Predictability and Performance*

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First draft: January 2017. This version: May 2017

Abstract

Different types of conditionally efficient returns have the same conditional performance, but may differ considerably in their unconditional performance. This paper analyzes five types of conditionally efficient strategies, with a focus on how return predictability drives the differences in their unconditional Sharpe and Sortino ratios, and their coefficients of asymmetry and kurtosis. We provide formulas that decompose these measures into interpretable components that are driven by the maximum conditional Sharpe ratio. We find strong differences between the five types of strategies across several combinations of mean and variance predictability.

Keywords: Asymmetry, Conditionally efficient returns, Kurtosis, Mean-variance framework, Sharpe ratio, Sortino ratio.

JEL: G11, G12.

*We would like to thank seminar participants at Queens College CUNY for helpful comments and suggestions. Of course, the usual caveat applies. Financial support from the PSC-CUNY Research Awards (Peñaranda) is acknowledged.

1 Introduction

Risk and risk premia of returns change over time, and investors can exploit their time variation by means of conditionally efficient (CE) portfolio strategies, which yield the maximum conditional Sharpe ratio. These strategies can be understood as an extension of the classic mean-variance framework of Markowitz (1952) that considers return predictability. Hansen and Richard (1987) developed the corresponding theoretical framework, and Ferson and Siegel (2001) used that framework to guide portfolio choice.

Importantly, all CE returns are constructed by scaling a common optimal combination of risky assets. The time variation of this scale does not affect the conditional performance of CE returns, but it is critical for their unconditional performance. This paper analyzes five relevant types of CE returns, or equivalently five different choices of this time-varying scale. We focus on how return predictability drives their differences in terms of unconditional Sharpe ratios (Sharpe, 1994) and Sortino ratios (Sortino and Forsey, 1996), and their coefficients of asymmetry and kurtosis.

We use the residual Sharpe ratio[1] of Peñaranda (2016) to decompose the unconditional Sharpe ratio of a portfolio return as its residual ratio penalized by the coefficient of determination in the forecasting regression of the return. We also use the residual ratio to decompose the unconditional Sortino ratio of a portfolio return as its residual ratio penalized by a covariance term. In particular, the covariance between the return conditional variance and the conditional semivariance of the standardized forecast error. We use these decompositions to understand the performance differences between types of CE returns.

The first two types of CE returns that we study are specially relevant for practitioners who construct portfolios with a constant target for the conditional variance or mean of their portfolio return. We denote them CE1 and CE2 returns, respectively. The next two types that we study yield the maximum residual and unconditional Sharpe ratios. We denote them CE3 and CE4 returns, respectively. The final CE returns that we study, denoted CE5, maximize the unconditional Sortino ratio among CE returns.

These five types of returns yield the same maximum conditional Sharpe ratio, but they may have very different unconditional Sharpe and Sortino ratios. We find that the performance ratios depend on different properties of a single variable, the maximum conditional Sharpe ratio. In fact, a constant maximum conditional Sharpe ratio is the condition that makes these CE subsets

[1] Unlike traditional Sharpe ratios, residual ratios penalize the average conditional variance of the portfolio return instead of its total variance. The average conditional variance can be interpreted as the residual variance in the forecasting regression of the portfolio return.

equivalent.[2] Importantly, this condition does not mean lack of predictability, because there is time-variation in both risk premia and risk that can be compatible with this condition. This condition does not mean that predictability is irrelevant either, because the Sharpe ratio of these strategies can still be higher than the maximum one obtained from fixed weight (FW) strategies.

The five types of returns also have the same conditional coefficients of asymmetry and kurtosis, but they may differ considerably in their unconditional counterparts. We decompose the unconditional coefficients into different sources of asymmetry and kurtosis that are driven by the maximum conditional Sharpe ratio.

We find strong differences between the five types of strategies across several combinations of mean and variance predictability, even though there is a single risky asset in our examples. CE1 returns, which keep a constant variance target, yield higher performance ratios than FW returns, but obviously lower than CE3, CE4 or CE5 returns. The asymmetry of CE1 returns is similar to FW returns, but they remove the kurtosis that is derived from time variation in the conditional variance of the risky asset return. In fact, CE1 returns are usually the CE return with the lowest kurtosis. Interestingly, keeping a constant mean target yields very different results. In our examples, the performance ratios of CE2 returns may be much worse than FW returns, or may not even exist.

By definition, CE3 and CE4 returns are optimal with respect to the residual and Sharpe ratios, respectively. In our examples CE3 and CE4 returns are not too different in terms of these ratios, but they are more different in terms of the Sortino ratio, which is higher for CE3 returns. Moreover, these two returns can be very different in terms of asymmetry and kurtosis, with CE3 returns generating more positive skewness and excess kurtosis. However, CE5 returns may have the most extreme behavior. They yield the highest Sortino ratio among CE returns by definition, but sometimes jointly with extreme values of the coefficients of asymmetry and kurtosis, and very low residual and Sharpe ratios.

The rest of the paper is organized as follows. Section 2 reviews conditionally efficient returns, and develops our decompositions of unconditional Sharpe and Sortino ratios. Section 3 shows our propositions on the performance properties of five subsets of conditionally efficient returns, while Section 4 studies their asymmetry and kurtosis. Section 5 illustrates and quantifies our results by means of a single risky return, and Section 6 concludes. Proofs and auxiliary results are relegated to appendices.

[2] The equivalence of CE5 returns also requires a second condition related to the conditional distribution of the forecast errors of CE returns, as stated in point 2 of Corollary 1.

2 Predictability and Performance

The investment set has a safe asset with gross return $R_{0,t+1}$ known at t, and some risky assets with excess return vector $\mathbf{r}_{t+1}$. The payoff from t to $t+1$ of a unit cost portfolio, or gross return, is

$$R_{p,t+1} = R_{0,t+1} + r_{p,t+1}, \qquad r_{p,t+1} = \mathbf{w}_t'\mathbf{r}_{t+1},$$

where $r_{p,t+1}$ denotes the portfolio excess return, and the vector $\mathbf{w}_t$ denotes the wealth fraction that is invested in each risky asset. In our notation, an object with subindex t is some function of the information at t. For instance, the conditional first and second moments of the excess returns are denoted

$$\boldsymbol{\mu}_t = E_t\left(\mathbf{r}_{t+1}\right), \qquad \boldsymbol{\Sigma}_t = Var_t\left(\mathbf{r}_{t+1}\right).$$

We assume that $\boldsymbol{\mu}_t$ has at least one nonzero entry and $\boldsymbol{\Sigma}_t$ is nonsingular with probability one to simplify the exposition and avoid trivial settings.

The conditional Sharpe ratio of an excess return $r_{p,t+1}$ is defined by

$$S_{pt} = \frac{E_t\left(r_{p,t+1}\right)}{Var_t^{1/2}\left(r_{p,t+1}\right)} = \frac{\mathbf{w}_t'\boldsymbol{\mu}_t}{\mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t}.$$

When investors are concerned about asymmetric returns, they may prefer to measure risk with the semivariance instead of the variance, and measure performance with the Sortino ratio instead of the Sharpe ratio. The semivariance, or lower partial moment of order 2, penalizes only returns below a reference point or threshold. If we use the safe asset return as the return threshold,[3] or equivalently a zero threshold for excess returns, then we can define the conditional Sortino ratio as

$$\mathfrak{S}_{pt} = \frac{E_t\left(r_{p,t+1}\right)}{E_t^{1/2}\left[r_{p,t+1}^2 I\left(r_{p,t+1} \leq 0\right)\right]},$$

where $I(A)$ is the indicator function that returns 1 if A is true and 0 otherwise.

2.1 Conditionally Efficient Returns

Conditionally efficient (CE) returns are the counterpart of the textbook mean-variance efficient returns when investors exploit return predictability. These returns achieve the maximum conditional Sharpe ratio with the portfolio weights

$$\mathbf{w}_{ct} = \omega_t \boldsymbol{\varphi}_t, \qquad \boldsymbol{\varphi}_t = \boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t, \tag{1}$$

[3] Pedersen and Satchell (2002) study the theoretical foundations of this measure, and advocate the use of the safe asset return as the threshold.

for some chosen scale ω_t that depends on information., while $\boldsymbol{\varphi}_t$ represents the optimal combination of risky assets. Equivalently, the CE excess returns can be expressed as

$$r_{c,t+1} = \omega_t r^*_{t+1}, \qquad r^*_{t+1} = \boldsymbol{\varphi}'_t \mathbf{r}_{t+1},$$

where r^*_{t+1} represents the excess return from the optimal combination of risky assets. The conditional mean and variance of r^*_{t+1} are equal, and denoted

$$\mathcal{S}^2_t = E_t\left(r^*_{t+1}\right) = Var_t\left(r^*_{t+1}\right) = \boldsymbol{\mu}'_t \boldsymbol{\Sigma}^{-1}_t \boldsymbol{\mu}_t. \tag{2}$$

Our assumptions on $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ imply that $\mathcal{S}^2_t$ is different from zero with probability one.

In the following, we will focus on scales $\omega_t > 0$. The choice $\omega_t = 0$ is equivalent to holding the safe asset, and it is also efficient, but its Sharpe ratio is not defined. As we change the scale ω_t to obtain different CE returns, the conditional mean and variance that we achieve are

$$E_t\left(r_{c,t+1}\right) = \omega_t \mathcal{S}^2_t, \qquad Var_t\left(r_{c,t+1}\right) = \omega^2_t \mathcal{S}^2_t,$$

but we always obtain the same conditional Sharpe ratio, which is the positive square root of (2)

$$S_{ct} = \frac{E_t\left(r_{c,t+1}\right)}{Var^{1/2}_t\left(r_{c,t+1}\right)} = \frac{E_t\left(r^*_{t+1}\right)}{Var^{1/2}_t\left(r^*_{t+1}\right)} = \mathcal{S}_t.$$

Equivalently, (2) denotes the square of the maximum conditional Sharpe ratio. For instance, if the conditional correlations across returns are zero, then $\mathcal{S}^2_t = \sum_{i=1}^n \mu^2_{ti}/\sigma^2_{ti}$ and $\mathcal{S}_t = +\left(\sum_{i=1}^n \mu^2_{ti}/\sigma^2_{ti}\right)^{1/2}$. If there is only one risky return, then $\mathcal{S}^2_t = \mu^2_t/\sigma^2_t$ and $\mathcal{S}_t = |\mu_t|/\sigma_t$.

CE returns are also equivalent with respect to other measures of conditional performance. All CE returns yield the same conditional Sortino ratio[4] , the ratio provided by the optimal combination of risky assets

$$\mathfrak{S}_{ct} = \frac{E_t\left(r_{c,t+1}\right)}{E^{1/2}_t\left[r^2_{c,t+1} I\left(r_{c,t+1} \leq 0\right)\right]} = \frac{E_t\left(r^*_{t+1}\right)}{E^{1/2}_t\left[r^{*2}_{t+1} I\left(r^*_{t+1} \leq 0\right)\right]}.$$

2.2 Unconditional Sharpe Ratio Decomposition

We can always decompose an excess return as

$$r_{p,t+1} = E_t\left(r_{p,t+1}\right) + e_{p,t+1} = E_t\left(r_{p,t+1}\right) + Var^{1/2}_t\left(r_{p,t+1}\right) u_{p,t+1}, \tag{3}$$

where $e_{p,t+1}$ is the forecast error with zero conditional mean, and $u_{p,t+1}$ is the standardized forecast error with unit conditional variance. Similarly, we can decompose the deviation of the excess return with respect to its unconditional mean as

$$r_{p,t+1} - E\left(r_{p,t+1}\right) = d_{pt} + e_{p,t+1}, \tag{4}$$

[4] Unlike the conditional Sharpe ratio of CE returns, this conditional Sortino ratio does not need to be the maximum one that can be achieved from the vector $\mathbf{r}_{t+1}$. Of course, if this vector is conditionally Gaussian, then CE returns are also optimal with respect to this measure.

where

$$d_{pt} = E_t\left(r_{p,t+1}\right) - E\left(r_{p,t+1}\right)$$

is the deviation of the conditional mean with respect to its average. Using this notation, we can decompose the return unconditional variance as

$$Var\left(r_{p,t+1}\right) = E\left(r_{p,t+1} - E\left(r_{p,t+1}\right)\right)^2$$
$$= E\left(e_{p,t+1}^2\right) + E\left(d_{pt}^2\right) = E\left(Var_t\left(r_{p,t+1}\right)\right) + Var\left(E_t\left(r_{p,t+1}\right)\right).$$

In Section 4 we provide similar decompositions for higher order moments, where additional cross-moments appear.

The performance of a portfolio strategy is often evaluated by a Sharpe ratio computed from the historical mean and variance of its excess return, that is, from unconditional moments. The unconditional Sharpe ratio of an excess return $r_{p,t+1}$ is the square root of

$$S_p^2 = \frac{E^2\left(r_{p,t+1}\right)}{Var\left(r_{p,t+1}\right)} = \frac{E^2\left(E_t\left(r_{p,t+1}\right)\right)}{E\left(Var_t\left(r_{p,t+1}\right)\right) + Var\left(E_t\left(r_{p,t+1}\right)\right)}$$
$$= \frac{E^2\left(\mathbf{w}_t'\boldsymbol{\mu}_t\right)}{E\left(\mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t\right) + Var\left(\mathbf{w}_t'\boldsymbol{\mu}_t\right)}.$$

Unconditional Sharpe ratios penalize the two sources of return variance, the mean of the conditional variance and the variance of the conditional mean. The former captures the variance of the forecast error, while the latter captures the time variation in the forecast. In Section 4 we provide similar decompositions for higher order moments, where additional cross-moments appear.

We can decompose the squared unconditional Sharpe ratio as

$$S_p^2 = \mathbb{S}_p^2\left(1 - R_p^2\right), \tag{5}$$

where

$$\mathbb{S}_p^2 = \frac{E^2\left(r_{p,t+1}\right)}{E\left(Var_t\left(r_{p,t+1}\right)\right)},$$
$$R_p^2 = \frac{Var\left(E_t\left(r_{p,t+1}\right)\right)}{Var\left(r_{p,t+1}\right)} = \frac{Var\left(E_t\left(r_{p,t+1}\right)\right)}{E\left(Var_t\left(r_{p,t+1}\right)\right) + Var\left(E_t\left(r_{p,t+1}\right)\right)}.$$

The ratio $\mathbb{S}_p^2$ is the square of the residual Sharpe ratio, defined in Peñaranda (2016) as a Sharpe ratio that measures risk with $E\left(Var_t\left(r_{p,t+1}\right)\right)$ instead of $Var\left(r_{p,t+1}\right)$. The second component is the coefficient of determination of $r_{p,t+1}$, a measure of predictability that considers the fraction of the time-variation in $r_{p,t+1}$ that is due to the time-variation in the forecast $E_t\left(r_{p,t+1}\right)$.

