the interaction of this ranking system and the biased agents would coincide with the outcome that would have emerged with the desired ranking system had the agents been unbiased. Assume, for instance, that PGoogle believes that the best search results would be obtained if agents' sincere opinions were aggregated using the PageRank method (CES parameter $r=0$). However, Google is concerned that due to its success, agents are biased toward its published ranking. By using a ranking based on the economy with a negative $r$, Goggle can move closer to the desired ranking. In the case of a linear bias, the appropriate CES parameter to use is $r=-1$.

## 8. Concluding comments

8.1. **Preference relations over an arbitrary number of goods.** In the general formulation of the exchange economy the utility function is a vector $u=(u^1,u^2,...,u^n)$ where each $u^k$ is a symmetric utility function. The symmetry of the utility functions guaranties that all goods are treated in the same fashion, so that an item's name does not affect its ranking. We did not, however, make any restrictions on the relationship between the different $u^k$'s, i.e., the utility functions from consuming different number of goods may be very different. As a result, changes in the graph may lead to undesired outcomes. For example, the share of $i$'s budget spent on $j$ may increase if we add an outgoing link from $i$ to another node because adding the link changes $i$'s utility function.

While we did not impose on the $u^k$'s any restriction beyond symmetry, we have focussed our discussion on CES utility functions. These induce similar consumption patterns across all consumers, regardless of the number of goods they consume. Defining a restriction on $u=(u^1,u^2,...,u^n)$ that would generate a natural relationship between different $u^k$'s is not straightforward. One condition that would naturally link between different $u^k$'s is separability:

Let $\succeq$ be a reflexive complete order over $\mathbb{R}^n_+$. We say that $\succeq$ is *separable*[12] (Debreu, 1960) if for every $k=1,...,n-1$, $x^{(k)},y^{(k)}\in\mathbb{R}^k_+$, $x^{(n-k)},y^{(n-k)}\in\mathbb{R}^k_+$ the following holds

$$(x^{(k)},x^{(n-k)})\succeq(y^{(k)},x^{(n-k)}) \text{ if and only if } (x^{(k)},y^{(n-k)})\succeq(y^{(k)},y^{(n-k)}).$$

Most commonly used utility functions, such as minimum, Cobb-Douglas and CES induce separable preference orders, and if symmetric, induce symmetric separable preference order.

8.2. **Economies and rankings.** The question remains open as to what ranking schemes are economy-based. In other words, what are the conditions that characterize the ranking schemes generated by an exchange economy. This question can be refined further by restricting the generating economies to those governed by a smaller set of utility functions, such as separable or symmetric CES utility functions.

[12]We found no paper that characterizes utility functions representing separable preference orders. We did find however, an extensive study of preference orders representable by separably additive utility functions. The latter, in particular, are separable.

Another important question concerns the relationship between the economic properties of the utility function, such as the CES parameter $\beta$ (which captures the extent to which goods are substitutes or complements of each other) and the ranking scheme they generate.

The answers to these questions lie beyond the scope of the current paper and are left for a future project.

8.3. **Monotonicity and zero prices.** Proposition 4 states that for any CES utility function with $\beta \neq \infty$, the price of $i$ in quasi-equilibrium is positive if and only if $i \in V'$. This fact results from the property that in quasi-equilibrium if $p_j > 0$ and $\pi_{ji} = 1$, then necessarily $p_i > 0$. This property is related, but not equivalent to, monotonicity.

For $x, y \in \mathbb{R}^d$ we say that $x > y$, if $x$ is coordinate-wise greater than or equal to $y$ and in at least one coordinate the inequality is strict. Also, $x >> y$, if $x$ is coordinate-wise greater than $y$. A utility function $u$ is *monotonic* if $x > y$ implies $u(x) > u(y)$. Note that CES functions are not monotonic. Monotonicity is violated when one of $x$'s coordinates equals 0. However, if $x > y$ and $x > \overrightarrow{0}$ ($\overrightarrow{0}$ being the zero vector), then $u(x) > u(y)$ when $u$ is CES with $\beta \neq \infty$. It implies that when $p_i > 0$, the budget of $i$ is positive and he consumes a basket $y >> \overrightarrow{0}$ for which $x > y$ implies $u(x) > u(y)$. This is the property that the proof of Proposition 4 hinges on. Furthermore, this is precisely the property that makes the difference between $\beta \neq \infty$ and $\beta = \infty$ (i.e., the minimum utility function). When $u = Min$, it might happen that $y$ is an optimal basket subject to a positive budget constraint, $x > y$ and yet $u(x) = u(y)$.

We summarize this comment with the following corollary.

**Corollary 1.** *Assume that the network's graph is connected. Suppose that the economy is governed by a utility function $u$ with the property that if a consumer's budget is positive and $y$ is an optimal basket, then $x > y$ implies $u(x) > u(y)$. Let $p$ be a quasi-equilibrium system. Then, $p_i > 0$ if and only if $i \in V'$.*

8.4. **Employing other economic concepts.** We dealt so far with ranking of individual nodes. A direction for further application of these ideas is to ranking cluster of nodes. This gives rise to using ideas from the theory of (international) trade between countries (that stand for clusters).

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