The Design of Refinancing Contracts

Santiago Forte Arcos* and J. Ignacio Peña†

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Abstract

This paper introduces the concept of refinancing contract. Its design appears as an alternative to the classical assumption of new equity issued to finance the payment of corporate debt. Dividend rates, maturities, and nominal debt payments, are modeled as part of the contract. We also describe credit spreads and debt risk as a function of the firm characteristics, the risk free interest rate, and the specific contract selected. It is finally shown that the “lagged effect” of the risk free interest rate on the credit spreads on corporate bonds found by Guha, Hiris and Visviki [11], the positive autocorrelation in the change of credit quality documented by Altman and Kao [1], and the range of firm values in which KMV Corporation [5] finds typically to be the default point, are all consistent with the refinancing assumption.

Keywords: Debt refinancing, debt structure, credit spreads, debt risk, default and migration probabilities.

Yield Classification: G13, G21, G28, G32, G35.

* Santiago Forte is a Ph.D. student at Universidad Carlos III de Madrid (sforte@emp.uc3m.es). Forte acknowledges financial support from MEC Grant Ref: AP2000-1327.
† J. Ignacio Peña is Professor of Finance and Accounting at Universidad Carlos III de Madrid (ypenya@eco.uc3m.es). Peña acknowledges financial support from MEC Grant Ref: PB98-0030. The authors appreciate helpful comments from the participants in the IX Foro de Finanzas at Pamplona, specially from Carmen Ansotegui.
1 Introduction:

A typical assumption in the analysis of corporate debt is that payments are financed by issuing additional equity. For those models that do not include a perpetuity, and for which nominal debt is given, this implies that corporate debt eventually disappears, and with it the credit risk of the firm. This is the case in Merton [16], and Geske [9]. Giving that firms use to keep a considerable amount of debt as part of its capital structure, this result is clearly unsatisfactory. Considering a perpetuity, as in Black and Cox [2], avoids this problem. This possibility however presents two main objections: First, corporate debt do have a finite maturity. Second, and less obvious, the bankruptcy-triggering firm value derived in this case is constant, giving that the coupon is. As long as this threshold value does not grow as the firm value does, the result is that the default probability tends to zero as time goes by.

Nevertheless, issuing new equity is not the only option the firm has. We introduce the concept of refinancing contract, and prove that whenever the firm has to satisfy a debt payment, designing a contract of this type is feasible under the same conditions it is feasible to issue new equity, while keeping both, equity and debtholders wealth, unchanged. We also show that under this new alternative, stable corporate debt structure (with finite maturity) can be presumed. This means that the number of future payments, the weight of these payments as a proportion over total nominal debt, and the time spread between them, can be assumed constant. Dividend rates, maturities, and nominal debt payments, are modeled as part of the contract. We also describe credit spreads and debt risk (measured according to Merton [16] by the instantaneous debt return volatility) as a function of the firm characteristics, the risk free interest rate, and the specific contract selected. Both, the credit spread and the debt risk, respond in the same way to changes in those variables and parameters that affect them, which is not always the case in Merton’s paper. Finally, the ”lagged effect” of the riskfree interest rate on the credit spreads on corporate bonds found by Guha, Hiris and Visviki [11], the positive autocorrelation in the change of credit quality documented by Altman and Kao [1], and the range of firm values in which KMV Corporation [5] finds typically to be the default point, are all shown to be consistent with

\footnote{This will allow us to consider for instance that the firm always maintains short and long-term debt, while keeping the ratio between them constant.}
the refinancing assumption.

The following assumptions will be made along the paper:

**A1:** There are no taxes, problems concerning indivisibility, bankruptcy costs, transactions costs, or agency costs.

**A2:** Trading takes place continuously.

**A3:** There exits a riskless asset with constant interest rate \( r \), that applies for borrowing and lending, and for any maturity.

**A4:** Every individual acts as if she can buy or sell as much of any security as she wishes without affecting the market price.

**A5:** Individuals may take short positions in any security, including the riskless asset, and receive the proceeds of the sale. Restitution is required for payouts made to securities held short.

**A6:** Modigliani-Miller Theorem obtains, that is, the firm value is independent of its capital structure.

**A7:** The firm value, \( V \), follows the diffusion process given by

\[
dV = (\mu - \delta)V dt + \sigma V dz
\]  

where \( \mu \) is the expected rate of return on \( V \), \( \delta \) is the constant rate of firm value which is paid to equity holders as dividends, \( \sigma \) is the volatility of the rate of return which will be assumed to be constant, and \( z \) is a standard Brownian motion.

The remainder of the article is organized as follows: Section 2 presents how a refinancing contract with an arbitrary number of future debt payments \( n \), can be designed under the same conditions it is feasible to issue new equity to pay the debt. Section 3 discusses the specific cases of \( n = 1 \) and \( n = 2 \). Assumptions made in section 2 are formally proved to hold for these cases. Credit spread and debt risk are described as a function of the firm characteristics, the riskfree interest rate, and the specific contract selected. Results are related to those derived in Merton [16], and some empirical evidence is shown to be consistent with the refinancing assumption. Finally section 4 offers some conclusions and proposes further research.
2 The General Case

No assumption is made at this moment about the profile of nominal payments that constitute the corporate debt. We assume however that at least a certain debt payment has to be satisfied at some future period $\tau > t$, and that this, and any posterior debt payment, is to be financed by issuing additional equity. Under these conditions the equity and debt values will be a function of the firm value and time. Denote the equity value as $S(V,t)$, and the debt value as $F(V,t)$.

**Definition 1:** A refinancing contract between the firm and the debtholders at $\tau$, is a vector\(^2\) $\Theta := (\delta, \Psi, \Upsilon) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, with $n < \infty$, by which:

a) The firm, which is assumed to maximize equity holders wealth, promises (under limited liability) the payment of $\Psi$ at $\Upsilon$, that is, the payment of $\psi_i$ at $\tau_i$, where $\psi_i \in \Psi, \tau_i \in \Upsilon, i = 1, ..., n$, and $\tau_1 > \tau$.

b) The firm also restricts itself to apply a dividend rate equal to $\delta$, and loses the right to issue new debt. These restrictions apply until $\Theta$ has been canceled, either by satisfying nominal payments regularly (issuing new equity), or by means of a posterior debt refinancing contract.

c) The debtholders renounce to $F(V,\tau)$.

We say that $\Theta$ is feasible, if and only if the firm and the debtholders are willing to sign $\Theta$. The set of feasible $\Theta$ is denoted by $\Theta^f$.

**Lemma:** Let $S(V,\Theta,\tau)$ and $F(V,\Theta,\tau)$ denote the equity and debt value at $\tau$ when the value of the firm is $V$, the debt profile consists on the payment of $\Psi$ at $\Upsilon$, and the dividend rate is $\delta$. $\Theta \in \Theta^f$ if and only if $S(V,\Theta,\tau) = S(V,\tau)$, implying $S(V,\tau) > 0$ as a necessary condition for a feasible $\Theta$ to exist.

\(\sigma\) could be introduced as part of the contract. We assume however that it is implicitly prescribed not to alter the firm risk. Note also that a refinancing contract does not necessarily has to be signed with current debtholders.
Proof: $S(V, \Theta, \tau)$ is what equity holders get after signing $\Theta$, therefore they will be willing to sign if and only if $S(V, \Theta, \tau) \geq S(V, \tau)$. $F(V, \Theta, \tau)$ is what debtholders have after signing $\Theta$, therefore they will be willing to sign if and only if $F(V, \Theta, \tau) \geq F(V, \tau)$. At the same time $S(V, \Theta, \tau) + F(V, \Theta, \tau) = S(V, \tau) + F(V, \tau) = V$. $S(V, \Theta, \tau) = S(V, \tau)$ implies that $F(V, \Theta, \tau) = F(V, \tau)$ and $\Theta \in \Theta^f$. On the other hand $\Theta \in \Theta^f$ implies $S(V, \Theta, \tau) \geq S(V, \tau)$. Suppose $S(V, \Theta, \tau) > S(V, \tau)$, then $F(V, \Theta, \tau) < F(V, \tau)$ and $\Theta \notin \Theta^f$, what is a contradiction, proving the first argument of the lemma. Finally $S(V, \Theta, \tau) > 0 \forall \Theta | \tau_1 > \tau^3$, implying $S(V, \tau) > 0$ as a necessary condition for a feasible $\Theta$ to exist.

Remark 1: Refinancing does not alter neither equity holders, nor debtholders wealth. This implies that $S(V, \Theta, \tau)$ and $F(V, \Theta, \tau)$ can be valued assuming that debt payments are to be financed issuing new equity, even if this never happens, that is, even if the firm always chooses to refinance its debt at maturity.

Assumption B1: Let $\Psi = \psi_1 \Phi$, where $\Phi$ is the $n$-dimensional vector which first element $\phi_1$ equals 1, and the remaining are some fixed values $\phi_i > 0$ \forall $i \geq 2$. $S(V, \Theta, \tau)$ is then assumed to be a continuous and strictly decreasing function in $\psi_1 (CSD (\psi_1))$, with $S(V, \Theta, \tau) |_{\psi_1=0} = V$, and $\lim_{\psi_1 \to \infty} S(V, \Theta, \tau) = E^{R. N} \int_\tau^{\tau_1} \delta V(s) e^{-r(s-\tau)} ds = V (1 - e^{-\delta \tau_1})$. $^3$

$\Psi = \psi_1 \Phi$ reflects the ratio between nominal debt payments. If these consist in a constant coupon for instance, then $\phi_i = 1 \forall i$. B1 asserts that the equity value is a continuous and strictly decreasing function in the nominal payments that equity holders have to satisfy. $S(V, \Theta, \tau) |_{\psi_1=0} = V$ recognizes that if there is no debt, then the equity holders own the firm. $\lim_{\psi_1 \to \infty} S(V, \Theta, \tau) = E^{R. N} \int_\tau^{\tau_1} \delta V(s) e^{-r(s-\tau)} ds$ indicates that as nominal debt tends to infinity, default at $\tau_1$ becomes unavoidable, and the unique value associated to equity is the value of the dividends that will be received until the first debt payment is required. Standard arguments allow us to use risk neutral valuation. It will be explicitly shown that B1 holds for $n = 1$ and $n = 2$.