All CE returns have the same squared conditional Sharpe ratio, but they may differ in their unconditional Sharpe ratio. The two components of the unconditional Sharpe ratio of CE returns are

$$\mathbb{S}_{\omega}^2 = \frac{E^2\left(\omega_t \mathcal{S}_t^2\right)}{E\left(\omega_t^2 \mathcal{S}_t^2\right)}, \qquad R_{\omega}^2 = \frac{Var\left(\omega_t \mathcal{S}_t^2\right)}{E\left(\omega_t^2 \mathcal{S}_t^2\right) + Var\left(\omega_t \mathcal{S}_t^2\right)},$$

and hence the unconditional Sharpe ratios of CE returns are equal to

$$S_{\omega}^2 = \mathbb{S}_{\omega}^2\left(1 - R_{\omega}^2\right) = \frac{E^2\left(\omega_t \mathcal{S}_t^2\right)}{E\left(\omega_t^2 \mathcal{S}_t^2\right) + Var\left(\omega_t \mathcal{S}_t^2\right)}.$$

Therefore, only two variables are relevant for this performance measure, the scale ω_t and the squared maximum conditional Sharpe ratio $\mathcal{S}_t^2$. The former is chosen by the investor, while the latter is given by the mean-variance properties of the vector $\mathbf{r}_{t+1}$.

2.3 Unconditional Sortino Ratio Decomposition

Following the decomposition of an excess return in (3), we can express the conditional semivariance of $r_{p,t+1}$ as

$$E_t\left[r_{p,t+1}^2 I\left(r_{p,t+1} \leq 0\right)\right] = Var_t\left(r_{p,t+1}\right) G_{pt},$$

where

$$G_{pt} = E_t\left[\left(S_{pt} + u_{p,t+1}\right)^2 I\left(u_{p,t+1} \leq -S_{pt}\right)\right],$$

is the conditional semivariance of the standardized forecast error $u_{p,t+1}$ with a threshold of $-S_{pt}$. This semivariance depends only on S_{pt} and the conditional distribution[5] of $u_{p,t+1}$. Importantly, given that distribution, G_{pt} decreases[6] with S_{pt}. It grows without bound as $S_{pt} \to -\infty$, and converges to zero as $S_{pt} \to +\infty$. If $r_{p,t+1}$ is conditionally symmetric, then $G_{pt} = 0.5$ at $S_{pt} = 0$.

We can always express the conditional Sortino ratio as the conditional Sharpe ratio divided by $G_{pt}^{1/2}$

$$\mathfrak{S}_{pt} = \frac{S_{pt}}{G_{pt}^{1/2}}.$$

Let us apply this expression to CE returns. All these returns have the same standardized forecast error, given by the optimal combination of risky assets

$$\frac{r_{c,t+1} - E_t\left(r_{c,t+1}\right)}{Var_t^{1/2}\left(r_{c,t+1}\right)} = \frac{r_{t+1}^* - E_t\left(r_{t+1}^*\right)}{Var_t^{1/2}\left(r_{t+1}^*\right)} = u_{t+1}^*,$$

[5] As an example, if $r_{p,t+1}$ is conditionally Gaussian, and $\Phi(\cdot)$ and $\phi(\cdot)$ denote the CDF and density of the standard normal, then

$$G_{pt} = \left(1 + S_{pt}^2\right)\Phi\left(-S_{pt}\right) - S_{pt}\phi\left(-S_{pt}\right).$$

[6] The corresponding derivative is

$$\frac{\partial G_{pt}}{\partial S_{pt}} = 2E_t\left[\left(S_{pt} + u_{p,t+1}\right) I\left(u_{p,t+1} \leq -S_{pt}\right)\right] < 0.$$

and the same conditional Sharpe ratio $\mathcal{S}_t$, and hence they share the same conditional semivariance of the standardized forecast error

$$\mathcal{G}_t = E_t\left[\left(\mathcal{S}_t + u^*_{t+1}\right)^2 I\left(u^*_{t+1} \leq -\mathcal{S}_t\right)\right],$$

and their common conditional Sortino ratio is[7]

$$\mathfrak{S}^2_{ct} = \frac{\mathcal{S}^2_t}{\mathcal{G}_t}.$$

However, different CE returns may have different unconditional Sortino ratios. These ratios are defined by the square root of

$$\mathfrak{S}^2_p = \frac{E^2\left(r_{p,t+1}\right)}{E\left[r^2_{p,t+1} I\left(r_{p,t+1} \leq 0\right)\right]}$$

for a zero threshold. Following our decomposition of the conditional semivariance, we can also decompose the unconditional semivariance in two terms

$$E\left[r^2_{p,t+1} I\left(r_{p,t+1} \leq 0\right)\right] = E\left[Var_t\left(r_{p,t+1}\right) G_{pt}\right]$$
$$= E\left(Var_t\left(r_{p,t+1}\right)\right) E\left(G_{pt}\right) + Cov\left(Var_t\left(r_{p,t+1}\right), G_{pt}\right),$$

and denote

$$C_p = \frac{Cov\left(Var_t\left(r_{p,t+1}\right), G_{pt}\right)}{E\left[Var_t\left(r_{p,t+1}\right) G_{pt}\right]}$$

the relative importance of the covariance term in the total semivariance.

Using this notation, we can decompose an unconditional Sortino ratio as follows

$$\mathfrak{S}^2_p = \frac{\mathbb{S}^2_p}{E\left(G_{pt}\right)}\left(1 - C_p\right). \tag{6}$$

The square of a Sortino ratio is like a squared residual ratio scaled by the average G_{pt} and penalized by C_p. Because of this last term, a return such that $Var_t\left(r_{p,t+1}\right)$ has a negative correlation with G_{pt}, or similarly a positive correlation with S_{pt}, will tend to have a higher Sortino ratio.

The two components of the unconditional Sortino ratio of CE returns are

$$\frac{\mathbb{S}^2_\omega}{E\left(\mathcal{G}_t\right)} = \frac{E^2\left(\omega_t \mathcal{S}^2_t\right)}{E\left(\omega^2_t \mathcal{S}^2_t\right) E\left(\mathcal{G}_t\right)}, \qquad C_\omega = \frac{Cov\left(\omega^2_t \mathcal{S}^2_t, \mathcal{G}_t\right)}{E\left(\omega^2_t \mathcal{S}^2_t \mathcal{G}_t\right)},$$

and hence the unconditional Sortino ratios of CE returns are equal to

$$\mathfrak{S}^2_\omega = \frac{\mathbb{S}^2_\omega}{E\left(\mathcal{G}_t\right)}\left(1 - C_\omega\right) = \frac{E^2\left(\omega_t \mathcal{S}^2_t\right)}{E\left(\omega^2_t \mathcal{S}^2_t \mathcal{G}_t\right)}.$$

[7] The conditional variance $Var_t\left(r_{p,t+1}\right)$ cannot be zero for a risky return because we assume that $\mathbf{\Sigma}_t$ is nonsingular with probability one. Similarly, we assume that the distribution of the vector $\mathbf{r}_{t+1}$ is such that $E_t\left[r^2_{p,t+1} I\left(r_{p,t+1} \leq 0\right)\right]$, and hence $\mathcal{G}_t$, is different from zero with probability one.

Therefore, only three variables are relevant for this performance measure, the scale ω_t, the squared maximum conditional Sharpe ratio $\mathcal{S}_t^2$, and the conditional semivariance $\mathcal{G}_t$. The first variable is chosen by the investor, while the second and third variables are given by the properties of the vector $\mathbf{r}_{t+1}$. Below we show how different choices of the scale ω_t yield different unconditional performances.

3 Performance of Conditionally Efficient Returns

This section studies five relevant choices of the scale ω_t in the CE portfolio weights (1) that define five different subsets of CE returns, or equivalently five portfolio strategies that achieve the maximum conditional Sharpe ratio $\mathcal{S}_t$. In this section we will focus on their differences in terms of unconditional Sharpe and Sortino ratios, and also provide the conditions for their equivalence.

3.1 Constant Risk Target

Many investors choose to target a fixed risk level for their investment. If we measure risk in terms of return variance, such investors are interested in CE returns with constant conditional variance target θ_1^2,

$$\mathbf{w}_t' \boldsymbol{\Sigma}_t \mathbf{w}_t = \theta_1^2.$$

We denote CE1 this subset of CE returns. Given the portfolio weights of CE returns in (1), with conditional variance $\omega_t^2 \mathcal{S}_t^2$, those investors should choose

$$\omega_{1t} = \frac{\theta_1}{\mathcal{S}_t}. \tag{7}$$

CE1 returns decrease their position in the optimal combination of risky assets r_{t+1}^* as $\mathcal{S}_t$ increases. Their conditional mean and variance are

$$E_t\left(\omega_{1t} r_{t+1}^*\right) = \theta_1 \mathcal{S}_t, \qquad Var_t\left(\omega_{1t} r_{t+1}^*\right) = \theta_1^2,$$

and hence the conditional Sharpe ratio of CE1 returns is $\mathcal{S}_t$, like any other CE return. However, CE1 returns differ from the rest of CE returns in other dimensions as the following proposition states.

Proposition 1 *Properties of the CE1 returns defined by portfolio weights (1) with (7):*

1. *They maximize $E_t\left(r_{p,t+1}\right)$ for a constant target $Var_t\left(r_{p,t+1}\right)$, that is, they solve*

$$\max_{\mathbf{w}_t} \mathbf{w}_t' \boldsymbol{\mu}_t \text{ subject to } \mathbf{w}_t' \boldsymbol{\Sigma}_t \mathbf{w}_t = \theta_1^2. \tag{8}$$

2. Their unconditional Sharpe ratio has the following components

$$\mathbb{S}_1^2 = E^2(\mathcal{S}_t), \qquad R_1^2 = \frac{Var(\mathcal{S}_t)}{1 + Var(\mathcal{S}_t)}, \tag{9}$$

and hence

$$S_1^2 = \mathbb{S}_1^2 \left(1 - R_1^2\right) = \frac{E^2(\mathcal{S}_t)}{1 + Var(\mathcal{S}_t)}. \tag{10}$$

3. Their unconditional Sortino ratio has the following components

$$\mathbb{S}_1^2 = E^2(\mathcal{S}_t), \qquad C_1 = 0,$$

and hence

$$\mathfrak{S}_1^2 = \frac{\mathbb{S}_1^2}{E(\mathcal{G}_t)}(1 - C_1) = \frac{E^2(\mathcal{S}_t)}{E(\mathcal{G}_t)}. \tag{11}$$

Point 2 shows that both $\mathbb{S}_1^2$ and S_1^2 increase in $E(\mathcal{S}_t)$, but S_1^2 also decreases with $Var(\mathcal{S}_t)$. The latter effect is due to R_1^2, the predictability in the CE1 return, increasing with $Var(\mathcal{S}_t)$. Point 3 shows that the CE1 returns have a zero covariance component in their Sortino ratio, which is simply their residual ratio divided by $E^{1/2}(\mathcal{G}_t)$.

3.2 Constant Mean Target

Some investors may strive to achieve a steady performance across different periods. This can be represented by CE returns with constant mean target θ_2,

$$\mathbf{w}_t' \boldsymbol{\mu}_t = \theta_2.$$

We denote CE2 this subset of CE returns. Given the portfolio weights of CE returns in (1), with conditional mean $\omega_t \mathcal{S}_t^2$, these investors should choose

$$\omega_{2t} = \frac{\theta_2}{\mathcal{S}_t^2}. \tag{12}$$

Like CE1 returns, CE2 returns decrease their position in the optimal combination of risky assets r_{t+1}^* as $\mathcal{S}_t$ increases. Their conditional mean and variance are

$$E_t\left(\omega_{2t} r_{t+1}^*\right) = \theta_2, \qquad Var_t\left(\omega_{2t} r_{t+1}^*\right) = \frac{\theta_2^2}{\mathcal{S}_t^2},$$

and hence the conditional Sharpe ratio of CE2 returns is $\mathcal{S}_t$, like any other CE return. However, CE2 returns differ from other CE returns in other dimensions as the following proposition states.

Proposition 2 *Properties of the CE2 returns defined by portfolio weights (1) with (12):*

1. They minimize $Var_t(r_{p,t+1})$ for a constant target $E_t(r_{p,t+1})$, that is, they solve

$$\min_{\mathbf{w}_t} \mathbf{w}_t' \boldsymbol{\Sigma}_t \mathbf{w}_t \text{ subject to } \mathbf{w}_t' \boldsymbol{\mu}_t = \theta_2. \tag{13}$$

2. Their unconditional Sharpe ratio has the following components

$$\mathbb{S}_2^2 = \frac{1}{E\left(\mathcal{S}_t^{-2}\right)}, \qquad R_2^2 = 0, \tag{14}$$

and hence

$$S_2^2 = \mathbb{S}_2^2\left(1 - R_2^2\right) = \frac{1}{E\left(\mathcal{S}_t^{-2}\right)}. \tag{15}$$

3. Their unconditional Sortino ratio has the following components

$$\mathbb{S}_2^2 = \frac{1}{E\left(\mathcal{S}_t^{-2}\right)}, \qquad C_2 = \frac{Cov\left(\mathcal{S}_t^{-2}, \mathcal{G}_t\right)}{E\left(\mathcal{S}_t^{-2}\mathcal{G}_t\right)},$$

and hence

$$\mathfrak{S}_2^2 = \frac{\mathbb{S}_2^2}{E\left(\mathcal{G}_t\right)}\left(1 - C_2\right) = \frac{1}{E\left(\mathcal{S}_t^{-2}\mathcal{G}_t\right)}. \tag{16}$$

Point 2 shows that $S_2^2 = \mathbb{S}_2^2$ decreases in $E\left(\mathcal{S}_t^{-2}\right)$. No other moment of $\mathcal{S}_t$ is relevant for these ratios. Of course, this proposition implicitly assumes that $E\left(\mathcal{S}_t^{-2}\right)$ exists but, even if $E\left(\mathcal{S}_t\right)$ and $E\left(\mathcal{S}_t^2\right)$ exist, this may not be the case. CE1 and CE2 returns satisfy different constraints, a constant conditional variance vs. a constant conditional mean, and hence there is not a natural ranking in their performance measures. If $\ln \mathcal{S}_t \sim N\left(a, b^2\right)$ for instance, then $\mathbb{S}_1^2 = \exp\left(2a + b^2\right) > \mathbb{S}_2^2 = \exp\left(2a - 2b^2\right)$ whenever $\mathcal{S}_t$ is not constant. However, the forecast time-variations in these returns are such that $R_1^2 > R_2^2 = 0$, and this could revert the ranking in terms of unconditional Sharpe ratios.

Point 3 shows that the CE2 returns should have a positive covariance term in their semivariance because both $\mathcal{S}_t^{-2}$ and $\mathcal{G}_t$ decrease with $\mathcal{S}_t$. They should not perform well with respect to the Sortino ratio, which is confirmed by our numerical examples below.

