Abstract

A Study of the Interaction of Insurance and Financial Markets: Efficiency and Full Insurance Coverage

Abstract

This paper studies the interaction between insurance and capital markets within a single but general framework. We analyze insurance and investment decisions as well as insurance and investment prices in competitive equilibrium. We determine a set of conditions for agents to optimally wish to purchase full coverage and demonstrate that they are satisfied in an efficient insurance market equilibrium, even if insurance prices carry a positive loading to compensate for undiversifiable risk. We are able to characterize agents' investment strategies and determine the equilibrium price of insurance contracts and firms. We show that insurance contracts are determined by their actuarial value plus a loading reflecting the aggregate price of risk. We also show that capital markets greatly enhance the risk sharing capacity of insurance markets and the scope of risks that are insurable because efficiency does not depend on the number of agents at risk, nor on risks being independent, nor on the preferences and endowments of agents at risk being the same.

"Consumers trading in both markets at once use the financial market to diversify their investment portfolio and use the insurance market to insure their personal risk. In ignoring trade in financial assets, the formal model in this paper bypasses an important aspect of consumer behavior under risk." Marshall (1974b, p675)

This paper provides a framework to study insurance without bypassing trading in financial assets. Thus we can study the way insurance and financial markets interact both in partial and general equilibrium. Using this framework we are able to establish a number of results on the demand for insurance, insurance pricing, asset demands and insurance market efficiency. In particular, we show that efficient insurance with frictionless financial markets implies, but is not equivalent to, agents purchasing private insurance contracts that provide them with full coverage even if insurance prices are not actuarially fair.

The most striking result in this paper is that under quite general circumstances agent's optimal insurance strategies are to purchase full coverage. Mossin(1968) established the well-known result that "if the [insurance] premium is actuarially unfavorable, then it will never be optimal to take full coverage". And yet we show that the purchase of full coverage will occur in equilibrium even if equilibrium insurance prices are actuarially unfavorable. The main difference between our result and Mossin's is the context in which the coverage decision is made. Mossin makes a statement about how agents behave when faced with an isolated insurance decision. We, on the other hand, consider the case where the agent faces insurance as well as other financial risks, and agents have access to a sufficiently rich set of financial instruments (and these may just be a riskless bond and a single risky portfolio). What we demonstrate is that under certain conditions agents will separate their insurance decisions from their financial decisions, eliminating their personal risk (via a full coverage contract) while participating in the market risk (via risky investments). Further, we prove that these conditions are satisfied in equilibrium and hence our results are valid quite generally.

A second result we obtain from this framework is that insurance markets can be very efficient institutions to attain optimal risk sharing if the economy has active financial markets. Specifically, agents can attain Pareto efficient consumption allocations simply by purchasing insurance and investing in insurance company shares and bonds even if risks are not independent and there is substantial aggregate risk (as in markets ensuring against catastrophic events). Thus agents act both as insureds (when they purchase private insurance) and as insurers (when they invest in risky insurance company stock). Furthermore, we provide a simple explicit description of how to construct the specific portfolio of insurance company shares agents will use and demonstrate that it will be an equally weighted portfolio of all insurance companies.

Finally, we can show that a number of important results demonstrated in different contexts in the literature can be seen to hold quite clearly in this unified framework. For example, as markets are frictionless and there are no agency costs reinsurance is redundant as put forward in Doherty and Tiniç(1981). In fact, as investors will always hold a fully diversified portfolio of all the insurance companies, the number of insurance companies is irrelevant (as long as they act competitively). Also, when there is aggregate risk arising from insurance activity, investors need to receive a positive risk premium to compensate for it. We show how this premium determines the price of risk for the economy and how this price of risk trickles down to private insurance prices in the form of a positive loading. This loading is determined not by the interaction of the risk aversions of insurance companies and reinsurers (as in many papers from Borch(1962) to Aase(2002)) but from insurance markets where companies are mere intermediaries (Marshall (1974b) refers to them as brokerage firms) modelled as financial assets priced in equilibrium. The technique of using equilibrium prices to determine insurance contract prices is applied in Ellickson and Penalva(1997), Aase(1999) or Schweitzer (2001). We show that an important property of this loading is that it does not depend on the insured's willingness to pay for insurance but on the market price for risk. We conclude by solving for the optimal investment decisions of agents with quite commonly used (HARA) utility functions. We show that in economies with agents that have HARA preferences the agent's optimal investment decisions will be static buy-and-hold strategies and we are able to compute what those strategies would be. The implication of this is that under the HARA assumption efficient risk sharing can be quite easily attained even in the presence of quite complex risks.