Assumption B2: Let $\Pi = (\tau_1 - \tau) \Lambda$, where $\Pi$ denotes the $n$-dimensional vector which first element $\pi_1$ equals $(\tau_1 - \tau)$, and $\pi_i$ equals $(\tau_i - \tau_{i-1})$ $^4$
\[ \forall i \geq 2. \quad \Lambda \] on the other hand denotes the \( n \)-dimensional vector which first element \( \eta_1 \) equals 1, and the remaining are some fixed values \( \eta_i > 0 \ \forall i \geq 2 \). \( S(V, \Theta, \tau) \) is then assumed to be a continuous and strictly increasing function in \( \tau_1 \) \((CSI(\tau_1))\), with \( \lim_{\tau_1 \rightarrow \tau} S(V, \Theta, \tau) = \text{Max} \{0, V - \sum_{i=1}^{n} \psi_i\} \), and \( \lim_{\tau_1 \rightarrow \infty} S(V, \Theta, \tau) = V \). Denote \( \hat{\psi}_1 \) the \( \psi_1 \) value such that \( S(V, \tau) = V - \hat{\psi}_1 \sum_{i=1}^{n} \phi_i \), that is, \( \hat{\psi}_1 = \frac{F(V, \tau)}{\sum_{i=1}^{n} \phi_i} \).

\( \Pi = (\tau_1 - \tau) \Lambda \) describes the timing spread of payments. If \( \eta_i = 1 \ \forall i \) for instance, then time between one debt payment and the following is always the same. As \( \tau_1 \) tends to \( \tau \), new corporate debt tends to consist in a single payment satisfied at \( \tau \). As \( \tau_1 \) tends to infinity, the present value of future debt payments, that is, \( F(V, \Theta, \tau) \) tends to zero, and the equity value tends to the firm value. Finally, we may denote \( \Delta \equiv (n, \Phi, \Lambda) \) the vector that describes the corporate debt structure that results from a given refinancing contract \( \Theta \) and \( \Theta^f \mid \Delta \) the subset of \( \Theta^f \) that satisfies some given corporate debt structure \( \Delta \).

**Assumption B3:** \( S(V, \Theta, \tau) \) is a continuous and strictly increasing function in \( \delta \) \((CSI(\delta))\), with \( \lim_{\delta \rightarrow \infty} S(V, \Theta, \tau) = V \).

\[ \lim_{\delta \rightarrow \infty} S(V, \Theta, \tau) = V \] reflects that for any \( \tau_1 > \tau \), in the limit case of \( \delta = \infty \), the equity holders liquidate the firm before any debt payment can be required. Note also that \( S(V, \Theta, \tau) \mid_{\delta=0} \) coincides with the case presented in Geske [9]. B3 will also be proved for \( n = 1 \) and \( n = 2 \).

**Definition 2:** The sequence \( \alpha - \beta - \gamma \) is an order of choice in \( \delta, \psi_1 \) and \( \tau_1 \).

**Remark 2:** An order of choice in \( \delta, \psi_1 \) and \( \tau_1 \) implies an order of choice in \( \delta, \Psi \) and \( \Upsilon \), given \( \Delta \).

**Theorem:** Suppose \( S(V, \tau) > 0 \), and let \( \varphi^\delta \equiv \mathbb{R}_+, \varphi^{\psi_1} \equiv \left( \hat{\psi}_1, \infty \right), \varphi^{\tau_1} \equiv (\tau, \infty) \). Consider any sequence \( \alpha - \beta - \gamma \), where \( \alpha \) is chosen in \( \varphi^\alpha \), and define \( \varphi^{\beta \alpha} \) as the subset of \( \varphi^\beta \) for which \( S(V, \Theta, \tau) = S(V, \tau) \) reaches a solution for at least one \( \gamma \in \varphi^n \), given \( \alpha \). For any \( \Delta \), \( \varphi^{\beta \alpha} \) is a non empty set. Moreover, for any \( \alpha \in \varphi^\alpha \), and \( \beta \in \varphi^{\beta \alpha} \), there is only one \( \gamma \in \varphi^\gamma \) such that \( S(V, \Theta, \tau) = S(V, \tau) \).

\[ V - \sum_{i=1}^{n} \psi_i = V - \psi_1 \sum_{i=1}^{n} \phi_i \]
Proof: We have six possible sequences $\alpha - \beta - \gamma$. Let analyze each one of them:

**Case 1:** $\tau_1 - \delta - \psi_1$

Given $\tau_1 \in \varphi^{\tau_1}$, $S(V, \Theta, \tau)$ is $CSD(\psi_1)$, with $\lim_{\psi_1 \rightarrow \hat{\psi}_1} S(V, \Theta, \tau) > S(V, \tau)$ $\forall \delta \in \varphi^\delta$, and $\lim_{\psi_1 \rightarrow \infty} S(V, \Theta, \tau) = V(1 - e^{-\delta T_1})$. Therefore, $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\psi_1 \in \varphi^{\psi_1}$, if and only if $V(1 - e^{-\delta T_1}) < S(V, \tau)$, if and only if $\delta < \frac{\ln[\frac{V_1}{\tau_1}]}{\Gamma}$. As a result, $\varphi^{\delta \mid \tau_1} \equiv \left[0, \frac{\ln[\frac{V_1}{\tau_1}]}{\Gamma}\right] \neq \emptyset$. Because $S(V, \Theta, \tau)$ is $CSD(\psi_1)$, we finally have that for any $\tau_1 \in \varphi^{\tau_1}$, and $\delta \in \varphi^{\delta \mid \tau_1}$, there is a unique $\psi_1 \in \varphi^{\psi_1}$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

**Case 2:** $\tau_1 - \psi_1 - \delta$

Given $\tau_1 \in \varphi^{\tau_1}$, $S(V, \Theta, \tau)$ is $CSI(\delta)$, with $\lim_{\delta \rightarrow \infty} S(V, \Theta, \tau) = V \forall \psi_1 \in \varphi^{\psi_1}$. Therefore, $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\delta \in \varphi^\delta$, if and only if $S(V, \Theta, \tau) \mid_{\delta=0} \leq S(V, \tau)$, if and only if $\psi_1 \geq \psi_1^{0, \tau_1}$, where $\psi_1^{0, \tau_1} > \hat{\psi}_1$ is the $\psi_1$ value such that $S(V, \Theta, \tau) \mid_{\delta=0} = S(V, \tau)^6$. As a result, $\varphi^{\psi_1 \mid \tau_1} \equiv [\psi_1^{0, \tau_1}, \infty) \neq \emptyset$. Because $S(V, \Theta, \tau)$ is $CSI(\delta)$, we finally have that for any $\tau_1 \in \varphi^{\tau_1}$, and $\psi_1 \in \varphi^{\psi_1 \mid \tau_1}$, there is a unique $\delta \in \varphi^\delta$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

**Case 3:** $\delta - \tau_1 - \psi_1$

Given $\delta \in \varphi^\delta$, $S(V, \Theta, \tau)$ is $CSD(\psi_1)$, with $\lim_{\psi_1 \rightarrow \hat{\psi}_1} S(V, \Theta, \tau) > S(V, \tau)$ $\forall \tau_1 \in \varphi^{\tau_1}$, and $\lim_{\psi_1 \rightarrow \infty} S(V, \Theta, \tau) = V(1 - e^{-\delta T_1})$. Therefore, $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\psi_1 \in \varphi^{\psi_1}$, if and only if $V(1 - e^{-\delta T_1}) < S(V, \tau)$, if and only if $\tau_1 < \tau + \frac{\ln[\frac{V_1}{\tau_1}]}{\delta}$. As a result, $\varphi^{\tau_1 \mid \delta} \equiv (\tau, \tau + \frac{\ln[\frac{V_1}{\tau_1}]}{\delta}) \neq \emptyset$. Because $S(V, \Theta, \tau)$ is $CSD(\psi_1)$, we finally have that for any $\delta \in \varphi^\delta$, and $\tau_1 \in \varphi^{\tau_1 \mid \delta}$, there is a unique $\psi_1 \in \varphi^{\psi_1}$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

$^6$B1 implies that $\psi_1^{0, \tau_1}$ exists and is unique, and joint with B2 also implies that $\psi_1^{0, \tau_1} > \psi_1$. 
Case 4: $\delta - \psi_1 - \tau_1$

Given $\delta \in \varphi^\delta$, $S(V, \Theta, \tau)$ is $CSI(\tau_1)$, with $\lim_{\tau_1 \to \tau} S(V, \Theta, \tau) = V - \psi_1 \sum_{i=1}^n \phi_i < S(V, \tau) \forall \psi_1 \in \varphi^\psi_1$, and $\lim_{\tau_1 \to \infty} S(V, \Theta, \tau) = V$. As a result, $\varphi^{\psi_1} \equiv \varphi^{\psi_1} \neq 0$. Because $S(V, \Theta, \tau)$ is $CSI(\tau_1)$, we finally have that for any $(\delta, \psi_1) \in \varphi^\delta \times \varphi^\psi_1$, there is a unique $\tau_1 \in \varphi^{\tau_1}$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