3.3 Maximum Residual Sharpe Ratio

Investors can also target a constant risk-return trade-off, and hence be interested in CE returns such that

$$\frac{\mathbf{w}_t'\boldsymbol{\mu}_t}{\mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t} = \frac{1}{\theta_3},$$

where we may think of θ_3 as a constant risk tolerance. We denote CE3 this subset of CE returns. Given the portfolio weights of CE returns in (1), with conditional mean $\omega_t\mathcal{S}_t^2$ and conditional variance $\omega_t^2\mathcal{S}_t^2$, those investors should choose

$$\omega_{3t} = \theta_3. \tag{17}$$

Unlike CE1 and CE2 returns, CE3 returns do not decrease their position in the optimal combination of risky assets r_{t+1}^* as $\mathcal{S}_t$ increases. Their conditional mean and variance are

$$E_t\left(\omega_{3t}r_{t+1}^*\right) = \theta_3\mathcal{S}_t^2, \qquad Var_t\left(\omega_{3t}r_{t+1}^*\right) = \theta_3^2\mathcal{S}_t^2,$$

and hence the conditional Sharpe ratio of CE3 returns is $\mathcal{S}_t$, like any other CE return. However, CE3 returns differ from other CE returns in other dimensions as the following proposition states.

Proposition 3 *Properties of the CE3 returns defined by portfolio weights (1) with (17):*

1. *They maximize a constant risk-return trade-off between* $Var_t\left(r_{p,t+1}\right)$ *and* $E_t\left(r_{p,t+1}\right)$

$$\max_{\mathbf{w}_t} \mathbf{w}_t'\boldsymbol{\mu}_t - \frac{1}{2\theta_3}\mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t, \tag{18}$$

   and they yield the maximum $\mathbb{S}_p^2$.

2. *Their unconditional Sharpe ratio has the following components*

$$\mathbb{S}_3^2 = E\left(\mathcal{S}_t^2\right), \qquad R_3^2 = \frac{Var\left(\mathcal{S}_t^2\right)}{E\left(\mathcal{S}_t^2\right) + Var\left(\mathcal{S}_t^2\right)}, \tag{19}$$

   and hence

$$S_3^2 = \mathbb{S}_3^2\left(1 - R_3^2\right) = \frac{E^2\left(\mathcal{S}_t^2\right)}{E\left(\mathcal{S}_t^2\right) + Var\left(\mathcal{S}_t^2\right)}. \tag{20}$$

3. *Their unconditional Sortino ratio has the following components*

$$\mathbb{S}_3^2 = E\left(\mathcal{S}_t^2\right), \qquad C_3 = \frac{Cov\left(\mathcal{S}_t^2, \mathcal{G}_t\right)}{E\left(\mathcal{S}_t^2\mathcal{G}_t\right)},$$

   and hence

$$\mathfrak{S}_3^2 = \frac{\mathbb{S}_3^2}{E\left(\mathcal{G}_t\right)}\left(1 - C_3\right) = \frac{E^2\left(\mathcal{S}_t^2\right)}{E\left(\mathcal{S}_t^2\mathcal{G}_t\right)}. \tag{21}$$

Point 2 shows that both $\mathbb{S}_3^2$ and S_3^2 increase in $E\left(\mathcal{S}_t^2\right)$, but S_3^2 also decreases with $Var\left(\mathcal{S}_t^2\right)$. The latter effect is due to R_3^2, the forecast time-variation in the CE3 return. Following the proposition, $\mathbb{S}_p^2 = E\left(\mathcal{S}_t^2\right)$ is the maximum value of a squared residual Sharpe ratio. This property was found by Peñaranda (2016), who called these returns residually efficient. From Proposition 3 and 1, we can easily see that $\mathbb{S}_3^2 \geq \mathbb{S}_1^2$, and from Proposition 3 and 2, we can easily see that $\mathbb{S}_3^2 \geq \mathbb{S}_2^2$. On the other hand, we can also see from those propositions that $\mathbb{S}_1^2$ and $\mathbb{S}_2^2$ are the maximum residual Sharpe ratios that can be achieved for a constant risk and a constant risk premium, respectively. Importantly, there could be cases where R_3^2 was high enough to make S_3^2 lower than the unconditional Sharpe ratios of CE1 and CE2 returns.

Point 3 of Proposition 3 shows that the CE3 returns should have a negative covariance term in their semivariance because $\mathcal{S}_t^2$ and $\mathcal{G}_t$ move in opposite directions. This effect, jointly with the fact that CE3 returns yield the maximum residual ratio, suggest that these returns should also perform better than CE1 and CE2 returns in terms of Sortino ratios. Our numerical examples below confirm this point.

3.4 Maximum Unconditional Sharpe Ratio

The fourth subset of CE returns that we study has a constant ratio of conditional mean to second moment

$$\frac{\mathbf{w}_t'\boldsymbol{\mu}_t}{\mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t + (\mathbf{w}_t'\boldsymbol{\mu}_t)^2} = \frac{\mathbf{w}_t'\boldsymbol{\mu}_t}{\mathbf{w}_t'\boldsymbol{\Gamma}_t\mathbf{w}_t} = \frac{1}{\theta_4},$$

where $\boldsymbol{\Gamma}_t$ denotes the uncentred conditional second moment of $\mathbf{r}_{t+1}$

$$\boldsymbol{\Gamma}_t = E_t\left(\mathbf{r}_{t+1}\mathbf{r}_{t+1}'\right) = \boldsymbol{\Sigma}_t + \boldsymbol{\mu}_t\boldsymbol{\mu}_t'.$$

We denote CE4 this subset of CE returns. Given the portfolio weights of CE returns in (1), with conditional mean $\omega_t\mathcal{S}_t^2$ and conditional second moment $\omega_t^2\mathcal{S}_t^2 + \omega_t^2\mathcal{S}_t^4$, this subset of CE returns requires

$$\omega_{4t} = \frac{\theta_4}{1+\mathcal{S}_t^2}. \tag{22}$$

Like CE1 and CE2 returns, but unlike CE3 returns, CE4 returns decrease their position in the optimal combination of risky assets r_{t+1}^* as $\mathcal{S}_t$ increases.

Let us define the ratio

$$\mathcal{U}_t = \frac{E_t^2\left(r_{t+1}^*\right)}{E_t\left(r_{t+1}^{*2}\right)} = \frac{\mathcal{S}_t^2}{1+\mathcal{S}_t^2}, \tag{23}$$

where we divide by the uncentred second moment instead of the variance. This ratio is also common across CE returns

$$\frac{E_t^2\left(r_{c,t+1}\right)}{E_t\left(r_{c,t+1}\right)} = \mathcal{U}_t.$$

The conditional mean and variance of CE4 returns are

$$E_t\left(\omega_{4t}r_{t+1}^*\right) = \theta_4\mathcal{U}_t, \qquad Var_t\left(\omega_{4t}r_{t+1}^*\right) = \theta_4^2\mathcal{U}_t\left(1-\mathcal{U}_t\right),$$

and hence the conditional Sharpe ratio of CE4 returns is $\mathcal{S}_t$, like any other CE return. However, CE4 returns differ from other CE returns in other dimensions as the following proposition states.

Proposition 4 *Properties of the CE4 returns defined by portfolio weights (1) with (22):*

1. *They maximize a constant risk-return trade-off between $E_t\left(r_{p,t+1}^2\right)$ and $E_t\left(r_{p,t+1}\right)$*

$$\max_{\mathbf{w}_t} \mathbf{w}_t'\boldsymbol{\mu}_t - \frac{1}{2\theta_4}\mathbf{w}_t'\boldsymbol{\Gamma}_t\mathbf{w}_t, \tag{24}$$

and they yield the maximum S_p^2.

2. *Their unconditional Sharpe ratio has the following components*

$$\mathbb{S}_4^2 = \frac{E^2\left(\mathcal{U}_t\right)}{E\left(\mathcal{U}_t\left(1-\mathcal{U}_t\right)\right)}, \qquad R_4^2 = \frac{Var\left(\mathcal{U}_t\right)}{E\left(\mathcal{U}_t\right)\left(1-E\left(\mathcal{U}_t\right)\right)}, \tag{25}$$

and hence

$$S_4^2 = \mathbb{S}_4^2\left(1-R_4^2\right) = \frac{E\left(\mathcal{U}_t\right)}{1-E\left(\mathcal{U}_t\right)}. \tag{26}$$

3. Their unconditional Sortino ratio has the following components

$$\mathbb{S}_4^2 = \frac{E^2(\mathcal{U}_t)}{E(\mathcal{U}_t(1-\mathcal{U}_t))}, \qquad C_4 = \frac{Cov(\mathcal{U}_t(1-\mathcal{U}_t), \mathcal{G}_t)}{E(\mathcal{U}_t(1-\mathcal{U}_t)\mathcal{G}_t)},$$

and hence

$$\mathfrak{S}_4^2 = \frac{\mathbb{S}_4^2}{E(\mathcal{G}_t)}(1 - C_4) = \frac{E^2(\mathcal{U}_t)}{E(\mathcal{U}_t(1-\mathcal{U}_t)\mathcal{G}_t)}. \tag{27}$$

Point 2 shows that S_4^2 increases in $E(\mathcal{U}_t)$, and no other information of $\mathcal{S}_t$ is relevant. However, this does not need to be the case with $\mathbb{S}_4^2$. Following the proposition, $S_p^2 = E(\mathcal{U}_t)/(1 - E(\mathcal{U}_t))$ is the maximum value of a squared unconditional Sharpe ratio. Jagannathan (1996) obtained a similar expression for the unconditional Sharpe ratio of unconditionally efficient returns when the safe asset return is constant over time. These returns were later studied by Ferson and Siegel (2001). However, if the safe asset return changes over time, then unconditionally efficient returns are not equivalent to CE4 returns. Peñaranda (2016) clarified this point, and denoted performance efficient this fourth type of CE returns.

We can also see from Proposition 2 that S_2 is the maximum unconditional Sharpe ratio that can be achieved for a constant mean target. In fact, we can be explicit about the inequality $S_4^2 \geq S_2^2$ because we can relate these ratios by means of $\mathcal{S}_t^{-2} = \mathcal{U}_t^{-2} - 1$. Then it is easy to see that

$$S_4^2 = \frac{E(\mathcal{U}_t)}{1 - E(\mathcal{U}_t)} > S_2^2 = \frac{1}{E(\mathcal{U}_t^{-2}) - 1}$$

whenever $\mathcal{S}_t$ is not constant.

Point 2 of Proposition 4 shows that, like CE3 returns, the CE4 returns should have a negative covariance term in their semivariance because $\mathcal{U}_t(1-\mathcal{U}_t)$ and $\mathcal{G}_t$ move in opposite directions.

3.5 Maximum Unconditional Sortino Ratio

CE returns have conditional mean $\omega_t \mathcal{S}_t^2$ and conditional semivariance $\omega_t^2 \mathcal{S}_t^2 \mathcal{G}_t$, where $\mathcal{G}_t$ is the conditional semivariance of the standardized forecast error of CE returns. The final subset of CE returns that we study, which we denote CE5, has the following constant risk-return trade-off

$$\frac{\omega_{5t}\mathcal{S}_t^2}{\omega_{5t}^2 \mathcal{S}_t^2 \mathcal{G}_t} = \frac{1}{\theta_5},$$

and therefore

$$\omega_{5t} = \frac{\theta_5}{\mathcal{G}_t}. \tag{28}$$

The conditional semivariance $\mathcal{G}_t$ decreases with $\mathcal{S}_t$ and hence, unlike the previous types of CE returns, CE5 returns increase their position in the optimal combination of risky assets r_{t+1}^* as $\mathcal{S}_t$ increases.

Let us define the ratio

$$\mathcal{V}_t = \frac{E_t^2\left(r_{t+1}^*\right)}{Var_t\left(r_{t+1}^*\right)\mathcal{G}_t} = \frac{\mathcal{S}_t^2}{\mathcal{G}_t},$$

which is the squared conditional Sortino ratio of r_{t+1}^*. This ratio is also common across CE returns

$$\frac{E_t^2\left(r_{c,t+1}\right)}{Var_t\left(r_{c,t+1}\right)\mathcal{G}_t} = \mathcal{V}_t.$$

The conditional mean and variance of CE5 returns are

$$E_t\left(\omega_{5t}r_{t+1}^*\right) = \theta_5\mathcal{V}_t, \qquad Var_t\left(\omega_{5t}r_{t+1}^*\right) = \theta_5^2\frac{\mathcal{V}_t}{\mathcal{G}_t},$$

and hence the conditional Sharpe ratio of CE5 returns is $\mathcal{S}_t$, like any other CE return. However, CE5 returns differ from other CE returns in other dimensions as the following proposition states.

Proposition 5 *Properties of the CE5 returns defined by portfolio weights (1) with (28):*

1. *They maximize a constant risk-return trade-off between* $Var_t\left(r_{c,t+1}\right)G_{pt}$ *and* $E_t\left(r_{c,t+1}\right)$ *among CE returns*

$$\max_{\omega_t} \omega_t\mathcal{S}_t^2 - \frac{1}{2\theta_5}\omega_t^2\mathcal{S}_t^2\mathcal{G}_t, \tag{29}$$

   and they yield the maximum $\mathfrak{S}_\omega^2$.

2. *Their unconditional Sharpe ratio has the following components*

$$\mathbb{S}_5^2 = \frac{E^2\left(\mathcal{V}_t\right)}{E\left(\mathcal{V}_t/\mathcal{G}_t\right)}, \qquad R_5^2 = \frac{Var\left(\mathcal{V}_t\right)}{E\left(\mathcal{V}_t/\mathcal{G}_t\right) + Var\left(\mathcal{V}_t\right)}, \tag{30}$$

   and hence

$$S_5^2 = \mathbb{S}_5^2\left(1 - R_5^2\right) = \frac{E^2\left(\mathcal{V}_t\right)}{E\left(\mathcal{V}_t/\mathcal{G}_t\right) + Var\left(\mathcal{V}_t\right)}. \tag{31}$$

3. *Their unconditional Sortino ratio has the following components*

$$\mathbb{S}_5^2 = \frac{E^2\left(\mathcal{V}_t\right)}{E\left(\mathcal{V}_t/\mathcal{G}_t\right)}, \qquad C_5 = \frac{Cov\left(\mathcal{V}_t/\mathcal{G}_t, \mathcal{G}_t\right)}{E\left(\mathcal{V}_t\right)},$$

   and hence

$$\mathfrak{S}_5^2 = \frac{\mathbb{S}_5^2}{E\left(\mathcal{G}_t\right)}\left(1 - C_5\right) = E\left(\mathcal{V}_t\right). \tag{32}$$

Like in the previous propositions, the properties of CE5 returns are driven by the maximum conditional Sharpe ratio $\mathcal{S}_t$. However, unlike the previous propositions, point 1 of Proposition 5 describes an optimality property of CE5 returns among CE returns, not all returns. The reason is that the optimality in the first point of the previous propositions was based on mean and variance properties, while the semivariance also involves the conditional distribution of the standardized forecast error. This distribution is the same across CE returns, as they share the same error u_{t+1}^*, but not necessarily across all returns.

Point 2 shows that both $\mathbb{S}_5^2$ and S_5^2 increase in $E(\mathcal{V}_t)$ and decrease with $E(\mathcal{V}_t/\mathcal{G}_t)$, but S_5^2 also decreases with $Var(\mathcal{V}_t)$. The latter effect is due to R_5^2, the forecast time-variation in the CE5 return.

Section 2.3 pointed out that returns such that $Var_t(r_{p,t+1})$ has a negative correlation with G_{pt}, or similarly a positive correlation with S_t, will tend to have a higher Sortino ratio. For CE5 returns, the conditional variance is a ratio with $\mathcal{S}_t^2$ in the numerator and $\mathcal{G}_t^2$ in the denominator, and hence there are two sources of negative correlation with $\mathcal{G}_t$. However, CE3 returns have a conditional variance that is driven only by $\mathcal{S}_t^2$, and hence there is only one source of negative correlation.

Following the proposition, $\mathfrak{S}_c^2 = E(\mathcal{V}_t)$ is the maximum value of a squared Sortino ratio among CE returns. For instance, from Proposition 3 and 5, we can easily see that $\mathfrak{S}_5^2 \geq \mathfrak{S}_3^2$. Of course, we can also see that $\mathbb{S}_5^2 \leq \mathbb{S}_3^2$ because CE3 returns yield the maximum residual Sharpe ratio.

3.6 Equivalence Conditions

The following table summarizes the previous results.

(Table 1: Properties of five types of CE returns)

The five subsets of CE returns that we have studied are different in general. For instance, as we computed above, a CE3 return yields a conditional mean $\theta_3\mathcal{S}_t^2$ and a conditional variance $\theta_3^2\mathcal{S}_t^2$. Therefore, a CE3 return cannot yield a constant conditional mean or a constant conditional variance unless $\mathcal{S}_t$ is constant. In general though, CE3 returns do not even satisfy the target constraint of the CE1 and CE2 returns that we defined.

We can study each pair of the five strategies to obtain equivalence conditions. For instance, we can find a scale of CE1 returns (7) equal to a particular scale of CE2 returns (12) if and only if $\mathcal{S}_t$ is constant. It turns out that we find a similar condition when we compare any other pair of those strategies, as the following corollary of Propositions 1-5 shows.

Corollary 1 *Equivalence of CE returns:*

1. *CE1, CE2, CE3, and CE4 returns are equivalent if and only if the maximum conditional Sharpe ratio $\mathcal{S}_t$ is constant. In this case, their common R^2 is zero, and their common unconditional Sharpe ratio is equal to the constant $\mathcal{S}_t$. Their common Sortino ratio is equal to $\mathcal{S}_t/E^{1/2}(\mathcal{G}_t)$.*

2. *CE5 returns are equivalent to the other four types of CE returns if and only if $\mathcal{S}_t$ and $\mathcal{G}_t$ are constant. In this case, CE5 returns have the properties commented in point 1, and the common Sortino ratio is equal to the constant $\mathcal{S}_t/\mathcal{G}_t^{1/2}$.*

CE5 returns require an additional condition because $\mathcal{G}_t$ is the conditional semivariance of u^*_{t+1} for a threshold $-\mathcal{S}_t$, which may still show time-variation when $\mathcal{S}_t$ is constant if the conditional distribution of u^*_{t+1} changes over time.

A constant $\mathcal{S}_t$ does not mean lack of predictability because there is time-variation in both risk premia $\boldsymbol{\mu}_t$ and risk $\boldsymbol{\Sigma}_t$ that can be compatible with this condition. A constant $\mathcal{S}_t$ does not mean that predictability is irrelevant either, because the Sharpe ratio of these strategies can still be higher than the maximum one obtained from fixed weight (FW) strategies, whose square is

$$S_0^2 = E(\mathbf{r}_{t+1})' [Var(\mathbf{r}_{t+1})]^{-1} E(\mathbf{r}_{t+1}) = E(\boldsymbol{\mu}_t)' [E(\boldsymbol{\Sigma}_t) + Var(\boldsymbol{\mu}_t)]^{-1} E(\boldsymbol{\mu}_t). \tag{33}$$

Let us illustrate this point with a single risky return with conditional mean μ_t and conditional variance σ_t^2. In this case, a constant $\mathcal{S}_t$

$$\mathcal{S}_t = k,$$

is equivalent to the conditional mean and the volatility of the risky return being proportional

$$\mu_t = k\sigma_t.$$

Let us denote CV the coefficient of variation of σ_t, which is equal to the coefficient of variation of μ_t in this setting,

$$CV = \frac{Var^{1/2}(\sigma_t)}{E(\sigma_t)} = \frac{Var^{1/2}(\mu_t)}{E(\mu_t)}.$$

In this context, following point 1 of Corollary 1, the CE returns in Propositions 1-4 yield

$$\mathbb{S}_i^2 = k^2, \qquad R_i^2 = 0,$$
$$S_i^2 = \mathbb{S}_i^2 \left(1 - R_i^2\right) = k^2, \qquad i = 1, 2, 3, 4.$$

However, FW strategies (which are also CE with a single risky return) provide

$$\mathbb{S}_0^2 = \frac{E^2(\mu_t)}{E\left(\sigma_t^2\right)} = \frac{k^2}{1 + CV^2},$$
$$R_0^2 = \frac{Var(\mu_t)}{E\left(\sigma_t^2\right) + Var(\mu_t)} = \frac{k^2 Var(\sigma_t)}{E^2(\sigma_t) + (1 + k^2) Var(\sigma_t)},$$
$$S_0^2 = \mathbb{S}_0^2 \left(1 - R_0^2\right) = \frac{E^2(\mu_t)}{E\left(\sigma_t^2\right) + Var(\mu_t)} = \frac{k^2 E^2(\sigma_t)}{E^2(\sigma_t) + (1 + k^2) Var(\sigma_t)}.$$

Note that the CE returns above are unpredictable in the sense of $R_i^2 = 0$, while that does not need to be the case for the original return because $R_0^2 \geq 0$. In this situation, unconditional and residual Sharpe ratios are equal for these CE returns, $\mathbb{S}_i^2 = S_i^2$, but not necessarily for FW returns, $\mathbb{S}_0^2 \geq S_0^2$.

The residual Sharpe ratio of CE1 to CE4 returns is higher than the ratio of the FW returns whenever $Var(\sigma_t) > 0$, and hence also $Var(\mu_t) > 0$, i.e., whenever there is predictability in the risky return. Moreover, the gap between both Sharpe ratios is given by CV

$$\frac{k^2}{\mathbb{S}_0^2} = 1 + CV^2.$$

The higher this coefficient of variation, which we can associate to higher predictability in the risky return (R_0^2 increases), the higher the gap between the performances. In any case, as k^2 grows without bound, both $\mathbb{S}_i^2 = k^2$ and $\mathbb{S}_0^2$ grow without bound.

Regarding unconditional Sharpe ratios, we can express the relationship between $S_i^2 = k^2$ and S_0^2 as

$$\frac{S_0^{-2} - k^{-2}}{1 + k^{-2}} = CV^2.$$

Once again, S_0^2 is lower than k^2 whenever $Var(\sigma_t) > 0$, and the gap between both Sharpe ratios is given by CV. As CV increases while we fix k^2, $\mathbb{S}_0^2$ decreases and R_0^2 increases, and both effects decrease S_0^2. On the other hand, S_0^2 converges to CV^{-2} as k^2 grows without bound. This is due to the fact that, while $R_i^2 = 0$, R_0^2 converges to one.

4 Higher Order Moments

We have seen that all CE returns are equivalent in terms of conditional Sharpe and Sortino ratios. This equivalence also holds in terms of conditional skewness and kurtosis, which are defined as

$$A_{pt} = \frac{E_t\left(r_{p,t+1} - E_t\left(r_{p,t+1}\right)\right)^3}{Var_t^{3/2}\left(r_{p,t+1}\right)},$$

and

$$K_{pt} = \frac{E_t\left(r_{p,t+1} - E_t\left(r_{p,t+1}\right)\right)^4}{Var_t^2\left(r_{p,t+1}\right)} - 3,$$

respectively.

Given that CE returns can be represented by $r_{c,t+1} = \omega_t r_{t+1}^*$ for different choices of ω_t with a common optimal combination of risky assets r_{t+1}^*, they all share the same conditional coefficients

$$A_{ct} = \frac{E_t\left(r_{c,t+1} - E_t\left(r_{c,t+1}\right)\right)^3}{Var_t^{3/2}\left(r_{c,t+1}\right)} = \frac{E_t\left(r_{t+1}^* - E_t\left(r_{t+1}^*\right)\right)^3}{Var_t^{3/2}\left(r_{t+1}^*\right)} = \mathcal{A}_t,$$
$$K_{ct} = \frac{E_t\left(r_{c,t+1} - E_t\left(r_{c,t+1}\right)\right)^4}{Var_t^2\left(r_{c,t+1}\right)} - 3 = \frac{E_t\left(r_{t+1}^* - E_t\left(r_{t+1}^*\right)\right)^4}{Var_t^2\left(r_{t+1}^*\right)} - 3 = \mathcal{K}_t,$$

which are given by the higher order conditional properties of r_{t+1}^*. However, different CE returns may differ considerably in their unconditional asymmetry and kurtosis, as we show in this section. In fact, they can show high unconditional coefficients even if that is not the case conditionally.[8]

4.1 Asymmetry

The unconditional coefficient of asymmetry of an excess return $r_{p,t+1}$ is

$$A_p = \frac{E\left(r_{p,t+1} - E\left(r_{p,t+1}\right)\right)^3}{Var^{3/2}\left(r_{p,t+1}\right)}.$$

Following the decomposition of an excess return in (3) and (4), the third moment in the previous numerator is equal to

$$\begin{aligned} E\left(r_{p,t+1} - E\left(r_{p,t+1}\right)\right)^3 &= E\left(e^3_{p,t+1}\right) + E\left(d^3_{pt}\right) + 3E\left(e^2_{p,t+1} d_{pt}\right) \qquad (34) \\ = E\left[E_t\left[r_{p,t+1} - E_t\left(r_{p,t+1}\right)\right]^3\right] &+ E\left[E_t\left(r_{p,t+1}\right) - E\left(r_{p,t+1}\right)\right]^3 \\ &+3Cov\left(Var_t\left(r_{p,t+1}\right), E_t\left(r_{p,t+1}\right)\right). \end{aligned}$$

The first component is the average of the conditional third moment of $r_{p,t+1}$, the second component is the third moment of $E_t\left(r_{p,t+1}\right)$, and the third component is three times the covariance between the conditional first and second moments of $r_{p,t+1}$. Even if $r_{p,t+1}$ is conditionally symmetric, and hence the first component is zero, the other two components may yield asymmetry in the unconditional distribution of $r_{p,t+1}$.

The third order moment of CE returns is

$$E\left(r_{c,t+1} - E\left(r_{c,t+1}\right)\right)^3 = E\left(\omega_t^3 \mathcal{S}_t^3 \mathcal{A}_t\right) + E\left[\omega_t \mathcal{S}_t^2 - E\left(\omega_t \mathcal{S}_t^2\right)\right]^3 + 3Cov\left(\omega_t^2 \mathcal{S}_t^2, \omega_t \mathcal{S}_t^2\right).$$

We expect the last two components to be nonnegative because $\omega_t > 0$ for risky CE returns, and hence their contribution to asymmetry should be nonnegative. The following corollary of Propositions 1-5 characterizes the asymmetry of the five types of CE returns that we study:

Corollary 2 *The unconditional coefficients of asymmetry of the five strategies defined in (7),*

[8] CE returns do not need to be optimal under general patterns of conditional asymmetry and kurtosis. The justification of mean-variance preferences under the expected utility paradigm was linked to elliptical distributions by Chamberlain (1983) and Owen and Rabinovitch (1983) in the Markowitz set-up without conditioning information. Therefore, once we consider return predictability, we can justify the optimality of CE returns when the vector $\mathbf{r}_{t+1}$ is conditionally elliptical. This family of distributions nests the normal distribution, and allows for conditional excess kurtosis, but not conditional asymmetry.

(12), (17), (22) ,and (28) are given by

$$A_1 [1 + Var(\mathcal{S}_t)]^{3/2} = E(\mathcal{A}_t) + E[\mathcal{S}_t - E(\mathcal{S}_t)]^3 ,$$

$$A_2 E^{3/2}\left(\mathcal{S}_t^{-2}\right) = E\left(\mathcal{S}_t^{-3}\mathcal{A}_t\right),$$

$$A_3 \left[E\left(\mathcal{S}_t^2\right) + Var\left(\mathcal{S}_t^2\right)\right]^{3/2} = E\left(\mathcal{S}_t^3\mathcal{A}_t\right) + E\left[\mathcal{S}_t^2 - E\left(\mathcal{S}_t^2\right)\right]^3 \\ +3Var\left(\mathcal{S}_t^2\right),$$

$$A_4 [E(\mathcal{U}_t)(1 - E(\mathcal{U}_t))]^{3/2} = E\left[\mathcal{U}_t^{3/2}(1-\mathcal{U}_t)^{3/2}\mathcal{A}_t\right] + E[\mathcal{U}_t - E(\mathcal{U}_t)]^3 \\ +3Cov(\mathcal{U}_t(1-\mathcal{U}_t), \mathcal{U}_t),$$

$$A_5 [E(\mathcal{V}_t/\mathcal{G}_t) + Var(\mathcal{V}_t)]^{3/2} = E\left[(\mathcal{V}_t/\mathcal{G}_t)^{3/2}\mathcal{A}_t\right] + E[\mathcal{V}_t - E(\mathcal{V}_t)]^3 \\ +3Cov(\mathcal{V}_t/\mathcal{G}_t, \mathcal{V}_t),$$

respectively.

CE1 returns do not have a covariance component in their coefficient of asymmetry, the asymmetry in $\mathcal{S}_t$ and the average $\mathcal{A}_t$ are the only sources of asymmetry. CE2 returns would not show asymmetry if there was no conditional asymmetry. However, CE3, CE4, and CE5 returns have two different sources of asymmetry on top of the conditional asymmetry term: the asymmetry in $\mathcal{S}_t^2$, $\mathcal{U}_t$, and $\mathcal{V}_t$, respectively, and a covariance component given by $Var\left(\mathcal{S}_t^2\right)$, $Cov(\mathcal{U}_t(1-\mathcal{U}_t), \mathcal{U}_t)$, and $Cov(\mathcal{V}_t/\mathcal{G}_t, \mathcal{V}_t)$, respectively. We expect CE3, CE4, and specially CE5 returns to have a positive contribution to skewness from these two sources that cannot be matched by CE1 and CE2. In all cases, apart from $\mathcal{A}_t$ itself, the asymmetry is driven by moments of the maximum conditional Sharpe ratio $\mathcal{S}_t$, or its functions $\mathcal{U}_t$ and $\mathcal{V}_t$.