Case 5: $\psi_1 - \tau_1 - \delta$

Given $\psi_1 \in \varphi^\psi_1$, $S(V, \Theta, \tau)$ is $CSI(\delta)$, with $\lim_{\delta \to \tau_1} S(V, \Theta, \tau) = V \forall \tau_1 \in \varphi^{\tau_1}$. Therefore, $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\delta \in \varphi^\delta$, if and only if $S(V, \Theta, \tau) \mid_{\delta=0} = S(V, \tau)$, if and only if $\tau_1 \leq \tau^{0, \psi_1}$, where $\tau^{0, \psi_1} > \tau$ is the $\tau_1$ value such that $S(V, \Theta, \tau) \mid_{\delta=0} = S(V, \tau^7)$. As a result, $\varphi^{\tau_1|\psi_1} \equiv \left(\tau, \tau^{0, \psi_1}\right) \neq 0$. Because $S(V, \Theta, \tau)$ is $CSI(\delta)$, we finally have that for any $\psi_1 \in \varphi^\psi_1$, and $\tau_1 \in \varphi^{\tau_1|\psi_1}$, there is a unique $\delta \in \varphi^\delta$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

Case 6: $\psi_1 - \delta - \tau_1$

Given $\psi_1 \in \varphi^\psi_1$, $S(V, \Theta, \tau)$ is $CSI(\tau_1)$, with $\lim_\tau \to \tau S(V, \Theta, \tau) = V - \psi_1 \sum_{i=1}^n \phi_i < S(V, \tau) \forall \delta \in \varphi^\delta$, and $\lim_{\tau \to \infty} S(V, \Theta, \tau) = V$. As a result, $\varphi^{\delta|\psi_1} \equiv \varphi^{\delta} \neq 0$. Because $S(V, \Theta, \tau)$ is $CSI(\tau_1)$, we finally have that for any $(\psi_1, \delta) \in \varphi^{\psi_1} \times \varphi^\delta$, there is a unique $\tau_1 \in \varphi^{\tau_1}$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

\textbf{Corollary 1:} $\Theta^F \neq \emptyset$ if and only if $S(V, \tau) > 0$.

\textbf{Proof:} This holds given lemma and theorem above.

\textbf{Corollary 2:} Suppose $S(V, \tau) > 0$, and fix $\Delta$. Then for a given $(\tau_1, \delta) \in \varphi^{\tau_1} \times \varphi^\delta$ such that $\delta T_1 < \ln \left[\frac{V}{T_1}\right]$ there exists one, and only one $\psi_1 \in \varphi^{\psi_1}$, such that the refinancing contract generated in this way is feasible.

\textbf{Proof:} This holds given the proof of case 1 and case 3.

\textsuperscript{7}B2 implies that $\tau^{0, \psi_1}$ exists and is unique. B2 also implies that $\tau^{0, \psi_1} > \tau$. 

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**Corollary 3:** Suppose $S(V, \tau) > 0$, and fix $\Delta$. Then for a given $(\tau_1, \psi_1) \in \varphi^{\tau_1} \times \varphi^{\psi_1}$ such that $S(V, \Theta, \tau) |_{\delta=0} \leq S(V, \tau)$, there exists one, and only one $\delta \in \varphi^\delta$, such that the refinancing contract generated in this way is feasible.

**Proof:** This holds given the proof of case 2 and case 5.

**Corollary 4:** Suppose $S(V, \tau) > 0$, and fix $\Delta$. Then for a given $(\delta, \psi_1) \in \varphi^\delta \times \varphi^{\psi_1}$ there exists one, and only one $\tau_1 \in \varphi^{\tau_1}$, such that the refinancing contract generated in this way is feasible.

**Proof:** This holds given the proof of case 4 and case 6.

Theorem above asserts that, whenever $S(V, \tau) > 0$, a feasible refinancing contract with any arbitrary capital structure can be generated, and also describes how can it be constructed. Although this feasible $\Theta$ is not unique for a given $\Delta$, and there is actually an infinite number of elements in $\Theta^f | \Delta$, not everything is possible. Choosing one element in $\Theta^f | \Delta$ could be seen as a matter of priority. Take for instance case 1: The maturity of the first payment, $\tau_1$, is freely chosen in the interval $(\tau, \infty)$, however, this election restricts the range of dividend rates, $\delta$, that can be selected in $[0, \infty)$, and the election of the dividend rate in the restricted interval, finally determines a unique first debt payment, $\psi_1$, in $\left(\hat{\psi}_1, \infty\right)$. On the other hand, corollary 1 implies that refinancing is feasible under the same conditions it is feasible to issue new equity to pay the debt. As a consequence, the default probability at $\tau$ will not depend on how one expect this payment will be financed. Note also that $S(V, t) > 0 \forall t < \tau$ given limited liability, what would allow the firm to refinance at any $t < \tau$. Remark 1 indicates that in this context refinancing does not alter neither equity holders, nor debtholders wealth, and therefore there is no incentives to refinance before maturity. But even at maturity both will be indifferent between one possibility and the other. Assuming that the firm refinances its debt has however a logical advantage: It does not imply that corporate debt eventually disappears, which is the case if one assumes that debt payments are financed with new equity. As we will see, under the refinancing assumption, an stable corporate debt structure can be presumed, while some empirical evidence finds theoretical support.

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8 This does not happen if the firm issues a perpetuity, but even in this case, as the firm value grows, the corporate debt tends to be a smaller part of total firm value.
3 Particular Cases:

3.1 $n = 1$

Although any possible initial debt structure could be considered, we will assume in this case that $n$ remains constant along time. This means that a single zero coupon bond, with nominal $\psi$ and maturity at $\tau$, is replaced by a single zero coupon bond, with some nominal $\psi_1$ and some maturity $\tau_1 > \tau$, whenever $S(V, \tau) > 0$.

In order to show that a feasible refinancing contract exists, we need to describe how $S(V, \Theta, \tau)$ is to be valued. Specifically, we need to find the equity value at $\tau$, when the corporate debt consists in the payment of $\psi_1$ at $\tau_1 > \tau$, and the dividend rate is $\delta$, that is, $S(V, \Theta, \tau)$ for $\Theta \equiv (\delta, \psi_1, \tau_1)$. $S(V, \Theta, \tau)$ has two sources of value. On one hand the value associated to the dividends that will be received between $\tau$ and $\tau_1$, $D(V, \tau_1)$, and on the other hand the option value, $O(V, \delta, \tau_1)$, that comes from the possibility of acquiring the firm at $\tau_1$ by paying $\psi_1$. Applying risk neutral valuation we find that

$$D(V, \tau) = V \left(1 - e^{-\delta \tau_1}\right)$$

and

$$O(V, \tau) = Ve^{-\delta \tau_1} N(d_1) - \psi_1 e^{-r \tau_1} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{V}{\psi_1}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) \tau_1}{\sigma \sqrt{\tau_1}}$$

$$d_2 = d_1 - \sigma \sqrt{\tau_1}$$

Finally

$^9N(\cdot)$ denotes the standard normal cumulative distribution function.
\[ S(V, \Theta, \tau) = V (1 - e^{-\delta \tau_1}) + Ve^{-\delta \tau_1}N(d_1) - \psi_1 e^{-r \tau_1}N(d_2) \] (2)

\[ S(V, \Theta, \tau) \] is clearly a continuous function in \( \psi_1 \), with\(^{10}\)

\[ S(V, \Theta, \tau)_{\psi_1} = -e^{-r \tau_1}N(d_2) < 0 \]

\[ S(V, \Theta, \tau) \mid_{\psi_1=0} = V \]

\[ \lim_{\psi_1 \to -\infty} S(V, \Theta, \tau) = V (1 - e^{-\delta \tau_1}) \]

and B1 holds. On the other hand, \( S(V, \Theta, \tau) \) is a continuous function in \( \tau_1 \), with\(^{11}\)

\[ S(V, \Theta, \tau)_{\tau_1} = \delta Ve^{-\delta \tau_1} [1 - N(d_1)] + \]

\[ +Ve^{-\delta \tau_1}f(d_1)\frac{\sigma}{\sqrt{T_1}} + r\psi_1 e^{-r \tau_1}N(d_2) > 0 \]

\[ \lim_{\tau_1 \to -\infty} S(V, \Theta, \tau) = V \]

\[ \lim_{\tau_1 \to +} S(V, \Theta, \tau) = \begin{cases} V - \psi_1 & \text{if } V > \psi_1 \\ 0 & \text{if } V \leq \psi_1 \end{cases} \]

\[ = \text{Max}\{0, V - \psi_1\} \]

and B2 also holds. Finally, \( S(V, \Theta, \tau) \) is a continuous function in \( \delta \), with

\[ S(V, \Theta, \tau)_{\delta} = T_1 Ve^{-\delta \tau_1} [1 - N(d_1)] > 0 \]

\[ \lim_{\delta \to -\infty} S(V, \Theta, \tau) = V \]

\(^{10}\) \( S(V, \Theta, \tau)_j \) denotes the first derivative of \( S(V, \Theta, \tau) \) with respect to \( j \).

\(^{11}\) \( f(\cdot) \) denotes the standard normal density function.
and our last condition B3 holds as well. Note that $\hat{\psi}_1 = \psi$, and therefore $\varphi^{\psi_1} \equiv (\psi, \infty)$.

$S(V, \Theta, \tau)_{\psi_1, \psi_1} > 0$, implying that $S(V, \Theta, \tau)$ is a strictly convex function in $\psi_1$. Figure 1 describes case 1 and case 2 for $n = 1$.