4.2 Kurtosis

The unconditional coefficient of (excess) kurtosis of an excess return $r_{p,t+1}$ is

$$K_p = \frac{E(r_{p,t+1} - E(r_{p,t+1}))^4}{Var^2(r_{p,t+1})} - 3.$$

Following the decomposition of an excess return in (3) and (4), the fourth moment in the previous numerator is equal to

$$\begin{aligned} E(r_{p,t+1} - E(r_{p,t+1}))^3 &= E\left(e_{p,t+1}^4\right) + E\left(d_{pt}^4\right) + 6E\left(e_{p,t+1}^3 d_{pt}\right) + 4E\left(e_{p,t+1}^2 d_{pt}^2\right) \qquad (35) \\ &= E\left[E_t[r_{p,t+1} - E_t(r_{p,t+1})]^4\right] + E[E_t(r_{p,t+1}) - E(r_{p,t+1})]^4 \\ +6Cov\Big(E_t[r_{p,t+1} - E_t(r_{p,t+1})]^3, E_t(r_{p,t+1})\Big) &+ 4E\left[Var_t(r_{p,t+1})[E_t(r_{p,t+1}) - E(r_{p,t+1})]^2\right]. \end{aligned}$$

The first component is the average of the conditional fourth moment of $r_{p,t+1}$, the second component is the fourth moment of $E_t(r_{p,t+1})$, the third component is six times the covariance between the conditional third and first moments of $r_{p,t+1}$, and the fourth component is four times the cross-moment between $Var_t(r_{p,t+1})$ and the squared deviations of $E_t(r_{p,t+1})$ with respect to its mean. If the return is conditionally symmetric and mesokurtic (e.g., the return is conditionally Gaussian), then the first term simplifies to $3E\left(Var_t^2(r_{p,t+1})\right)$ and the third term becomes zero, but there are still two more potential sources of unconditional kurtosis.

The fourth order moment of CE returns is

$$E\left(r_{c,t+1} - E_t\left(r_{c,t+1}\right)\right)^4 = E\left[\omega_t^4 \mathcal{S}_t^4 \left(\mathcal{K}_t + 3\right)\right] + E\left[\omega_t \mathcal{S}_t^2 - E\left(\omega_t \mathcal{S}_t^2\right)\right]^4$$
$$+6Cov\left(\omega_t^3 \mathcal{S}_t^3 \mathcal{A}_t, \omega_t \mathcal{S}_t^2\right) + 4E\left[\omega_t^2 \mathcal{S}_t^2 \left(\omega_t \mathcal{S}_t^2 - E\left(\omega_t \mathcal{S}_t^2\right)\right)^2\right].$$

The following corollary of Propositions 1-5 characterizes the kurtosis of the five types of CE returns that we study:

Corollary 3 *The unconditional coefficients of kurtosis of the five strategies defined in (7), (12), (17), (22) ,and (28) are given by*

$$(K_1 + 3)\left[1 + Var\left(\mathcal{S}_t\right)\right]^2 = E\left(\mathcal{K}_t + 3\right) + E\left[\mathcal{S}_t - E\left(\mathcal{S}_t\right)\right]^4$$
$$+6Cov\left(\mathcal{A}_t, \mathcal{S}_t\right) + 4Var\left(\mathcal{S}_t\right),$$

$$(K_2 + 3)\, E^2\left(\mathcal{S}_t^{-2}\right) = E\left[\mathcal{S}_t^{-4}\left(\mathcal{K}_t + 3\right)\right],$$

$$(K_3 + 3)\left[E\left(\mathcal{S}_t^2\right) + Var\left(\mathcal{S}_t^2\right)\right]^2 = E\left[\mathcal{S}_t^4\left(\mathcal{K}_t + 3\right)\right] + E\left[\mathcal{S}_t^2 - E\left(\mathcal{S}_t^2\right)\right]^4$$
$$+6Cov\left(\mathcal{S}_t^3 \mathcal{A}_t, \mathcal{S}_t^2\right) + 4E\left[\mathcal{S}_t^2\left[\mathcal{S}_t^2 - E\left(\mathcal{S}_t^2\right)\right]^2\right],$$

$$(K_4 + 3)\left[E\left(\mathcal{U}_t\right)\left(1 - E\left(\mathcal{U}_t\right)\right)\right]^2 = E\left[\mathcal{U}_t^2\left(1 - \mathcal{U}_t\right)^2\left(\mathcal{K}_t + 3\right)\right] + E\left[\mathcal{U}_t - E\left(\mathcal{U}_t\right)\right]^4$$
$$+6Cov\left(\mathcal{U}_t^{3/2}\left(1 - \mathcal{U}_t\right)^{3/2}\mathcal{A}_t, \mathcal{U}_t\right) + 4E\left[\mathcal{U}_t\left(1 - \mathcal{U}_t\right)\left[\mathcal{U}_t - E\left(\mathcal{U}_t\right)\right]^2\right],$$

$$(K_5 + 3)\left[E\left(\mathcal{V}_t/\mathcal{G}_t\right) + Var\left(\mathcal{V}_t\right)\right]^2 = E\left[\left(\mathcal{V}_t/\mathcal{G}_t\right)^2\left(\mathcal{K}_t + 3\right)\right] + E\left[\mathcal{V}_t - E\left(\mathcal{V}_t\right)\right]^4$$
$$+6Cov\left(\left(\mathcal{V}_t/\mathcal{G}_t\right)^{3/2}\mathcal{A}_t, \mathcal{V}_t\right) + 4E\left[\left(\mathcal{V}_t/\mathcal{G}_t\right)\left[\mathcal{V}_t - E\left(\mathcal{V}_t\right)\right]^2\right],$$

respectively.

The first kurtosis component, the one driven by conditional kurtosis, is given by the cross-moment of $\mathcal{K}_t + 3$ and the square of 1, $\mathcal{S}_t^{-2}$, $\mathcal{S}_t^2$, $\mathcal{U}_t\left(1 - \mathcal{U}_t\right)$ and $\mathcal{V}_t/\mathcal{G}_t$ for CE1, CE2, CE3, CE4 and CE5 returns, respectively. CE2 returns do not have additional sources of kurtosis. The second kurtosis component is given by the fourth moment of $\mathcal{S}_t$, $\mathcal{S}_t^2$, $\mathcal{U}_t$ and $\mathcal{V}_t$ for CE1, CE3, CE4 and CE5 returns, respectively. The third kurtosis component is given by the covariances of a scaled $\mathcal{A}_t$ and $\mathcal{S}_t$, $\mathcal{S}_t^2$, $\mathcal{U}_t$ and $\mathcal{V}_t$ for CE1, CE3, CE4 and CE5 returns, respectively. The fourth kurtosis component is given by $Var(\mathcal{S}_t)$ for CE1 returns, the cross-moment of $\mathcal{S}_t^2$ and $\left[\mathcal{S}_t^2 - E\left(\mathcal{S}_t^2\right)\right]^2$ for CE3 returns, the cross-moment of $\mathcal{U}_t(1-\mathcal{U}_t)$ and $[\mathcal{U}_t - E(\mathcal{U}_t)]^2$ for CE4 returns, and the cross-moment of $\mathcal{V}_t/\mathcal{G}_t$ and $[\mathcal{V}_t - E(\mathcal{V}_t)]^2$ for CE5 returns. In all cases, apart from $\mathcal{K}_t$ and $\mathcal{A}_t$ and themselves, the kurtosis is driven by moments of the maximum conditional Sharpe ratio $\mathcal{S}_t$, or its functions $\mathcal{U}_t$ and $\mathcal{V}_t$.

5 Examples with a Single Risky Return

This section illustrates the properties of CE returns by means of some examples with a single risky excess return r_{t+1}. Following the decomposition (3), we can always decompose this excess return as

$$r_{t+1} = \mu_t + e_{t+1} = \mu_t + \sigma_t u_{t+1},$$

where μ_t is the conditional mean given information at t, and $e_{t+1} = \sigma_t u_{t+1}$ is the forecast error, with σ_t^2 being the conditional variance, and u_{t+1} having zero mean and unit variance.

The next sections study the performance of CE returns for different combinations of dynamics models for μ_t and σ_t. We simulate $500,000$ excess returns in each design. We calibrate the DGP of r_{t+1} to well known properties of the monthly excess return on the US stock market. The unconditional Sharpe ratio is fixed to 0.14, or equivalently an annualized value of 0.5. In each design, the skewness and excess kurtosis of r_{t+1} are set close to -0.5 and 2, respectively. We accommodate the skewness and kurtosis that is not explained by the dynamic properties of the DGP by generating u_{t+1} from a mixture of normal distributions. The autocorrelation in r_{t+1} is set to zero by means of the appropriate negative correlation between the mean and return socks.

The optimal combination of risky assets is simply $r_{t+1}^* = r_{t+1}\mu_t/\sigma_t^2$ in this setting, and the maximum conditional Sharpe ratio is equal to the conditional Sharpe ratio of the risky return, whose square is $\mathcal{S}_t^2 = \mu_t^2/\sigma_t^2$. The optimal weights in (1) evaluated at a particular ω_t are equal to $\omega_t\mu_t/\sigma_t^2$, and they become

$$w_{1t} = \frac{\theta_1 \ell_t}{\sigma_t}, \qquad w_{2t} = \frac{\theta_2}{\mu_t}, \qquad w_{3t} = \frac{\theta_3 \mu_t}{\sigma_t^2},$$
$$w_{4t} = \frac{\theta_4 \mu_t}{\sigma_t^2 + \mu_t^2}, \qquad w_{5t} = \frac{\theta_5 \mu_t}{\sigma_t^2 \mathcal{G}_t},$$

for each one of the five CE returns that we study, where ℓ_t is a variable that only takes values ± 1 depending on the sign of μ_t, and $\sigma_t^2 \mathcal{G}_t$ is the conditional semivariance of r_{t+1}.

We compare these strategies between them, and with a FW strategy

$$w_{0t} = \theta_0$$

which is also CE when there is only a single risky return.

5.1 Time-varying mean

In this section we focus on the mean predictability of r_{t+1}, keeping the conditional variance constant. We first consider that μ_t follows a Gaussian AR(1). Table 2 reports the corresponding results when the autocorrelation of the mean process is 0.8, and the coefficient of determination in the forecasting regression is 0.05. This coefficient value is relatively high for monthly returns, but helps to clarify the differences across portfolio strategies. The results for lower values of the coefficient of determination show similar patterns, albeit weaker because of the weaker mean predictability. These results are available upon request from the authors.

The performance measures are reported for FW and four types of CE returns. CE2 returns are not reported because their moments are not well defined in this design. These returns scale r_{t+1} by $1/\mu_t$ and the latter has no well defined moments when the mean is normal. For instance, the Sharpe ratio of CE2 returns should be the square root of the inverse of $E\left(\mathcal{S}_t^{-2}\right) = \sigma^2 E\left(\mu_t^{-2}\right)$ when the conditional variance is constant, but this moment does not exist in this design.

(Table 2: Performance for a normal mean and constant variance)

The residual, Sharpe, and Sortino ratios of CE1, CE3, CE4, and CE5 returns are provided in Propositions 1, 3, 4, and 5, respectively. The only difference between CE1 and FW returns in this design is that CE1 returns switch from $+r_{t+1}$ to $-r_{t+1}$ depending on the sign of μ_t. Still, they increase considerably the residual ratio. Of course, the highest residual ratio is achieved by CE3 returns (the bold ratios report the maximum values), but CE4 returns yield a close value. CE5 returns lay in between those two returns and CE1 returns.

The decomposition (5) shows that the Sharpe ratio of a portfolio strategy can be understood as penalizing the residual ratio by the mean predictability of the strategy. We can see that CE1 returns show half the coefficient of determination of FW returns, but the rest of CE returns suffer from a higher coefficient. Still, the superiority of their residual ratios is such that their Sharpe ratios are much higher than for FW returns. Of course, CE4 returns have a similar residual ratio to CE3 returns but a lower R^2, which translates into the highest Sharpe ratio, but not far from CE3 returns. CE5 returns lay in between those two returns and CE1 returns.

The decomposition (6) shows that the Sortino ratio of a portfolio strategy can be understood as penalizing the residual ratio by a covariance term. In particular, the covariance between the return conditional variance and a decreasing function of $\mathcal{S}_t$. In this design, the conditional variance of r_{t+1} is constant, and hence FW returns have a zero covariance term. CE1 returns have a zero covariance term by construction because they are defined by a constant variance target. Therefore, the ordering of their Sortino ratios is the same as their residual ratios, being much higher for CE1 returns.

Still, the Sortino ratio of CE1 returns is lower than the rest of CE returns. CE3, CE4, and CE5 yield a negative covariance term because their conditional variances are positively correlated to $\mathcal{S}_t$. As expected, this term is specially negative for CE5 returns, which compensates their lower residual ratio, and finally they yield the maximum Sortino ratio. It is also expected that the covariance term is more negative for CE3 returns than CE4 returns, and hence CE3 returns yield the second highest value of the Sortino ratio.

Equation (34) decomposes the third moment of a return into three components. The first one $E\left(e_{p,t+1}^3\right)$ is the average conditional third moment, and in this design it is the only source of asymmetry for r_{t+1}, or equivalently FW returns. CE1 returns have a nonzero but negligible second source of asymmetry $E\left(d_{pt}^3\right)$, the asymmetry in the conditional mean of the portfolio return.

However, the asymmetry of the conditional mean is positive for the rest of CE returns, specially CE3 and CE5 returns, as expected from Corollary 2. These two returns have an even stronger third source of positive asymmetry $3E\left(e_{p,t+1}^2 d_{pt}\right)$, the covariance between their conditional variance and mean, as both are positively related to $\mathcal{S}_t$. This effect is similar to the negative covariance term of the Sortino ratio commented above. These two positive sources of asymmetry more than offset the more negative conditional asymmetry of these returns, and we find a strongly positive total asymmetry, unlike with CE1 and FW returns. Like the ranking of Sortino ratios, CE5 show the highest skewness, followed by CE3 returns.

Equation (35) decomposes the fourth moment of a return into four components. The first one $E\left(e_{p,t+1}^4\right)$ is the average conditional fourth moment, and it is the main source of excess kurtosis for r_{t+1}, or equivalently FW returns. These returns have a negligible second and third components, $E\left(d_{pt}^4\right)$ and $6E\left(e_{p,t+1}^3 d_{pt}\right)$, related to the kurtosis of its conditional mean (μ_t is normal in this design) and the covariance between this mean and the return conditional asymmetry, respectively. However, the fourth source of kurtosis $4E\left(e_{p,t+1}^2 d_{pt}^2\right)$, the cross-moment between the return conditional variance and the squared deviations of the conditional mean with respect to its average, has a nonnegligible positive contribution. CE1 returns have a higher first source of kurtosis, but a lower fourth source. The net effect is a slightly higher excess kurtosis than FW returns. This will not be the case once we introduce variance predictability in other designs.

Once again, the rest of CE returns, and specially CE3 and CE5 returns, have stronger extra sources of kurtosis, which follows from Corollary 3. CE5 returns have strongly positive second and fourth sources, and a strongly negative third source. The net effect, jointly with a highly positive first source, is a high excess kurtosis. CE3 returns show similar signs in their four sources, but less extreme values, and they yield the next value of kurtosis.

We also consider a dynamic model where $\ln \mu_t$ follows a Gaussian AR(1), in which case CE2 returns have well defined performance measures. Table 3 reports the corresponding performance results when the autocorrelation of the mean process is 0.8, and the coefficient of determination in the forecasting regression is 0.03. We lower this coefficient with respect to Table 2 because CE3 returns become more extreme in designs with a log-normal mean. These returns scale r_{t+1} by μ_t/σ_t^2, which is log-normally distributed and hence can show high skewness and kurtosis if $\ln\left(\mu_t/\sigma_t^2\right)$ has a relatively high variance. Moreover, the performance of CE5 returns is even more sensitive to that variance because they actually scale CE3 returns by $1/\mathcal{G}_t$, which increases with μ_t/σ_t. For that reason, we do not report CE5 returns when the mean is log-normal. But the reader should keep in mind that, with respect to CE3 returns, their residual and Sharpe ratios will be lower, while their Sortino ratio and their coefficients of asymmetry and kurtosis will be higher.[9]

(Table 3: Performance for a log-normal mean and constant variance)

CE1 and FW returns coincide in this constant variance design because μ_t cannot be negative. The properties of CE2 returns were stated in Proposition 2. Their Sharpe ratio is equal to their residual ratio because, by definition, their R^2 is zero. Importantly, CE2 returns seem much worse than FW returns in terms of these ratios. They also seem much worse in terms of the Sortino ratio because, in addition to a low residual ratio, they are penalized by a positive covariance term, due to both $\mathcal{S}_t^{-2}$ and $\mathcal{G}_t$ decreasing with $\mathcal{S}_t$.

Regarding the ratios of CE3 and CE4 returns, we find similar patterns to Table 2. A noteworthy feature is that, even though the R^2 of r_{t+1} is lower in Table 3, the R^2 of CE3 returns is much higher in this table. Similarly, the covariance term of the Sortino ratio is much more negative. This feature is another dimension of the commented sensitivity of CE3 returns in this design, which is confirmed by the coefficients of asymmetry and kurtosis. We find high coefficient values in this design, and their main component is the asymmetry and kurtosis of their conditional mean, which is driven by the log-normal $\mathcal{S}_t^2$.

The asymmetry of CE2 returns is much more negative than FW returns in this design, and they also show much more excess kurtosis.[10] This is true even though there is only one nonzero component in the coefficients of CE2 returns. The conditional mean of these returns is constant

[9] For instance, in this particular design, their residual and Sharpe ratios are as low as 0.012 and 0.002, respectively. However, their Sortino ratio becomes 2.333.

[10] There is a small positive contribution of the positive asymmetry of μ_t to the asymmetry of r_{t+1}, or FW returns, which is not the case in the normal mean design of Table 2. Similarly, the contribution of the kurtosis of μ_t to the kurtosis of r_{t+1} is slightly more relevant than in Table 2.

by construction, and therefore only the average conditional asymmetry and kurtosis drive the corresponding coefficients.

5.2 Time-varying variance

In this section we focus on the variance predictability of r_{t+1}, keeping the conditional mean constant. In particular, $\ln \sigma_t^2$ follows a Gaussian AR(1). Table 4 reports the corresponding results when the autocorrelation of $\ln \sigma_t^2$ is 0.9, and the coefficient of variation of σ_t^2 is 1, which is relatively high for monthly returns, but helps to clarify the differences across portfolio strategies. Once again, the results for lower values of the coefficient of variation show similar patterns, albeit weaker because the variance predictability becomes weaker. These results are also available upon request from the authors.

(Table 4: Performance for constant mean and a log-normal variance)

In this table, both CE2 and CE5 appear simultaneously, although CE2 are actually equivalent to FW returns in this design. Another difference with respect to the previous tables is that the R^2 of FW returns is zero because there is no mean predictability. At the same time, there is a positive covariance term in the Sortino ratio of FW returns due to the variance predictability.

CE1 returns improve the residual, Sharpe, and Sortino ratios with respect to FW returns. The penalization in their Sharpe ratio for their mean predictability is negligible, while the covariance penalization in their Sortino ratio is exactly zero because their conditional variance is constant by construction.

The patterns in the performance ratios of CE3, CE4, and CE5 are similar to Table 2. In fact, CE3 and CE4 returns are very similar in terms of these ratios in this design. However, these returns seem less similar in terms of skewness and kurtosis. In particular, CE5 returns yield a much higher positive skewness and excess kurtosis. Once again, CE3 returns lay in between CE5 and CE4 returns in terms of higher order moments, but being much closer to the latter.

The excess kurtosis results have some special features with respect to Table 2. The fourth source of kurtosis $4E\left(e_{p,t+1}^2 d_{pt}^2\right)$ is zero for FW returns, while it is slightly positive for CE1 returns. In any case, the main point is that their average conditional kurtosis is so low that their excess kurtosis is around zero. CE1 returns scale r_{t+1} by $1/\sigma_t$, and hence remove the kurtosis that a time-varying volatility generates.

5.3 Time-varying mean and variance

In this section we model jointly the mean and variance predictability of r_{t+1}. We first consider that μ_t and $\ln \sigma_t^2$ follow a Gaussian AR(1). Once again, we set the autocorrelation of the mean process to 0.8, and the autocorrelation of $\ln \sigma_t^2$ to 0.9. We lower the R^2 of r_{t+1} to 0.01 with respect to Table 2, and the coefficient of variation of σ_t^2 to 0.8 with respect to Table 4, because now we have two sources of predictability. Table 5 displays the corresponding results. Panel A reports the results for a correlation of -0.1 between mean and log-variance shocks, while Panel B reports the results for a correlation of 0.1. Panel A represents slightly better mean-variance opportunities than independent mean and variance shocks, and Panel B represents slightly worse opportunities. A design with a zero correlation, and hence independence between the mean and variance predictability, yields results in between Panel A and Panel B. Similarly, a design with a more negative correlation strengths the patterns in Panel A, while a design with a more positive correlation weakens the patterns in Panel B. CE2 returns are not reported because their moments are not well defined for a normal mean.

(Table 5: Performance for a normal mean and a log-normal variance)

In general, we find a combination of the properties in Tables 2 and 4, which separated the effects of mean and variance predictability. The main novelty in Panel A[11] is in the performance of CE5 returns. Their Sharpe ratio is lower than for CE1 returns,[12] while the gap in Sortino ratios between CE5 returns and the rest increases, due to the stronger negative covariance term. Similarly, the skewness and kurtosis of CE5 returns becomes much more higher than in Tables 2 and 4.

The performance of the different returns is closer to each other in Panel B. The skewness and kurtosis of CE5 returns are still very high but less than in Panel A.

Finally, we consider a dynamic model where $\ln \mu_t$ follows a Gaussian AR(1), in which case CE2 returns have well defined performance measures. Table 6 reports the corresponding performance results for the same mean and variance parameters of Table 5. Panel A reports the results for a correlation of -0.1 between log-mean and log-variance shocks, while Panel B reports the results for a correlation of 0.1. Once again, a design with a zero correlation yields results in between Panel A and Panel B, a design with a more negative correlation strengths the patterns in Panel A, and a design with a more positive correlation weakens the patterns in Panel B. The performance of CE5 returns is not reported but the reader should keep in mind that, with respect to CE3 returns, their residual and Sharpe ratios will be lower, while their Sortino ratio and their coefficients of asymmetry and kurtosis will be higher.[13]

[11] A less relevant novelty is that FW returns show a small contribution of the third sources of skewness and kurtosis.

[12] Like in Table 4 with variance predictability, CE1 returns do not show excess kurtosis.

[13] In this particular design, their residual and Sharpe ratios are as low as 0.014 and 0.004 in Panel A, respectively. However, their Sortino ratio becomes 1.616. The results are slightly less extreme in Panel B, where the residual, Sharpe, and Sortino ratios are 0.040, 0.015, and 0.516, respectively.

(Table 6: Performance for a log-normal mean and a log-normal variance)

In general, we find a combination of the properties in Tables 3 and 4, which separated the effects of mean and variance predictability. Like in Table 3, both panels of Table 6 show that CE2 returns seem much worse than FW returns in terms of performance ratios. Similarly, the asymmetry of CE2 returns is much more negative than FW returns, and they also show much more excess kurtosis. Of course, CE2 and FW returns coincide in Table 4.

Comparing Tables 5 and 6, the skewness and kurtosis of CE3 returns become much higher with a log-normal mean,. This was also the case with Tables 2 and 3. Nevertheless, there is a difference with respect to Table 3. The main source does not need to be the asymmetry and kurtosis of their conditional mean.

6 Conclusions

Our main contribution is the analysis of five relevant types of conditionally efficient (CE) strategies, with a focus on how return predictability drives the differences in their unconditional Sharpe and Sortino ratios, and their coefficients of asymmetry and kurtosis. The first two types of CE returns keep a constant target for the conditional variance or mean. We denote them CE1 and CE2 returns, respectively. The second two types of CE returns yield the maximum residual and unconditional Sharpe ratios. We denote them CE3 and CE4 returns, respectively. The final CE returns that we study, denoted CE5, maximize the unconditional Sortino ratio among CE returns.

We provide formulas that decompose the performance measures into interpretable components, and show that the maximum conditional Sharpe ratio drives these measures. We find strong differences between the five types of strategies across several combinations of mean and variance predictability, even though there is a single risky asset in our examples. We also compare the CE returns to fixed weight (FW) strategies.

CE1 returns yield higher performance ratios than FW returns, but obviously lower than CE3, CE4 or CE5 returns. The asymmetry of CE1 returns is similar to FW returns, but they remove the kurtosis that is derived from time variation in the conditional variance of the risky asset return. In fact, they are usually the CE return with the lowest kurtosis. Interestingly, the performance ratios of CE2 returns may be much worse than FW returns, or may not even exist.

By definition, CE3 and CE4 returns are optimal with respect to the residual and Sharpe ratios, respectively. In our examples CE3 and CE4 returns are not too different in terms of these ratios, but they are more different in terms of the Sortino ratio, which is higher for CE3 returns. Moreover, they can be very different in terms of asymmetry and kurtosis, with CE3 returns having higher coefficients. However, CE5 returns have the most extreme behavior. They yield the highest Sortino ratio among CE returns by definition, but sometimes jointly with very high coefficients of asymmetry and kurtosis, and very low residual and Sharpe ratios.

There are interesting avenues of future research that we plan to study. The maximum drawdown (see, e.g., Grossman and Zhou, 1993) is an important risk measure for investment managers and it would be interesting to study CE returns from that perspective. Our analysis has implicitly assumed that the investor observes the true conditional means and variances of returns, and a more realistic analysis would consider the possibility of noisy signals. For instance, along the lines of the real-time Bayesian frameworks of Avramov and Chordia (2006) and Johannes, Korteweg, and Polson (2014). Of course, we could also consider leverage constraints and transaction costs to make the analysis more realistic. At the empirical level, our examples where developed for monthly stock returns, and it would be interesting to study other return frequencies and asset classes with different predictability properties.

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Appendices

A Proofs

A.1 Proof of Proposition 1

1) This is the optimal choice because, for any $\mathbf{w}_t$ such that $\mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t = \theta_1^2$,

$$\begin{aligned}(\omega_{1t}\boldsymbol{\varphi}_t - \mathbf{w}_t)' \boldsymbol{\Sigma}_t (\omega_{1t}\boldsymbol{\varphi}_t - \mathbf{w}_t) &= \theta_1^2 + \theta_1^2 - 2\frac{\theta_1}{\mathcal{S}_t}\mathbf{w}_t'\boldsymbol{\mu}_t \\ &= 2\frac{\theta_1}{\mathcal{S}_t}\left(\omega_{1t}\boldsymbol{\varphi}_t'\boldsymbol{\mu}_t - \mathbf{w}_t'\boldsymbol{\mu}_t\right) \geq 0,\end{aligned}$$

which is strictly positive for any realization of the information at t unless $\mathbf{w}_t = \omega_{1t}\boldsymbol{\varphi}_t$.

2) The decomposition of their unconditional Share ratio is

$$\mathbb{S}_1^2 = \frac{E^2\left(\omega_{1t}\mathcal{S}_t^2\right)}{E\left(\omega_{1t}^2\mathcal{S}_t^2\right)} = \frac{E^2\left(\theta_1\mathcal{S}_t\right)}{\theta_1^2} = E^2\left(\mathcal{S}_t\right),$$

$$R_1^2 = \frac{Var\left(\omega_{1t}\mathcal{S}_t^2\right)}{E\left(\omega_{1t}^2\mathcal{S}_t^2\right) + Var\left(\omega_{1t}\mathcal{S}_t^2\right)} = \frac{Var\left(\theta_1\mathcal{S}_t\right)}{\theta_1^2 + Var\left(\sigma_e\mathcal{S}_t\right)} = \frac{Var\left(\mathcal{S}_t\right)}{1 + Var\left(\mathcal{S}_t\right)}.$$

3) Their unconditional Sortino ratio follows from $\mathbb{S}_1^2$ above and

$$C_1 = \frac{Cov\left(\omega_{1t}^2\mathcal{S}_t^2, \mathcal{G}_t\right)}{E\left(\omega_{1t}^2\mathcal{S}_t^2\mathcal{G}_t\right)} = \frac{Cov\left(\theta_1^2, \mathcal{G}_t\right)}{E\left(\theta_1^2\mathcal{G}_t\right)} = 0,$$

which completes the proof. □

A.2 Proof of Proposition 2

1) This is the optimal choice because, for any $\mathbf{w}_t$ such that $\mathbf{w}_t'\boldsymbol{\mu}_t = \theta_2$,

$$\begin{aligned}(\omega_{2t}\boldsymbol{\varphi}_t - \mathbf{w}_t)' \boldsymbol{\Sigma}_t (\omega_{2t}\boldsymbol{\varphi}_t - \mathbf{w}_t) &= \frac{\theta_2^2}{\mathcal{S}_t^2} + \mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t - 2\frac{\theta_2^2}{\mathcal{S}_t^2} \\ &= \mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t - (\omega_{2t}\boldsymbol{\varphi}_t)' \boldsymbol{\Sigma}_t (\omega_{2t}\boldsymbol{\varphi}_t) \geq 0,\end{aligned}$$

which is strictly positive for any realization of the information at t unless $\mathbf{w}_t = \omega_{2t}\boldsymbol{\varphi}_t$.