Consider first case 1: For a given $\tau_1 \in (\tau, \infty)$, $\varphi^{\psi_1|\tau_1} \equiv [0, \delta_2)$. Both, 0 and $\delta_1$, lead to a unique $\psi_1 \in (\psi, \infty)$ such that $S(V, \Theta, \tau) = S(V, \tau)$. For $\delta_2$ however, $S(V, \Theta, \tau) = S(V, \tau)$ only in the limit case of $\psi_1 = \infty$, and there is not a finite $\psi_1$ large enough to make $\Theta$ feasible.

Case 2 can also be analyzed using Figure 1: For a given $\tau_1 \in (\tau, \infty)$, $\varphi^{\psi_1|\tau_1} \equiv [\psi_0^{0,\tau_1}, \infty)$. Both, $\psi_0^{0,\tau_1}$ and $\psi_1^{0,\tau_1}$, lead to a unique $\delta \in [0, \infty)$ such that $S(V, \Theta, \tau) = S(V, \tau)$. For any $\psi_1 < \psi_0^{0,\tau_1}$ however, any $\delta \in [0, \infty)$ leads to $S(V, \Theta, \tau) > S(V, \tau)$.

![Figure 1: Case 1 and case 2 for $n = 1$.](image-url)
An interesting aspect is that the credit spread \((C.S)\) on corporate debt that results from refinancing, will depend on the specific contract chosen in the feasible set \(\Theta \mid n = 1\); A feasible set that at the same time depends on the current firm value, the current nominal debt, the firm return volatility, and the riskfree interest rate. We know that for any \(\Theta \in \Theta \mid n = 1\), \(S(V, \Theta, \tau) = S(V, \tau)\), what can be expressed as \(V - \psi_1 e^{-RT_1} = V - \psi\), where \(\bar{R}\) is the interest rate associated to the new corporate debt. Then it is straightforward to show that

\[
R - r = \frac{\ln \left( \frac{\psi_1}{\psi} \right)}{T_1} - r
\]

Although \(\delta\) does not explicitly appear in the expression above, it does through its influence on \(\psi_1\) and \(T_1\).

For a given \((V, \psi, \sigma, r)\), the \(C.S\) will be a function of \((\delta, \psi_1, \tau_1)\). We have seen however that only two of these three elements are "freely" chosen. Consider for instance \((\delta, \tau_1)\) are selected according to the restriction imposed by corollary 2, then \(C.S = C.S \left( V, \psi, \sigma, r, \delta, \tau_1 \mid \delta T_1 < \ln \left( \frac{\psi}{\psi} \right) \right)\). In order to make some comparative statics with respect to the \(C.S\), we need to derive how \(\psi_1\) depends on \(\left( V, \psi, \sigma, r, \delta, \tau_1 \mid \delta T_1 < \ln \left( \frac{\psi}{\psi} \right) \right)\). Let define \(\Gamma(V, \Theta, \tau) = S(V, \Theta, \tau) - S(V, \tau)\). Then \(\Theta \in \Theta \mid n = 1\) if and only if \(\Gamma(V, \Theta, \tau) = 0\), and the derivative of \(\psi_1\) with respect to variable or parameter \(j\) will be given by

\[
\left( \psi_1 \right)_j = -\frac{\Gamma(V, \Theta, \tau)_j}{\Gamma(V, \Theta, \tau)_{\psi_1}}
\]

what leads to the following results:

\[
\left( \psi_1 \right)_V = -\frac{e^{-\delta T_1} [1 - N(d_1)]}{e^{r \tau_1} N(d_2)} < 0
\]

\[
\left( \psi_1 \right)_\psi = \frac{1}{e^{r \tau_1} N(d_2)} > 0
\]

\[
\left( \psi_1 \right)_\sigma = \frac{Ve^{-\delta T_1} f(d_1) \sqrt{T_1}}{e^{r \tau_1} N(d_2)} > 0
\]
\[
(\psi_1)_r = \psi_1 T_1 > 0
\]
\[
(\psi_1)_\delta = \frac{T_1 V e^{-\delta T_1 [1 - N(d_1)]}}{e^{-r T_1} N(d_2)} > 0
\]
\[
(\psi_1)_{\tau_1} = \frac{\delta V e^{-\delta T_1 [1 - N(d_1)]} + V e^{-\delta T_1} f(d_1) - \sigma \sqrt{T_1} + r \psi_1 e^{-r T_1} N(d_2)}{e^{-r T_1} N(d_2)} > 0
\]

and therefore
\[
C.S_V = \frac{(\psi_1)_V}{\psi_1 T_1} < 0
\]
\[
C.S_\psi = \frac{(\psi_1)_{\psi_1} \psi - \psi_1}{\psi_1 \psi T_1} < 0
\]
\[
C.S_\sigma = \frac{(\psi_1)_{\sigma}}{\psi_1 T_1} > 0
\]

The lower \( V \), or the higher \( \psi \), the lower the credit quality of the firm and the higher the \( C.S \) that it has to face to refinance.

\[
C.S_r = 0
\]

It is also reasonable to observe that the higher the firm risk, the higher the credit spread on the firm debt.

The credit spread is independent of the riskfree interest rate. This result may appear inconsistent with that derived in Merton [16], where the credit spread results a decreasing function of the riskfree interest rate. In our case however, we are analyzing the credit spread of a new issued debt at the moment it is issued, and with the goal of refinancing current debt. This is not the case in Merton [16]. As we will see later on, the debt risk, measured by the debt return volatility, will neither depend on the riskfree interest rate.
\[ C.S_\delta = \frac{(\psi_1)_\delta}{\psi_1 T_1} > 0 \]

Higher dividend rate means lower expected firm value growth and higher default probability, what leads to a higher credit spread.

\[ C.S_{\tau_1} = \frac{(\psi_1)_{\tau_1} - \psi_1 R}{\psi_1 T_1} > 0 \]

The statement \( C.S_{\tau_1} > 0 \) is not algebraically derived, but by means of simulations below. It again may seem inconsistent with the results in Merton [16]. It must be pointed out the substantial difference in the analysis of the time dependence followed by Merton and the one we drive here (not only the inclusion of a dividend rate): In fact he sets the so called ”quasi debt-to-firm value ratio” constant. In order to keep that ratio equal to a fixed \( q \) for a given firm value and interest rate, \( \psi_1 \) should be determined as \( qVe^{rT} \). In our case, however, we impose that the implied \( \psi_1 \) value is consistent with a feasible refinancing contract, what makes the ratio \( q \) to move from values below 1 to values above 1 for different maturities\(^{12}\).

Figure 2 represents the \( C.S \) as a function of the maturity date for different firm values. The base case in this and other simulations is \( V = 100, \psi = 50, \sigma = 0.2, r = 0.05 \) and \( \delta = 0.03 \). It seems clear that for reasonable parameter values, the \( C.S \) is an increasing function in \( \tau_1 \). The closer the firm value is to the default point \( \psi \), the lower the credit quality of the firm, and the higher the \( C.S \) that it has to face to refinance. At the same time, the closer the firm value is to the default point \( \psi \), the lower the maturity date sustainable in a refinancing contract given the restriction \( \delta T_1 < \ln \left( \frac{V}{\psi} \right) \).

Figure 3 provides the credit spread as a function of the maturity date for different current nominal debt payments. We can interpret the effect of a lower \( \psi \), in the same way we interpreted the effect of a higher \( V \).

Figure 4 again represents the \( C.S \) as a function of the maturity date, but now alternative dividend rates are considered. As before, the \( C.S \) appears to be an increasing function of the time to maturity chosen. It also reflects

\(^{12}\)Merton finds that the sign of \( C.S_{\tau_1} \) depends on whether \( q \) is higher, equal or lower than 1.
Figure 2: \( C.S \) as a function of the maturity date for different firm values. Base case: \( \psi = 50, \sigma = 0.2, r = 0.05 \) and \( \delta = 0.03 \).

Figure 3: \( C.S \) as a function of the maturity date for different current nominal debt payments. Base case: \( V = 100, \sigma = 0.2, r = 0.05 \) and \( \delta = 0.03 \).
the huge effect that the dividends have on the credit spread, and again this tend to infinity as $\delta T_1$ approaches $\ln \left( \frac{V}{\psi} \right)$. Note that for $\delta = 0$ any maturity appears feasible, and that even in this case that can be more reasonably compared with the analysis of Merton [16], the credit spread is monotonically increasing in the maturity date.

Figure 4: $C.S$ as a function of the maturity date for different dividends rates. Base case: $V = 100, \psi = 50, \sigma = 0.2$ and $r = 0.05$.

Figure 5 plots the $C.S$ as a function of the maturity date for different firm return volatilities, showing that it is an increasing function of the firm risk.

At this point it seems natural to follow Merton’s arguments and ask oneself whether or not the credit spread reflects the debt risk, that is: Can we say that the debt of firm $A$ is riskier (at the time of refinancing) than that of firm $B$, just because firm $A$ is facing a higher credit spread?. In order to answer this question we will use the same risk measure proposed by Merton: The instantaneous debt return volatility. As Merton indicates, a necessary condition for this relation to hold is that both, the credit spread and the debt return volatility, respond in the same way to changes in those variables and parameters that affect them. If we denote this variable as $\sigma_F$, standard arguments give that $\sigma_F F = F_V \sigma V$, what finally leads to $\sigma_F =$
Figure 5: \(C.S\) as a function of the maturity date for different firm return volatilities. Base case: \(V = 100, \psi = 50, r = 0.05\) and \(\delta = 0.03\).

\[
\frac{\psi}{\psi} \sigma e^{-\delta T_1} \left[ 1 - N(d_1) \right].
\]

In the same way happened with \((C.S)_{\tau_1}\), it is not possible to derive the sign of \((\sigma_F)_j\) algebraically. Simulations for reasonable values are presented below\textsuperscript{13}. Figures 6-9 repeat the exercise made in Figures 2-5, but now with the debt return volatility in the vertical axes.