2) The decomposition of their unconditional Share ratio is

$$\mathbb{S}_2^2 = \frac{E^2\left(\omega_{2t}\mathcal{S}_t^2\right)}{E\left(\omega_{2t}^2\mathcal{S}_t^2\right)} = \frac{\theta_2^2}{E\left(\theta_2^2\mathcal{S}_t^{-2}\right)} = \frac{1}{E\left(\mathcal{S}_t^{-2}\right)},$$

$$R_1^2 = \frac{Var\left(\omega_{2t}\mathcal{S}_t^2\right)}{E\left(\omega_{2t}^2\mathcal{S}_t^2\right) + Var\left(\omega_{2t}\mathcal{S}_t^2\right)} = \frac{0}{E\left(\theta_2^2\mathcal{S}_t^{-2}\right) + 0} = 0.$$

3) Their unconditional Sortino ratio follows from $\mathbb{S}_2^2$ above and

$$C_2 = \frac{Cov\left(\omega_{2t}^2\mathcal{S}_t^2, \mathcal{G}_t\right)}{E\left(\omega_{2t}^2\mathcal{S}_t^2\mathcal{G}_t\right)} = \frac{Cov\left(\theta_2^2\mathcal{S}_t^{-2}, \mathcal{G}_t\right)}{E\left(\theta_2^2\mathcal{S}_t^{-2}\mathcal{G}_t\right)} = \frac{Cov\left(\mathcal{S}_t^{-2}, \mathcal{G}_t\right)}{E\left(\mathcal{S}_t^{-2}\mathcal{G}_t\right)},$$

which completes the proof. □

A.3 Proof of Proposition 3

1) Let us define the criterion

$$U_t(\mathbf{w}_t) = \mathbf{w}_t' \boldsymbol{\mu}_t - \frac{1}{2\theta_3} \mathbf{w}_t' \boldsymbol{\Sigma}_t \mathbf{w}_t,$$

which evaluated at CE3 returns becomes

$$U_t(\omega_{3t} \boldsymbol{\varphi}_t) = \frac{\theta_3 \mathcal{S}_t^2}{2}.$$

For any $\mathbf{w}_t$,

$$U_t(\omega_{3t} \boldsymbol{\varphi}_t) - U_t(\mathbf{w}_t) = \frac{1}{2\theta_3} (\omega_{3t} \boldsymbol{\varphi}_t - \mathbf{w}_t)' \boldsymbol{\Sigma}_t (\omega_{3t} \boldsymbol{\varphi}_t - \mathbf{w}_t) \geq 0,$$

which is strictly positive for any realization of the information at t unless $\mathbf{w}_t = \omega_{3t} \boldsymbol{\varphi}_t$.

Moreover, if we take the expectation of the previous inequality, then we have

$$E[U_t(\omega_{3t} \boldsymbol{\varphi}_t) - U_t(\mathbf{w}_t)] \geq 0.$$

By definition of $U_t(\mathbf{w}_t)$, this expectation is equal to

$$E[U_t(\omega_{3t} \boldsymbol{\varphi}_t) - U_t(\mathbf{w}_t)] = E\left[(\omega_{3t} \boldsymbol{\varphi}_t)' \boldsymbol{\mu}_t - \frac{1}{2\theta_3} (\omega_{3t} \boldsymbol{\varphi}_t)' \boldsymbol{\Sigma}_t (\omega_{3t} \boldsymbol{\varphi}_t)\right] - E\left[\mathbf{w}_t' \boldsymbol{\mu}_t - \frac{1}{2\theta_3} \mathbf{w}_t' \boldsymbol{\Sigma}_t \mathbf{w}_t\right].$$

If we apply this expression to a $\mathbf{w}_t$ with the same mean as $\omega_{3t} \boldsymbol{\varphi}_t$, then we find

$$E\left[\mathbf{w}_t' \boldsymbol{\Sigma}_t \mathbf{w}_t\right] \geq E\left[(\omega_{3t} \boldsymbol{\varphi}_t)' \boldsymbol{\Sigma}_t (\omega_{3t} \boldsymbol{\varphi}_t)\right].$$

Therefore, $\omega_{3t} \boldsymbol{\varphi}_t$ has a higher residual Sharpe ratio than $\mathbf{w}_t$. The maximization of the residual Sharpe ratio was proved in Peñaranda (2016) with projection methods.

2) The decomposition of their unconditional Sharpe ratio is

$$\mathbb{S}_3^2 = \frac{E^2(\omega_{3t} \mathcal{S}_t^2)}{E(\omega_{3t}^2 \mathcal{S}_t^2)} = \frac{E^2(\theta_3 \mathcal{S}_t^2)}{E(\theta_3^2 \mathcal{S}_t^2)} = E(\mathcal{S}_t^2),$$

$$R_3^2 = \frac{Var(\omega_{3t} \mathcal{S}_t^2)}{E(\omega_{3t}^2 \mathcal{S}_t^2) + Var(\omega_{3t} \mathcal{S}_t^2)} = \frac{Var(\theta_3 \mathcal{S}_t^2)}{E(\theta_3^2 \mathcal{S}_t^2) + Var(\theta_3 \mathcal{S}_t^2)} = \frac{Var(\mathcal{S}_t^2)}{E(\mathcal{S}_t^2) + Var(\mathcal{S}_t^2)}.$$

3) Their unconditional Sortino ratio follows from $\mathbb{S}_3^2$ above and

$$C_3 = \frac{Cov(\omega_{3t}^2 \mathcal{S}_t^2, \mathcal{G}_t)}{E(\omega_{3t}^2 \mathcal{S}_t^2 \mathcal{G}_t)} = \frac{Cov(\theta_3^2 \mathcal{S}_t^2, \mathcal{G}_t)}{E(\theta_3^2 \mathcal{S}_t^2 \mathcal{G}_t)} = \frac{Cov(\mathcal{S}_t^2, \mathcal{G}_t)}{E(\mathcal{S}_t^2 \mathcal{G}_t)},$$

which completes the proof. $\square$

A.4 Proof of Proposition 4

1) Let us define the criterion

$$V_t(\mathbf{w}_t) = \mathbf{w}_t'\boldsymbol{\mu}_t - \frac{1}{2\theta_4}\mathbf{w}_t'\boldsymbol{\Gamma}_t\mathbf{w}_t,$$

and note that the portfolio weights in (22) can be written as

$$\omega_{4t}\boldsymbol{\varphi}_t = \theta_4\boldsymbol{\Gamma}_t^{-1}\boldsymbol{\mu}_t$$

because

$$\boldsymbol{\Gamma}_t^{-1}\boldsymbol{\mu}_t = \left(\boldsymbol{\Sigma}_t + \boldsymbol{\mu}_t\boldsymbol{\mu}_t'\right)^{-1}\boldsymbol{\mu}_t = \left(\boldsymbol{\Sigma}_t^{-1} - \frac{1}{1+\boldsymbol{\mu}_t'\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t}\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t\boldsymbol{\mu}_t'\boldsymbol{\Sigma}_t^{-1}\right)\boldsymbol{\mu}_t = \boldsymbol{\varphi}_t - \mathcal{U}_t\boldsymbol{\varphi}_t.$$

The criterion evaluated at these weights becomes

$$V_t(\omega_{4t}\boldsymbol{\varphi}_t) = \frac{\theta_4\boldsymbol{\mu}_t'\boldsymbol{\Gamma}_t^{-1}\boldsymbol{\mu}_t}{2} = \frac{\theta_4\mathcal{U}_t}{2}.$$

For any $\mathbf{w}_t$,

$$V_t(\omega_{4t}\boldsymbol{\varphi}_t) - V_t(\mathbf{w}_t) = \frac{1}{2\theta_4}\left(\omega_{4t}\boldsymbol{\varphi}_t - \mathbf{w}_t\right)'\boldsymbol{\Gamma}_t\left(\omega_{4t}\boldsymbol{\varphi}_t - \mathbf{w}_t\right) \geq 0,$$

which is strictly positive for any realization of the information at t unless $\mathbf{w}_t = \omega_{4t}\boldsymbol{\varphi}_t$.

To prove that CE2 returns maximize S^2, let us take the expectation of the previous inequality

$$E\left[V_t(\omega_{4t}\boldsymbol{\varphi}_t) - V_t(\mathbf{w}_t)\right] \geq 0.$$

By definition of $V_t(\mathbf{w}_t)$, this expectation is equal to

$$E\left[V_t(\omega_{4t}\boldsymbol{\varphi}_t) - V_t(\mathbf{w}_t)\right] = E\left[\left(\omega_{4t}\boldsymbol{\varphi}_t\right)'\boldsymbol{\mu}_t - \frac{1}{2\theta_4}\left(\omega_{4t}\boldsymbol{\varphi}_t\right)'\boldsymbol{\Gamma}_t\left(\omega_{4t}\boldsymbol{\varphi}_t\right)\right] - E\left[\mathbf{w}_t'\boldsymbol{\mu}_t - \frac{1}{2\theta_4}\mathbf{w}_t'\boldsymbol{\Gamma}_t\mathbf{w}_t\right].$$

If we apply this expression to a $\mathbf{w}_t$ with the same mean as $\left(\omega_{4t}\boldsymbol{\varphi}_t\right)'$, then we find

$$E\left[\mathbf{w}_t'\boldsymbol{\Gamma}_t\mathbf{w}_t\right] \geq E\left[\left(\omega_{4t}\boldsymbol{\varphi}_t\right)'\boldsymbol{\Gamma}_t\left(\omega_{4t}\boldsymbol{\varphi}_t\right)\right].$$

The portfolio $\omega_{4t}\boldsymbol{\varphi}_t$ has a lower unconditional uncentred second moment. As both returns have the same unconditional mean, then $\omega_{4t}\boldsymbol{\varphi}_t$ has a lower unconditional variance, and a higher unconditional Sharpe ratio. The maximization of the unconditional Sharpe ratio was proved in Peñaranda (2016) with projection methods.

2) The decomposition of their unconditional Share ratio is

$$\mathbb{S}_4^2 = \frac{E^2\left(\omega_{4t}\mathcal{S}_t^2\right)}{E\left(\omega_{4t}^2\mathcal{S}_t^2\right)} = \frac{E^2\left(\theta_4\mathcal{U}_t\right)}{E\left(\theta_4^2\mathcal{U}_t\left(1-\mathcal{U}_t\right)\right)} = \frac{E^2\left(\mathcal{U}_t\right)}{E\left(\mathcal{U}_t\left(1-\mathcal{U}_t\right)\right)},$$

$$R_4^2 = \frac{Var\left(\omega_{4t}\mathcal{S}_t^2\right)}{E\left(\omega_{4t}^2\mathcal{S}_t^2\right) + Var\left(\omega_{4t}\mathcal{S}_t^2\right)} = \frac{Var\left(\theta_4\mathcal{U}_t\right)}{E\left(\theta_4^2\mathcal{U}_t\left(1-\mathcal{U}_t\right)\right) + Var\left(\theta_4\mathcal{U}_t\right)} = \frac{Var\left(\mathcal{U}_t\right)}{E\left(\mathcal{U}_t\right)\left(1-E\left(\mathcal{U}_t\right)\right)},$$

3) Their unconditional Sortino ratio follows from $\mathbb{S}_4^2$ above and

$$C_4 = \frac{Cov\left(\omega_{4t}^2\mathcal{S}_t^2, \mathcal{G}_t\right)}{E\left(\omega_{4t}^2\mathcal{S}_t^2\mathcal{G}_t\right)} = \frac{Cov\left(\theta_4^2\mathcal{U}_t\left(1-\mathcal{U}_t\right), \mathcal{G}_t\right)}{E\left(\theta_4^2\mathcal{U}_t\left(1-\mathcal{U}_t\right)\mathcal{G}_t\right)} = \frac{Cov\left(\mathcal{U}_t\left(1-\mathcal{U}_t\right), \mathcal{G}_t\right)}{E\left(\mathcal{U}_t\left(1-\mathcal{U}_t\right)\mathcal{G}_t\right)},$$

which completes the proof. □

A.5 Proof of Proposition 5

1) Let us define the criterion

$$W_t(\omega_t) = \omega_t \mathcal{S}_t^2 - \frac{1}{2\theta_5}\omega_t^2 \mathcal{S}_t^2 \mathcal{G}_t,$$

which evaluated at CE5 returns becomes

$$W_t(\omega_{5t}) = \frac{\theta_5 \mathcal{S}_t^2}{2\mathcal{G}_t}.$$

For any ω_t

$$W_t(\omega_{5t}) - W_t(\omega_t) = \frac{1}{2\theta_5}\mathcal{S}_t^2 \mathcal{G}_t (\omega_{5t} - \omega_t)^2 \geq 0,$$

which is strictly positive for any realization of the information at t unless $\omega_t = \omega_{5t}$.

Moreover, if we take the expectation of the previous inequality, then we have

$$E[W_t(\omega_{5t}) - W_t(\omega_t)] \geq 0.$$

By definition of $W_t(\omega_t)$, this expectation is equal to

$$E[W_t(\omega_{5t}) - W_t(\omega_t)] = E\left[\omega_{5t}\mathcal{S}_t^2 - \frac{1}{2\theta_5}\omega_{5t}^2 \mathcal{S}_t^2 \mathcal{G}_t\right] - E\left[\omega_t \mathcal{S}_t^2 - \frac{1}{2\theta_5}\omega_t^2 \mathcal{S}_t^2 \mathcal{G}_t\right].$$

If we apply this expression to an ω_t with the same mean as ω_{5t}, then we find

$$E\left(\omega_t^2 \mathcal{S}_t^2 \mathcal{G}_t\right) \geq E\left(\omega_{5t}^2 \mathcal{S}_t^2 \mathcal{G}_t\right).$$

The choice ω_{5t} has a lower unconditional semivariance, and a higher unconditional Sortino ratio.

2) The decomposition of their unconditional Sharpe ratio is

$$\mathbb{S}_5^2 = \frac{E^2\left(\omega_{5t}\mathcal{S}_t^2\right)}{E\left(\omega_{5t}^2 \mathcal{S}_t^2\right)} = \frac{E^2\left(\theta_5 \frac{\mathcal{S}_t^2}{\mathcal{G}_t}\right)}{E\left(\theta_5^2 \frac{\mathcal{S}_t^2}{\mathcal{G}_t^2}\right)} = \frac{E^2(\mathcal{V}_t)}{E(\mathcal{V}_t/\mathcal{G}_t)},$$

$$R_5^2 = \frac{Var\left(\omega_{5t}\mathcal{S}_t^2\right)}{E\left(\omega_{5t}^2\mathcal{S}_t^2\right) + Var\left(\omega_{5t}\mathcal{S}_t^2\right)} = \frac{Var\left(\theta_5 \frac{\mathcal{S}_t^2}{\mathcal{G}_t}\right)}{E\left(\theta_5^2 \frac{\mathcal{S}_t^2}{\mathcal{G}_t^2}\right) + Var\left(\theta_5 \frac{\mathcal{S}_t^2}{\mathcal{G}_t}\right)} = \frac{Var(\mathcal{V}_t)}{E(\mathcal{V}_t/\mathcal{G}_t) + Var(\mathcal{V}_t)}.$$