In all cases it seems clear that the higher the maturity date, the higher (ceteris paribus) both, the \(C.S\) and \(\sigma_F\). Figures 2 and 6 also indicates that the lower the firm value the higher the credit spread, consistent with a higher debt risk. Comparing Figures 3 and 7 it appears that the effect of the current nominal debt on the credit spread is also consistent with the effect on the debt return volatility. Figures 4 and 8 consider different dividends rates. Both, the \(C.S\) and \(\sigma_F\) are an increasing function of \(\delta\). Finally Figures 5 and 9 reflect that the higher the firm risk, the higher the debt risk and the credit spread. Previous analysis is not enough however to argue that the credit spread fully reflects the debt risk (measured by \(\sigma_F\)), that is, it is not enough to argue that the \(C.S\) is a monotonically increasing function in \(\sigma_F\). As Merton [16] indicates, the fact that both, the \(C.S\) and \(\sigma_F\), respond in the

\textsuperscript{13}It is possible however to prove that \(\sigma_F\) is independent of the riskfree interest rate using \((\psi_1)_r\) to get \((\sigma_F)_r = 0\).
Figure 6: $\sigma_F$ as a function of the maturity date for different firm values. Base case: $\psi = 50$, $\sigma = 0.2$, $r = 0.05$ and $\delta = 0.03$.

Figure 7: $\sigma_F$ as a function of the maturity date for different current nominal debt payments. Base case: $V = 100$, $\sigma = 0.2$, $r = 0.05$ and $\delta = 0.03$. 
Figure 8: $\sigma_F$ as a function of the maturity date for different dividends rates. Base case: $V = 100, \psi = 50, \sigma = 0.2$ and $r = 0.05$.

Figure 9: $\sigma_F$ as a function of the maturity date for different firm return volatilities. Base case: $V = 100, \psi = 50, r = 0.05$ and $\delta = 0.03$.  

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same way to changes in those variables and parameters that affect them, is just a necessary condition for this relation to hold. If we were to answer the question of whether or not this relation takes place, we should say no. As an example consider the final value in Figures 4 and 8 for $\delta = 0.03$ and $\delta = 0.05$. The credit spread for $\delta = 0.03$ and $T_1 = 22$ is 0.07555, lower than the credit spread for $\delta = 0.05$ and $T_1 = 13$ which is 0.08069. However, the debt return volatility for $\delta = 0.03$ and $T_1 = 22$ is 0.18555, higher than the debt return volatility for $\delta = 0.05$ and $T_1 = 13$ which is 0.17739. Similar examples appear when two different maturities and firm values are considered, or different maturities and different firm risks, etc. The consequence is that we cannot say that the debt of firm A is riskier than the debt of firm B (at the time of refinancing) just because $A$ pays an interest rate higher than $B$. Can we at least say something about the relation between the C.S and $\sigma_F$ within a firm? That is, for a given $(V, \psi, \sigma, r)$, does the interest rate fully represent the debt risk derived from the specific refinancing contract chosen? The answer again is no because what figures 4 and 8 consider is precisely this case: $(V, \psi, \sigma, r)$ is given, but there is not a monotonic relation between C.S and $\sigma_F$. It is nevertheless feasible to argue that the credit spread is an increasing function of the debt risk (within a firm) when one of the three elements in $(\delta, \psi_1, \tau_1)$ is fixed. Figures 4 and 8 make clear that for a given maturity date, higher credit spread is associated to a higher debt risk. It is also clear that for a given dividend rate, higher credit spread implies a higher debt return volatility. What about fixing $\psi_1$? Figure 10 and 11 represent this case. For a given $\psi_1$ the credit spread is an increasing function of the debt risk.

Previous analysis implies that for $\delta = 0$, higher debt risk is associated to higher credit spread. This is not the case in the comparative statics performed in Merton [16].

**Implications of the Refinancing Assumption for $n = 1$.**

As has been pointed out, the credit spread will not depend on the riskfree interest rate. Nevertheless, this null influence holds only in the short run. The new nominal debt payment $\psi_1$ is an increasing function of the riskfree interest rate. As a consequence, the credit spread that the firm will have to face at $\tau_1$ to refinance $\psi_1$ will tend to be higher the higher the riskfree interest at $\tau$, given that this risk premium is an increasing function of the nominal debt to be refinanced. This "lagged effect" is consistent with the evidence provided by Guha, Hiris and Visviki [11]. They find that credit
Figure 10: $C.S$ as a function of $\psi_1$ for different dividends rates. Base case: $V = 100, \psi = 50, \sigma = 0.2$ and $r = 0.05$.

Figure 11: $\sigma_F$ as a function of $\psi_1$ for different dividends rates. Base case: $V = 100, \psi = 50, \sigma = 0.2$ and $r = 0.05$. 
spreads on bonds rated by Moody’s, are positively correlated with the two years lagged long-term Government Bond Yield.

On the other hand, the default probability at $\tau_1$ will also depend on $(V, \psi, \sigma, r, \delta, \tau_1 | \delta T_1 < \ln \left( \frac{V}{\psi} \right))$, and specifically on $V$. This probability will be

$$ P(\tau_1) = 1 - N(s_2) $$

where

$$ s_2 = \ln \left( \frac{V}{\psi} \right) + \left( \mu - \delta - \frac{\sigma^2}{2} \right) T_1 \sigma \sqrt{T_1} $$

Given $(\delta, \tau_1)$, we have seen that $\psi_1$ is a strictly decreasing function in $V$. At the same time

$$ (\psi_1)_{VV} = \left[ \frac{e^{-\delta T_1} f(d_1)}{e^{-r T_1} N(d_2)} - (\psi_1)_V \frac{f(d_2)}{N(d_2)} \right] \left[ \frac{\psi_1 - V(\psi_1)_V}{\sigma \sqrt{T_1} V \psi_1} \right] > 0 $$

implies also that $\psi_1$ is a strictly convex function in $V$. Finally $\Gamma(V, \Theta, \tau) = 0$ requires $\lim_{V \to 0} \psi_1 = \infty$ and $\lim_{V \to \infty} \psi_1 = \psi e^{r T_1}$. Figure 12 represents $\psi_1$ as a function of $V$ for a given $(\psi, \sigma, r, \delta, \tau_1 | \delta T_1 < \ln \left( \frac{V}{\psi} \right))$.

The representation of $\psi_1$ as a function of $V$ will help us to interpret the dependence of $P(\tau_1)$ on $V$:

$$ P(\tau_1)_V = -f(s_2) \left[ \frac{\psi_1 - V(\psi_1)_V}{\psi_1 V \sigma \sqrt{T_1}} \right] $$

Observe that a variation in $V$ has two different effects on the default probability at $\tau_1$, that go in the same direction: There is a direct effect given that a higher (lower) firm value at $\tau$, implies a higher (lower) expected value at $\tau_1$, and therefore a lower (higher) $P(\tau_1)$. This is what would appear in the standard case of Merton [16] in which $(\psi_1)_V = 0$. But now there is also an
indirect effect through \( \psi_1 \); A higher (lower) firm value implies a lower (higher) \( \psi_1 \), and therefore a lower (higher) \( P(\tau_1) \). This indirect or multiplier effect is not symmetric, as Figure 12 indicates. The same variation in \( V \) has a higher effect on \( \psi_1 \) when this variation is negative than when it is positive, and is also higher the lower the credit quality of the firm, that is, the closer is \( V \) to \( \psi \) at \( \tau \). Altman and Kao [1] find positive autocorrelation in S&P downgrades and upgrades for high-yield bonds, being this autocorrelation stronger for the case of downgrades. The asymmetric effect we describe here goes in the same direction. For those firms that suffer credit distress, refinancing the debt makes their credit quality to become even worst, given that they have to face increasing interest rates. This could tend to accelerate the lose of credit quality, and finally bring bankrupt, unless a positive firm value change inverts the process, or a substantial part of the corporate debt is replaced with new equity.
3.2 $n = 2$

We have analyzed the case in which the firm always maintains a single zero coupon bond as corporate debt. The main implications derived from assuming that the firm refines its debt in terms of credit spreads and default probabilities appear in this simple case. Exploring the situation in which the firm always refines with $n = 2$ is interesting however for several reasons: First, it can be seen as a simplification to short and long-term debt, what better represents the debt structure of a firm. Second, it incorporates the fact that equityholders do not only care about the debt that currently matures at the time of deciding whether or not satisfying it, but also about all future debt remaining. This makes for instance the current bankruptcy-triggering firm value to diverge from the current nominal debt payment, something that does not happen with $n = 1$. Giving that refinancing is feasible under the same conditions it is feasible to issue new equity, the set of refinancing contracts will also be affected.

For the sake of clarity we establish the main results of this subsection as propositions, leaving all the formal proofs in the appendix.

**Proposition 1:** $B1$-$B3$ hold for $n = 2$, making a refinancing contract with $n = 2$ feasible whenever $S(V, \tau) > 0$.

Proposition 1 does not assume any specific initial debt structure. However, we may think in a model in which the firm maintains an stable corporate debt structure with short and long-term debt, keeping the ratio short-term debt/long-term debt, and the time spread between them, constant. These can be associated to the specific industry in which the firm operates.

Assume also the firm does not pay dividends. Including this component of a refinancing contract would complicate the analysis, while most of the effects of this variable have been already described. The stable corporate debt structure translates into a vector $\Delta \equiv (n, \Phi, \Lambda)$, where $n = 2$, $\Phi = (1, \phi)$ and $\Lambda = (1, \eta)$; We will also assume a permanent $T_1$ (always feasible for $\delta = 0$). At any time the firm has to pay the short-term debt, it refines its total debt under $(\Delta, T_1)$.

The following proposition implies that the positive autocorrelation in the change of credit quality also holds in this case.
Proposition 2: Let $\bar{V}$ and $\bar{V}_1$ denote the bankruptcy-triggering firm value at $\tau$ and $\tau_1$ respectively. Then $\bar{V}_1$ is a strictly decreasing and strictly convex function in $V^{14}$, with $\lim_{V \to \bar{V}} = \infty$ and $\lim_{V \to \infty} = \bar{V} e^{r_1}$.

Proposition 2 leads to figure 13.

![Figure 13: $\bar{V}_1$ as a function of $V$.](image)

Figure 13: $\bar{V}_1$ as a function of $V$.

The shape of $\bar{V}_1$ as a function of $V$ is analogous to the one we found for $\psi_1$ under $n = 1$, and translates to the same effect on the evolution of the default probability. In fact, this (real world) probability at $\tau_1$ will be

$$P(\tau_1) = 1 - N(n_2)$$

where

$$n_2 = \frac{\ln \left( \frac{V}{\bar{V}} \right) + \left( \mu - \frac{\sigma^2}{2} \right) T_1}{\sigma \sqrt{T_1}}$$

and the same arguments we gave for $n = 1$ with respect to the asymmetric effect of a firm value change at $\tau$, also holds for $n = 2$.

$^{14}$This refers to the firm value at the time of refinancing $\tau$. 
On the other hand, refinancing makes $\bar{V}_1$ to be the new bankruptcy-triggering firm value, a critical threshold that will evolve over time as the firm refines its debt repeatedly. Although no explicit solution for it can be provided, it can be shown that it belongs to the same range in which KMV Corporation finds typically to be the default point.

**Proposition 3:** Let $\psi_1$ and $\psi_2$ be the new short and long-term debt resulting from refinancing at $\tau$, and let $T$ be the time spread between these payments. Then for any $\sigma \in (0, \infty)$, $\bar{V}_1 \in (\psi_1, \psi_1 + \psi_2 e^{-rT})$. Moreover $\lim_{\sigma \to 0} \bar{V}_1 = \psi_1 + \psi_2 e^{-rT}$ and $\lim_{\sigma \to \infty} \bar{V}_1 = \psi_1$.

With a database over 100,000 company-years of data and over 2,000 incidents of default or bankruptcy, KMV has found that firms generally default when the firm value lies somewhere between short-term debt and total debt in nominal terms\textsuperscript{15}. Clearly, $\bar{V}_1 \in (\psi_1, \psi_1 + \psi_2 e^{-rT})$ implies that $\bar{V}_1 \in (\psi_1, \psi_1 + \psi_2)$.

### 4 Conclusions:

The process of designing a refinancing contract has been described under the Modigliani-Miller Theorem. As far as we know, there is no previous published work describing how a contract of this type can be constructed. It has also been shown that whenever the firm has to satisfy a debt payment, designing a refinancing contract is feasible under the same conditions it is feasible to issue new equity, while keeping both, equity and debtholders wealth, unchanged. The fact that agents’ wealth is not altered through refinancing, makes the valuation formulas derived under the assumption of new equity issued still appropriate. Presuming that the firm refinances its debt has however an advantage: It allows us to consider an stable corporate debt structure. This means that the number of future payments, the weight of these payments as a proportion over total nominal debt, and the time spread between them, can be assumed constant, which is not possible under Merton or Geske’s original models. As an special case, it is feasible to derive a model in which the firm maintains an stable debt structure formed by short and long-term debt.

\textsuperscript{15}Crosbie, Peter J. [5].
Dividend rates, maturities, and nominal debt payments, have been modeled as part of the contract. At the same time, the credit spread and the debt risk on the new corporate debt contained in the contract, have been described as a function of the firm characteristics, the risk free interest rate, and the specific contract selected. Both appear as independent functions of the risk free interest rate (in the short-run), and increasing functions of the maturity date. We also show that they always behave in the same way with respect to changes in those variables and parameters that affect them. The fact that we analyze credit spread and debt risk for new issued debt, and with the goal of refinancing current debt, explains why these results are not the same than those in Merton [16].

Finally, some empirical evidence, namely, the "lagged effect" of the risk-free interest rate on the credit spreads on corporate bonds found by Guha, Hiris and Visviki [11], the positive autocorrelation in the change of credit quality documented by Altman and Kao [1], and the range of firm values in which KMV Corporation [5] finds typically to be the default point, is shown to be consistent with our refinancing assumption.

Under the Modigliani-Miller Theorem, there is no incentives to refinance, even at maturity. However, this assumption seems to bring more reasonable results than those derived under the assumption of new equity issued. Including the trade-off between the tax benefits on debt, and the default costs, could justify why firms should choose this alternative. Further research could also extend the proof of assumptions B1-B3 to an arbitrary number of payments in the contract.

5 Appendix:

5.1 Proof of proposition 1:

We need to derive what is the equity value when the debt structure of the firm is composed by short-term debt $\psi_1$, and long-term debt $\psi_2$; $\tau_1$ and $\tau_2$ are respectively short and long-term debt maturities. Suppose first an asset $C(V,t)$ with two sources of value: On one hand it pays the firm dividends between $\tau_1$ and $\tau_2$. On the other hand it gives the right to buy the firm at $\tau_2$ by paying $\psi_2$. In the same way we derived (2), it can be shown that at any $t \leq \tau_1$
\[ C(V,t) = V e^{-rT_1} (1 - e^{-\delta T}) + V e^{-\delta T_2} N(b_1) - \psi_2 e^{-rT_2} N(b_2) \]

where

\[
\begin{align*}
    b_1 &= \ln \left( \frac{V}{\psi_2} \right) + \left( r - \delta + \frac{\sigma^2}{2} \right) T_2 \\
    b_2 &= b_1 - \sigma \sqrt{T_2} \\
    T_1 &= \tau_1 - t \\
    T_2 &= \tau_2 - t \\
    T &= \tau_2 - \tau_1
\end{align*}
\]

If current and/or new potential equity holders pay \( \psi_1 \) at \( \tau_1 \), they acquire precisely this asset. As long as they have the option of refusing this payment

\[ S(V,\Theta,\tau_1) = \text{Max} \{ 0, C(V,\tau_1) - \psi_1 \} \]

\( C(V,\tau_1) \) is a strictly increasing function in \( V \), with \( \lim_{V \to 0} C(V,\tau_1) = 0 \) and \( \lim_{V \to \infty} C(V,\tau_1) = \infty \), therefore there exists a unique \( \bar{V}_1 \in (0, \infty) \) such that \( S(V,\Theta,\tau_1) > 0 \ \forall \ V > \bar{V}_1 \). This will be the implicit solution to

\[ S(\bar{V}_1,\Theta,\tau_1) = \bar{V}_1 (1 - e^{-\delta T}) + \bar{V}_1 e^{-\delta T} N(c_1) - \psi_2 e^{-rT} N(c_2) - \psi_1 = 0 \]

where

\[
\begin{align*}
    c_1 &= \ln \left( \frac{\bar{V}_1}{\psi_2} \right) + \left( r - \delta + \frac{\sigma^2}{2} \right) T \\
    c_2 &= c_1 - \sigma \sqrt{T}
\end{align*}
\]
At $\tau < \tau_1$ the equity also finds two sources of value. On one hand the value associated to dividends received from $\tau$ to $\tau_1$, $D(V, \tau)$; On the other hand the option value that appears due to the possibility of buying the described asset at $\tau_1$, $O(V, \tau)$. The first is

$$D(V, \tau) = V \left(1 - e^{-\delta \tau_1}\right)$$

The option value will be

$$O(V, \tau) = Ve^{-\delta \tau_1} \left(1 - e^{-\delta T}\right) N(a_1) + Ve^{-\delta T_2} N_2(a_1, b_1; \rho) - \psi_2 e^{-r T_2} N_2(a_2, b_2; \rho) - \psi_1 e^{-r T_1} N(a_2)$$

where

$$a_1 = \ln \left(\frac{V}{\bar{V}}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) T_1$$

$$a_2 = a_1 - \sigma \sqrt{T_1}$$

$$\rho = \sqrt{T_1 / T_2}$$

Finally

$$S(V, \Theta, \tau) = O(V, \tau) + D(V, \tau)
= V \left(1 - e^{-\delta \tau_1}\right) + Ve^{-\delta \tau_1} \left(1 - e^{-\delta T}\right) N(a_1) + Ve^{-\delta T_2} N_2(a_1, b_1; \rho) - \psi_2 e^{-r T_2} N_2(a_2, b_2; \rho) - \psi_1 e^{-r T_1} N(a_2)$$

This expression can be derived following the methodology applied to the valuation of compound options. For a detailed exposition see Kwok [13].

$N_2(a, b; \rho)$ represents the cumulative standard bivariate normal distribution function, with integration limits $a$ and $b$, and correlation coefficient $\rho$. 

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16This expression can be derived following the methodology applied to the valuation of compound options. For a detailed exposition see Kwok [13].
(3) converges to the expression given in Geske [9] for two periods as $\delta$ tends to zero. The first term on the r.h.s. is the value of the dividends that will be received from $\tau$ to $\tau_1$, the second is the value of the dividends that will be received from $\tau_1$ to $\tau_2$ if the firm does not default at $\tau_1$, and the last three terms represent the compound option on the firm.