3) Their unconditional Sortino ratio follows from $\mathbb{S}_5^2$ above and

$$C_5 = \frac{Cov\left(\omega_{5t}^2 \mathcal{S}_t^2, \mathcal{G}_t\right)}{E\left(\omega_{5t}^2 \mathcal{S}_t^2 \mathcal{G}_t\right)} = \frac{Cov\left(\theta_5^2 \frac{\mathcal{S}_t^2}{\mathcal{G}_t^2}, \mathcal{G}_t\right)}{E\left(\theta_5^2 \frac{\mathcal{S}_t^2}{\mathcal{G}_t^2}\mathcal{G}_t\right)} = \frac{Cov(\mathcal{V}_t/\mathcal{G}_t, \mathcal{G}_t)}{E(\mathcal{V}_t)},$$

which completes the proof. □

Table 1: Properties of five types of CE returns

	CE1	CE2	CE3	CE4	CE5
ω_{it}	$\frac{\theta_1}{\mathcal{S}_t}$	$\frac{\theta_2}{\mathcal{S}_t^2}$	θ_3	$\frac{\theta_4}{1+\mathcal{S}_t^2}$	$\frac{\theta_5}{\mathcal{G}_t}$
$E_t\left(\omega_{it} r_{t+1}^*\right)$	$\theta_1 \mathcal{S}_t$	θ_2	$\theta_3 \mathcal{S}_t^2$	$\theta_4 \mathcal{U}_t$	$\theta_5 \mathcal{V}_t$
$Var_t\left(\omega_{it} r_{t+1}^*\right)$	θ_1^2	$\frac{\theta_2^2}{\mathcal{S}_t^2}$	$\theta_3^2 \mathcal{S}_t^2$	$\theta_4^2 \mathcal{U}_t\left(1-\mathcal{U}_t\right)$	$\theta_5^2 \frac{\mathcal{V}_t}{\mathcal{G}_t}$
$\mathbb{S}_i^2$	$E^2\left(\mathcal{S}_t\right)$	$\frac{1}{E\left(\mathcal{S}_t^{-2}\right)}$	$E\left(\mathcal{S}_t^2\right)$	$\frac{E^2(\mathcal{U}_t)}{E(\mathcal{U}_t(1-\mathcal{U}_t))}$	$\frac{E^2(\mathcal{V}_t)}{E(\mathcal{V}_t/\mathcal{G}_t)}$
S_i^2	$\frac{E^2(\mathcal{S}_t)}{1+Var(\mathcal{S}_t)}$	$\frac{1}{E\left(\mathcal{S}_t^{-2}\right)}$	$\frac{E^2\left(\mathcal{S}_t^2\right)}{E\left(\mathcal{S}_t^2\right)+Var\left(\mathcal{S}_t^2\right)}$	$\frac{E(\mathcal{U}_t)}{1-E(\mathcal{U}_t)}$	$\frac{E^2(\mathcal{V}_t)}{E(\mathcal{V}_t/\mathcal{G}_t)+Var(\mathcal{V}_t)}$
$\mathfrak{S}_i^2$	$\frac{E^2(\mathcal{S}_t)}{E(\mathcal{G}_t)}$	$\frac{1}{E\left(\mathcal{S}_t^{-2}\mathcal{G}_t\right)}$	$\frac{E^2\left(\mathcal{S}_t^2\right)}{E\left(\mathcal{S}_t^2\mathcal{G}_t\right)}$	$\frac{E^2(\mathcal{U}_t)}{E(\mathcal{U}_t(1-\mathcal{U}_t)\mathcal{G}_t)}$	$E\left(\mathcal{V}_t\right)$

Note: This table displays the scale associated to each type of conditionally efficient (CE) return studied in Propositions 1-5, and their conditional mean and variance. The table also displays their residual, Sharpe, and Sortino ratios.

Table 2: Performance for a normal mean and constant variance

	FW	CE1	CE3	CE4	CE5
$\mathbb{S}_p$	0.148	0.220	**0.273**	0.271	0.265
R_p^2	0.050	0.025	0.119	0.092	0.182
S_p	0.144	0.217	0.256	**0.258**	0.240
C_p	0.000	0.000	-0.307	-0.246	-0.453
$\mathfrak{S}_p$	0.226	0.336	0.476	0.462	**0.489**
$E\left(e^3\right)/\sigma^3$	-0.427	-0.444	-0.594	-0.551	-0.756
$E\left(d^3\right)/\sigma^3$	0.000	0.004	0.105	0.048	0.349
$3E\left(e^2 d\right)/\sigma^3$	0.000	0.000	1.232	0.852	2.458
A_p	-0.427	-0.440	0.743	0.349	2.051
$E\left(e^4\right)/\sigma^4$	4.476	4.710	10.855	8.539	22.548
$E\left(d^4\right)/\sigma^4$	0.007	0.002	0.175	0.053	1.252
$6E\left(e^3 d\right)/\sigma^4$	0.000	0.000	-2.442	-1.390	-7.919
$4E\left(e^2 d^2\right)/\sigma^4$	0.189	0.099	1.866	0.856	8.143
K_p	1.672	1.811	7.454	5.058	21.024

Note: This table displays several properties of fixed weight (FW) returns, and the types of conditionally efficient (CE) returns studied in Propositions 1, 3, 4, and 5, when there is a singly risky asset with a normal mean and constant variance. The autocorrelation of the mean is 0.8, and the coefficient of determination of the corresponding forecasting regression is 0.05. First, the table displays the residual, Sharpe, and Sortino ratios of these returns, with the bold ratios reporting the maximum values. The table also displays the coefficient of determination in the decomposition of Sharpe ratios (5) and the covariance term in the decomposition of Sortino ratios (6). These two decompositions are applied to CE returns in the commented propositions. Second, the table displays the coefficients of skewness and kurtosis of these returns, decomposed in several terms following (34) and (35), respectively. These two decompositions are applied to CE returns in Corollaries 1-2.

Table 3: Performance for a log-normal mean and constant variance

	FW&CE1	CE2	CE3	CE4
$\mathbb{S}_p$	0.147	0.039	**0.227**	0.211
R_p^2	0.030	0.000	0.548	0.176
S_p	0.144	0.039	0.152	**0.191**
C_p	0.000	0.132	-0.797	-0.357
$\mathfrak{S}_p$	0.213	0.053	0.442	0.357
$E\left(e^3\right)/\sigma^3$	-0.415	-1.532	-0.468	-0.580
$E\left(d^3\right)/\sigma^3$	0.024	0.000	13.340	0.357
$3E\left(e^2 d\right)/\sigma^3$	0.000	0.000	4.875	1.597
A_p	-0.392	-1.532	17.747	1.373
$E\left(e^4\right)/\sigma^4$	4.328	113.936	23.090	11.809
$E\left(d^4\right)/\sigma^4$	0.043	0.000	646.724	1.036
$6E\left(e^3 d\right)/\sigma^4$	0.000	0.000	-31.995	-3.337
$4E\left(e^2 d^2\right)/\sigma^4$	0.113	0.000	159.113	3.754
K_p	1.484	110.936	793.931	10.260

Note: This table displays several properties of fixed weight (FW) returns, and the types of conditionally efficient (CE) returns studied in Propositions 1-4, when there is a singly risky asset with a log-normal mean and constant variance. The autocorrelation of the mean is 0.8, and the coefficient of determination of the corresponding forecasting regression is 0.03. In this design, CE1 and FW returns coincide. First, the table displays the residual, Sharpe, and Sortino ratios of these returns, with the bold ratios reporting the maximum values of the first two ratios. The table also displays the coefficient of determination in the decomposition of Sharpe ratios (5) and the covariance term in the decomposition of Sortino ratios (6). These two decompositions are applied to CE returns in the commented propositions. Second, the table displays the coefficients of skewness and kurtosis of these returns, decomposed in several terms following (34) and (35), respectively. These two decompositions are applied to CE returns in Corollaries 1-2.

Table 4: Performance for constant mean and a log-normal variance

	FW&CE2	CE1	CE3	CE4	CE5
$\mathbb{S}_p$	0.144	0.186	**0.203**	0.202	0.197
R_p^2	0.000	0.006	0.038	0.030	0.090
S_p	0.144	0.186	0.199	**0.199**	0.188
C_p	0.078	0.000	-0.117	-0.097	-0.221
$\mathfrak{S}_p$	0.209	0.281	0.324	0.320	**0.329**
$E\left(e^3\right)/\sigma^3$	-0.500	-0.380	-0.466	-0.442	-0.799
$E\left(d^3\right)/\sigma^3$	0.000	0.001	0.029	0.014	0.924
$3E\left(e^2 d\right)/\sigma^3$	0.000	0.000	0.553	0.394	3.227
A_p	-0.500	-0.380	0.116	-0.034	3.353
$E\left(e^4\right)/\sigma^4$	5.025	2.470	4.545	3.805	58.169
$E\left(d^4\right)/\sigma^4$	0.000	0.000	0.057	0.013	41.098
$6E\left(e^3 d\right)/\sigma^4$	0.000	0.000	-0.975	-0.577	-56.599
$4E\left(e^2 d^2\right)/\sigma^4$	0.000	0.025	0.709	0.329	113.431
K_p	2.025	-0.504	1.336	0.570	153.099

Note: This table displays several properties of fixed weight (FW) returns, and the types of conditionally efficient (CE) returns studied in Propositions 1-5, when there is a singly risky asset with constant mean and a log-normal variance. The autocorrelation of the log-variance is 0.9, and the coefficient of determination of the variance is 1. In this design, CE2 and FW returns coincide. First, the table displays the residual, Sharpe, and Sortino ratios of these returns, with the bold ratios reporting the maximum values. The table also displays the coefficient of determination in the decomposition of Sharpe ratios (5) and the covariance term in the decomposition of Sortino ratios (6). These two decompositions are applied to CE returns in the commented propositions. Second, the table displays the coefficients of skewness and kurtosis of these returns, decomposed in several terms following (34) and (35), respectively. These two decompositions are applied to CE returns in Corollaries 1-2.

Table 5: Performance for a normal mean and a log-normal variance

	Panel A: $\rho = -0.1$					Panel B: $\rho = 0.1$				
	FW	CE1	CE3	CE4	CE5	FW	CE1	CE3	CE4	CE5
$\mathbb{S}_p$	0.145	0.183	**0.231**	0.229	0.206	0.145	0.179	**0.218**	0.217	0.205
R_p^2	0.010	0.019	0.121	0.083	0.355	0.010	0.015	0.092	0.068	0.219
S_p	0.144	0.183	0.217	**0.219**	0.166	0.144	0.177	0.208	**0.209**	0.181
C_p	0.060	0.000	-0.333	-0.252	-0.883	0.046	0.000	-0.269	-0.214	-0.542
$\mathfrak{S}_p$	0.213	0.280	0.404	0.388	**0.428**	0.213	0.268	0.369	0.359	**0.383**
$E\left(e^3\right)/\sigma^3$	-0.466	-0.381	-0.541	-0.491	-1.701	-0.503	-0.413	-0.581	-0.532	-1.188
$E\left(d^3\right)/\sigma^3$	0.000	0.004	0.217	0.065	10.725	0.000	0.002	0.125	0.046	2.660
$3E\left(e^2 d\right)/\sigma^3$	-0.018	0.000	1.475	0.888	15.925	0.019	0.000	1.210	0.789	6.832
A_p	-0.484	-0.378	1.151	0.462	24.949	-0.484	-0.411	0.755	0.303	8.304
$E\left(e^4\right)/\sigma^4$	4.949	2.958	8.497	5.929	376.952	5.016	3.011	8.030	5.952	107.025
$E\left(d^4\right)/\sigma^4$	0.000	0.002	0.835	0.094	756.671	0.000	0.001	0.371	0.061	92.419
$6E\left(e^3 d\right)/\sigma^4$	0.026	0.000	-3.267	-1.376	-443.284	-0.028	0.000	-2.700	-1.316	-104.261
$4E\left(e^2 d^2\right)/\sigma^4$	0.040	0.074	3.947	1.083	1182.622	0.040	0.061	2.526	0.869	208.830
K_p	2.015	0.034	7.012	2.729	1869.961	2.028	0.073	5.226	2.565	301.013

Note: This table displays several properties of fixed weight (FW) returns, and the types of conditionally efficient (CE) returns studied in Propositions 1, 3, 4, and 5, when there is a singly risky asset with a normal mean and a log-normal variance. The mean has an autocorrelation of 0.8, and the coefficient of determination is 0.01. The log-variance has an autocorrelation of 0.9, and the coefficient of variation of the variance is 0.8. Panel A reports the results for a correlation $\rho = -0.1$ between mean and log-variance shocks, while Panel B reports the results for a correlation $\rho = 0.1$. First, the table displays the residual, Sharpe, and Sortino ratios of these returns, with the bold ratios reporting the maximum values. The table also displays the coefficient of determination in the decomposition of Sharpe ratios (5) and the covariance term in the decomposition of Sortino ratios (6). These two decompositions are applied to CE returns in the commented propositions. Second, the table displays the coefficients of skewness and kurtosis of these returns, decomposed in several terms following (34) and (35), respectively. These two decompositions are applied to CE returns in Corollaries 1-2.

Table 6: Performance for a log-normal mean and a log-normal variance

	Panel A: $\rho = -0.1$					Panel B: $\rho = 0.1$				
	FW	CE1	CE2	CE3	CE4	FW	CE1	CE2	CE3	CE4
$\mathbb{S}_p$	0.145	0.177	0.078	**0.233**	0.226	0.145	0.170	0.084	**0.215**	0.211
R_p^2	0.010	0.022	0.000	0.290	0.122	0.010	0.017	0.000	0.197	0.097
S_p	0.144	0.175	0.078	0.197	**0.212**	0.144	0.169	0.084	0.193	**0.201**
C_p	0.057	0.000	0.145	-0.517	-0.293	0.043	0.000	0.129	-0.386	-0.245
$\mathfrak{S}_p$	0.212	0.267	0.108	0.432	0.386	0.211	0.254	0.117	0.378	0.351
$E\left(e^3\right)/\sigma^3$	-0.461	-0.375	-0.869	-0.524	-0.485	-0.497	-0.407	-0.835	-0.603	-0.532
$E\left(d^3\right)/\sigma^3$	0.002	0.010	0.000	2.912	0.168	0.002	0.006	0.000	1.228	0.121
$3E\left(e^2d\right)/\sigma^3$	-0.017	0.000	0.000	3.142	1.140	0.018	0.000	0.000	2.457	1.026
A_p	-0.475	-0.365	-0.869	5.530	0.823	-0.476	-0.401	-0.835	3.082	0.615
$E\left(e^4\right)/\sigma^4$	4.967	2.946	25.322	13.219	6.415	4.986	2.983	19.064	12.520	6.525
$E\left(d^4\right)/\sigma^4$	0.001	0.012	0.000	67.837	0.369	0.001	0.001	0.000	18.062	0.251
$6E\left(e^3d\right)/\sigma^4$	0.023	0.000	0.000	-13.057	-1.922	-0.027	0.000	0.000	-9.513	-1.895
$4E\left(e^2d^2\right)/\sigma^4$	0.036	0.088	0.000	42.898	2.040	0.043	0.067	0.000	21.084	1.702
K_p	2.028	0.045	22.322	107.897	3.902	2.003	0.055	16.064	39.153	3.583

Note: This table displays several properties of fixed weight (FW) returns, and the types of conditionally efficient (CE) returns studied in Propositions 1-4, when there is a singly risky asset with a log-normal mean and a log-normal variance. The mean has an autocorrelation of 0.8, and the coefficient of determination is 0.01. The log-variance has an autocorrelation of 0.9, and the coefficient of variation of the variance is 0.8. Panel A reports the results for a correlation $\rho = -0.1$ between log-mean and log-variance shocks, while Panel B reports the results for a correlation $\rho = 0.1$. First, the table displays the residual, Sharpe, and Sortino ratios of these returns, with the bold ratios reporting the maximum values of the first two ratios. The table also displays the coefficient of determination in the decomposition of Sharpe ratios (5) and the covariance term in the decomposition of Sortino ratios (6). These two decompositions are applied to CE returns in the commented propositions. Second, the table displays the coefficients of skewness and kurtosis of these returns, decomposed in several terms following (34) and (35), respectively. These two decompositions are applied to CE returns in Corollaries 1-2.