According to the notation used in B1 and B2, we could express $\psi_2$ as $\phi \psi_1$, and $T_2$ as $(1 + \eta) T_1$. Then

$$S(V, \Theta, \tau) = V (1 - e^{-\delta T_1}) + V e^{-\delta T_1} \left(1 - e^{-\delta \eta T_1}\right) N(a_1) +$$

$$+ V e^{-\delta (1+\eta) T_1} N_2(a_1, b_1; \rho) - \phi \psi_1 e^{-r(1+\eta) T_1} N_2(a_2, b_2; \rho) - \psi_1 e^{-r T_1} N(a_2)$$

where

$$b_1 = \frac{\ln \left( \frac{V}{\phi \psi_1} \right) + \left( r - \delta + \frac{\sigma^2}{2} \right) (1 + \eta) T_1}{\sigma \sqrt{(1 + \eta) T_1}}$$

$$b_2 = b_1 - \sigma \sqrt{(1 + \eta) T_1}$$

$$\rho = \sqrt{\frac{1}{1 + \eta}}$$

At the same time, $\bar{V}_1$ will be the firm value that satisfies

$$S(\bar{V}_1, \Theta, \tau_1) = \bar{V}_1 (1 - e^{-\delta \eta T_1}) + \bar{V}_1 e^{-\delta \eta T_1} N(c_1) - \phi \psi_1 e^{-r \eta T_1} N(c_2) - \psi_1 = 0$$

where

$$c_1 = \frac{\ln \left( \frac{\bar{V}_1}{\phi \psi_1} \right) + \left( r - \delta + \frac{\sigma^2}{2} \right) \eta T_1}{\sigma \sqrt{\eta T_1}}$$

$$c_2 = c_1 - \sigma \sqrt{\eta T_1}$$
Proving that B1 holds requires to obtain the sign of $S(V, \Theta, \tau)_{\psi_1}$. In the process of differentiating $N_2(a, b; \rho)$ it is useful to use the following expression when applying Liebnitz’s rule:

$$N_2(a, b; \rho) = \int_{-\infty}^{a} f(x) N\left(\frac{b - \rho x}{\sqrt{1 - \rho^2}}\right) dx$$

On the other hand, it must be taken into account that $\bar{V}_1$ will change with $\psi_1$. In fact

$$(\bar{V}_1)_{\psi_1} = -\frac{S(\bar{V}_1, \Theta, \tau_1)_{\psi_1}}{S(\bar{V}_1, \Theta, \tau_1)_{\bar{V}_1}} = \frac{[1 + \phi e^{-r(T_1)} N(c_2)]}{1 - e^{-\delta T_1} [1 - N(c_1)]} > 0$$

At the same time, it can be easily proved that $c_i = \frac{b_i - \rho a_i}{\sqrt{1 - \rho^2}}$ for $i = 1, 2$. All this information, joint with the following definitions

$$m = (a_1)_{\psi_1} = (a_2)_{\psi_1}$$
$$n = (b_1)_{\psi_1} = (b_2)_{\psi_1}$$

lead to

$$S(V, \Theta, \tau)_{\psi_1} = V e^{-\delta T_1} (1 - e^{-\delta T_1}) f(a_1) m +$$
$$+Ve^{-\delta(1+\eta)T_1} \left[ f(a_1) N(c_1) m + f(b_1) N\left(\frac{a_1 - \rho b_1}{\sqrt{1 - \rho^2}}\right) n \right] -$$
$$-\phi e^{-r(1+\eta)T_1} N_2(a_2, b_2; \rho) -$$
$$-\phi \psi_1 e^{-r(1+\eta)T_1} \left[ f(a_2) N(c_2) m + f(b_2) N\left(\frac{a_2 - \rho b_2}{\sqrt{1 - \rho^2}}\right) n \right] -$$
$$-e^{-r T_1} N(a_2) - \psi_1 e^{-r T_1} f(a_2) m$$
If we now apply the following identities:

\[
\frac{a_1 - \rho b_1}{\sqrt{1 - \rho^2}} = \frac{a_2 - \rho b_2}{\sqrt{1 - \rho^2}}
\]

\[
Ve^{-\delta T_1} f (a_1) = \bar{V}_1 e^{-r T_1} f (a_2)
\]

\[
Ve^{-\delta (1+\eta) T_1} f (b_1) = \phi \psi_1 e^{-r (1+\eta) T_1} f (b_2)
\]

we finally get

\[
S (V, \Theta, \tau) \psi_1 = e^{-r T_1} f (a_2) m S (\bar{V}_1, \Theta, \tau_1) -
\]

\[
- \phi e^{-r (1+\eta) T_1} N (a_2, b_2; \rho) - e^{-r T_1} N (a_2)
\]

\[
= - \left[ \phi e^{-r (1+\eta) T_1} N (a_2, b_2; \rho) + e^{-r T_1} N (a_2) \right] < 0
\]

\[S (V, \Theta, \tau)\] is then a continuous function in \(\psi_1\), with \(S (V, \Theta, \tau) \psi_1 < 0\). This, joint with

\[S (V, \Theta, \tau) \mid_{\psi_1=0} = V\]

\[\lim_{\psi_1 \to -\infty} S (V, \Theta, \tau) = V \left( 1 - e^{-\delta T_1} \right)\]

proves that assumption B1 holds.

On the other hand, \(S (V, \Theta, \tau)\) is a continuous function in \(\tau_1\), with\(^\text{17}\)

\[S (\bar{V}_1, \Theta, \tau_1) = \delta \eta \bar{V}_1 e^{-\delta \eta T_1} \left[ 1 - N (c_1) \right] + r \eta \phi e^{-r \eta T_1} N (c_2) + \phi \psi_1 e^{-r \eta T_1} f (c_2) \frac{\sigma}{2 \sqrt{\eta T_1}} > 0,\]

and the same arguments applied to \(S (V, \Theta, \tau) \psi_1\) to derive \(S (V, \Theta, \tau) \tau_1\).
\[ S(V, \Theta, \tau)_{\tau_1} = \delta V e^{-\delta T_1} [1 - N(a_1)] + 
\] 
\[ + \delta (1 + \eta) V e^{-\delta (1+\eta) T_1} [N(a_1) - N_2(a_1, b_1; \rho)] + 
\] 
\[ + r (1 + \eta) \phi \psi_1 e^{-r(1+\eta) T_1} N_2(a_2, b_2; \rho) + 
\] 
\[ + \phi \psi_1 e^{-r(1+\eta) T_1} f(a_2) N(c_2) \frac{\sigma}{2 \sqrt{T_1}} + 
\] 
\[ + \phi \psi_1 e^{-r(1+\eta) T_1} f(b_2) N(c_2) \frac{\sigma}{2 \sqrt{T_1}} > 0 
\]

\[ \lim_{\tau_1 \to \infty} S(V, \Theta, \tau) = V \]

\[ \lim_{\tau_1 \to \tau} S(V, \Theta, \tau) = \begin{cases} 
V - \psi_1 - \phi \psi_1 & \text{if } V > \psi_1 + \phi \psi_1 \\
0 & \text{if } V \leq \psi_1 + \phi \psi_1 
\end{cases} 
\]

\[ = \text{Max} \{0, V - \psi_1 - \phi \psi_1\} \]

proving that assumption B2 also holds. Finally, \( S(V, \Theta, \tau) \) is a continuous function in \( \delta \), with\(^{18}\)

\[ S(V, \Theta, \tau)_\delta = T_1 V e^{-\delta T_1} [1 - N(a_1)] + 
\]

\[ (1 + \eta) T_1 V e^{-\delta (1+\eta) T_1} [N(a_1) - N_2(a_1, b_1; \rho)] > 0 
\]

\[ \lim_{\delta \to \infty} S(V, \Theta, \tau) = V 
\]

and assumption B3 is satisfied.■

\(^{18}\)Use

\[ S(V_1, \Theta, \tau_1)_\delta = \eta T_1 \dot{V}_1 e^{-\delta \eta T_1} [1 - N(c_1)] > 0, \]

and again the same arguments applied to \( S(V, \Theta, \tau_1)_{\psi_1} \) to get \( S(V, \Theta, \tau_1)_\delta \).
It is possible to analyze graphically case 1 for \( n = 2 \). Let denote \( S (V, \Theta, \tau) \) simply as \( S \), and let show that \( S \) is strictly convex in \((\psi_1, \psi_2)\), where we do not impose the restriction \( \psi_2 = \phi \psi_1 \).

\[
S_{\psi_1} = -e^{-rT_1} N (a_2)
\]

\[
S_{\psi_2} = -e^{-rT_2} N (a_2, b_2; \rho)
\]

\[
S_{\psi_1 \psi_1} = \frac{e^{-rT_1} f (a_2)}{V_1 \sigma \sqrt{T_1} \{1 - e^{-\delta T_1} [1 - N (c_1)]\}} > 0
\]

\[
S_{\psi_2 \psi_2} = \frac{e^{-rT_2} f (a_2) [N (c_2)]^2}{V_1 \sigma \sqrt{T_1} \{1 - e^{-\delta T_1} [1 - N (c_1)]\}} + \frac{e^{-rT_2} f (b_2) N \left( \frac{a_2 - \rho b_2}{\sqrt{1 - \rho^2}} \right)}{\psi_2 \sigma \sqrt{T_2} \psi_2} > 0
\]

\[
S_{\psi_1 \psi_2} = \frac{e^{-rT_2} f (a_2) N (c_2)}{V_1 \sigma \sqrt{T_1} \{1 - e^{-\delta T_1} [1 - N (c_1)]\}}
\]

\[
S_{\psi_1 \psi_1} S_{\psi_2 \psi_2} - (S_{\psi_1 \psi_2})^2 = \frac{e^{-r(T_1 + T_2)} f (a_2) f (b_2) N \left( \frac{a_2 - \rho b_2}{\sqrt{1 - \rho^2}} \right)}{V_1 \psi_2 \sigma \sqrt{T_1} \sqrt{T_2} \sqrt{T_1} \{1 - e^{-\delta T_1} [1 - N (c_1)]\}} > 0
\]

The strict convexity of \( S \), joint with \( S \big|_{(\psi_1, \psi_2) = (0, 0)} = V \), and \( \lim_{(\psi_1, \psi_2) \to (\infty, \infty)} S = V \left(1 - e^{-\delta T_1}\right) \), leads to figure 14.
Figure 14: Case 1 for $n = 2$. 
5.2 Proof of proposition 2:

Let $\psi$ be the debt payment to be satisfied at $\tau$. Then

$$S(V, \tau) = VN(w_1) - \phi \psi e^{-r\eta T_1} N(w_2) - \psi$$

where

$$w_1 = \frac{\ln \left( \frac{V}{\psi} \right) + \left( r + \frac{\sigma^2}{2} \right) \eta T_1}{\sigma \sqrt{\eta T_1}}$$

$$w_2 = w_1 - \sigma \sqrt{\eta T_1}$$

and $S(V, \tau) > 0 \ \forall V > \bar{V}$, being $\bar{V}$ the implicit solution to $S(\bar{V}, \tau) = 0$. On the other hand

$$S(V, \Theta, \tau) = VN_2(k_1, l_1; \rho) - \phi \psi e^{-r(1+\eta)T_1} N_2(k_2, l_2; \rho) - \psi_1 e^{-rT_1} N(k_2)$$

where

$$k_1 = \frac{\ln \left( \frac{V}{\psi_1} \right) + \left( r + \frac{\sigma^2}{2} \right) T_1}{\sigma \sqrt{T_1}}$$

$$k_2 = k_1 - \sigma \sqrt{T_1}$$

$$l_1 = \frac{\ln \left( \frac{V}{\psi_1} \right) + \left( r + \frac{\sigma^2}{2} \right) (1 + \eta) T_1}{\sigma \sqrt{(1 + \eta) T_1}}$$

$$l_2 = l_1 - \sigma \sqrt{(1 + \eta) T_1}$$

$$\rho = \sqrt{\frac{1}{1 + \eta}}$$
and $\bar{V}_1$ is the implicit solution to

$$S (\bar{V}_1, \Theta, \tau_1) = \bar{V}_1 N (h_1) - \phi \psi_1 e^{-r\eta T_1} N (h_2) - \psi_1 = 0$$

with

$$h_1 = \ln \left( \frac{\bar{V}_1}{\psi_1} \right) + \left( r + \frac{\sigma^2}{2} \right) \eta T_1 \sqrt{\eta T_1}$$

$$h_2 = h_1 - \sigma \sqrt{\eta T_1}$$

If we define $\theta = \frac{\bar{V}_1}{\psi_1}$, then previous expressions can be written as

$$\theta N (h_1) - \phi e^{-r\eta T_1} N (h_2) - 1 = 0$$

where

$$h_1 = \ln \left( \frac{\theta}{\phi} \right) + \left( r + \frac{\sigma^2}{2} \right) \eta T_1 \sqrt{\eta T_1}$$

The result is that $\theta$ is a constant, that is, $\theta$ will not depend on the firm value at $\tau$ (although $\bar{V}_1$ and $\psi_1$ will). Note also that $\theta = \frac{\bar{V}_1}{\psi}$. We can express condition $\Gamma (V, \Theta, \tau) = S (V, \Theta, \tau) - S (V, \tau) = 0$ as

$$\Gamma (V, \Theta, \tau) = [V N_2 (k_1, l_1; \rho) - \bar{V}_1 \phi e^{-r(1+\eta)T_1} N_2 (k_2, l_2; \rho) - \bar{V}_1 \phi e^{-rT_1} N (k_2)] -$$

$$- [V N (w_1) - \bar{V}_1 \phi e^{-r\eta T_1} N (w_2) - \bar{V}_1 \frac{1}{\phi}] = 0$$

(4)

where
(4) represents $\bar{V}_1$ as a function of $V$. As $V$ tends to $\bar{V}$, $\bar{V}_1$ tends to infinity. At the same time, $\lim_{V \to \infty} \bar{V}_1 = \bar{V} e^{r T_1}$\textsuperscript{19}. It can also be proved that $\bar{V}_1$ is a strictly decreasing function in $V$. In fact\textsuperscript{20}

$$
\begin{align*}
(\bar{V}_1)_V &= -\frac{\Gamma(V, \Theta, \tau)_V}{\Gamma(V, \Theta, \tau)_{\bar{V}_1}} \\
&= -\frac{N(w_1) - N_2(k_1, l_1; \rho)}{\phi e^{-r(1+\eta)T_1} N_2(k_2, l_2; \rho) + \frac{1}{\sigma} e^{-r T_1} N(k_2)} < 0
\end{align*}
$$

Figure 15 represents $S(V, \tau)$ and $S(V, \Theta, \tau)$ as a function of $V$\textsuperscript{21}. Clearly, for any $V > \bar{V}$, $N(w_1) > N_2(k_1, l_1; \rho)$, given that these are the derivatives of $S(V, \tau)$ and $S(V, \Theta, \tau)$ with respect to $V$. This implies $(\bar{V}_1)_V < 0$. $(\bar{V}_1)_{VV} > 0$ follows from the fact that $\phi e^{-r(1+\eta)T_1} N_2(k_2, l_2; \rho) + \frac{1}{\sigma} e^{-r T_1} N(k_2)$ is a strictly increasing function in $V$, while $N(w_1) - N_2(k_1, l_1; \rho)$ is strictly decreasing $\forall V > \bar{V}$ as Figure 15 indicates, and actually tends to zero as $V$ grows.\textsuperscript{\*}

\textsuperscript{19}Note that this implies that $\lim_{V \to -\infty} \psi_1 = \psi e^{r T_1}$ and $\lim_{V \to -\infty} \phi \psi_1 = \phi \psi e^{r T_1}$, that is, as the default risk tends to zero, new debt payments tend to current debt payments capitalized at the riskfree interest rate.

\textsuperscript{20}Use the arguments in Appendix 5.1 to derive $S(V, \Theta, \tau)_V$ and $S(V, \Theta, \tau)_{\bar{V}_1}$.

\textsuperscript{21}$x = V - \bar{V} \phi e^{-r T_1} - \bar{V} \frac{1}{\sigma}$ and $y = V - \bar{V}_1 \phi e^{-r(1+\eta)T_1} - \bar{V}_1 \frac{1}{\sigma} e^{-r T_1}$. $\bar{V} \phi e^{-r T_1} + \bar{V} \frac{1}{\sigma} < \bar{V}_1 \phi e^{-r(1+\eta)T_1} + \bar{V}_1 \frac{1}{\sigma} e^{-r T_1}$ given $\bar{V}_1 > \bar{V} e^{r T_1}$. 

\textsuperscript{\*}
5.3 Proof of proposition 3:
Let \( Z(V) = VN(g_1) - \psi_2 e^{-rT}N(g_2) - \psi_1 \)
where
\[
g_1 = \frac{\ln \left( \frac{V}{\psi_2} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
\]
\[
g_2 = g_1 - \sigma \sqrt{T}
\]
then \( \bar{V}_1 \equiv V \mid Z(V) = 0 \).
Suppose first \( \sigma \in (0, \infty) \) and \( V = \psi_1 \), then
\[
Z(\psi_1) = \psi_1 N(g_1) - \psi_2 e^{-rT}N(g_2) - \psi_1
\]
\[
= -\psi_1 \left[ 1 - N(g_1) \right] - \psi_2 e^{-rT}N(g_2) < 0
\]
this, joint with $Z_V = N(g_1) > 0$, implies that $\bar{V}_1 > \psi_1$ $\forall \sigma \in (0, \infty)$. We have that $\lim_{\sigma \to \infty} N(g_1) = 1$ and $\lim_{\sigma \to \infty} N(g_2) = 0$, therefore $\lim_{\sigma \to \infty} Z(V) = V - \psi_1 = 0 \Leftrightarrow V = \psi_1$, proving $\lim_{\sigma \to \infty} \bar{V}_1 = \psi_1$.

On the other hand, consider $V = \psi_1 + \psi_2 e^{-rT}$, then

$$Z(\psi_1 + \psi_2 e^{-rT}) = -\psi_1 [1 - N(g_1)] + \psi_2 e^{-rT} [N(g_1) - N(g_2)]$$

$$\lim_{\sigma \to 0} Z(\psi_1 + \psi_2 e^{-rT}) = 0$$

because

$$\lim_{\sigma \to 0} N(g_1 \mid V = \psi_1 + \psi_2 e^{-rT}) = \lim_{\sigma \to 0} N(g_2 \mid V = \psi_1 + \psi_2 e^{-rT}) = 1.$$ As a result, $\bar{V}_1 = \psi_1 + \psi_2 e^{-rT}$ in this limit case. Given that $Z_\sigma > 0$ we also have that $Z(\psi_1 + \psi_2 e^{-rT}) > 0$ $\forall \sigma \in (0, \infty)$. This, joint again with $Z_V = N(g_1) > 0$, implies $\bar{V}_1 < \psi_1 + \psi_2 e^{-rT}$ $\forall \sigma \in (0, \infty)$, and concludes the proof.

References