In looking at the related literature, we know only of one reference we have come across recently that states something closely related to the second part of our result on full insurance, namely that people purchase full coverage in an efficient equilibrium. In studying the information costs of different insurance market mechanisms in a very stylized setting, Kihlstrom and Pauly (1971) state: “Persons who bear part of the total loss might be thought of as having a kind of split personality in which they make a certain payment in return for coverage which does not depend on total loss but in which they hold “stock” in an insurance “firm” which makes their final wealth positions vary with the total loss” (quotes in the original). In our framework we do not need to use quotes to refer to stocks of insurance firms as they are explicitly included and the result is shown to hold quite generally. To some economists this result may seem like the most natural way, even the obvious way, to attain equilibrium consumption allocations and yet we have not found other references in the insurance literature. Also, our analysis goes further in that we identify conditions under which equilibrium implies full insurance demand and show that these conditions could also hold out of equilibrium.

A further contribution of this paper is the framework of analysis itself. This framework is designed to incorporate both the problem of private insurance and the problem of investing in financial markets together and to study how they interact. Other papers have looked into the interaction between insurance and investment decisions using alternative methods. Some have studied the individual insurance and investment decision problem: Smith and Mayers (1983), Eeckhoudt et al (1997) and Somerville (2004), while others take a more general equilibrium perspective: Penalva(2001). Our framework is related to the one in Penalva (2001). It differs in that the one we develop here is specifically designed to study insurance problems. In particular, we do not allow dynamic trading in insurance contracts – agents are restricted to buy-an-hold strategies – we strip out intermediate consumption dates and we introduce actual insurance (stock) companies into the model. Also, the focus of our analysis differs greatly: Penalva(2001) looks for conditions that will ensure efficiency while we are more interested in the consequences of efficiency for insurance markets and on the specific details of insurance prices, and agent's insurance demand and investment decisions. Nevertheless, we also make statements on insurance market efficiency which are related to those in Penalva(2001). Our results differ in that the financial assets traded in standard insurance markets as defined in this paper are different from the ones in Penalva(2001) (specially we do not allow dynamic trading in insurance contracts) and in the notion of equilibrium we use.

We are also not the first to study the problem of the efficiency of insurance markets. Borch(1962) and Wilson(1968) establish the 'mutuality principle': under conditions of uncertainty a Pareto optimal consumption allocation is characterized by individual consumption allocations that depend on uncertainty only through the aggregate level of income. Marshall(1974a) discusses how this (mutuality) approach to insurance provides a solution to the provision of catastrophic insurance that cannot be obtained from a reserves-based approach (which relies on the Law of Large Numbers) – an argument repeated in different forms in the literature and one that motivates our analysis of the role of financial markets as an institution to implement efficient risk-sharing. Many papers have studied alternative risk-sharing mechanisms that embody this mutuality principle. One mechanism is to build aggregate risk sharing into insurance industry structure, through the institution of mutual insurance companies – Dionne and Doherty (1993) uses this argument to explain the high proportion of mutuals relative to stock companies in the insurance sector, despite their higher capital costs as documented in Harrington Niehaus

(2002). Another alternative is to have aggregate risk incorporated into insurance contract design (Cummins and Mahul (2001), Cass Chichilninsky and Wu (1996)). Risk-sharing can also take place via secondary markets such as reinsurance (Borch (1984), Doherty Tiniç (1981), Froot (2001), Jaffee Russell (1997), Zanjani (2002)) and other financial markets (Ellickson Penalva (1997), Harrington Niehaus (1999), Aase (2001), Christensen et al (2001), Penalva(2001)). We take the latter approach and our results show that optimal insurance can be efficiently provided by insurance companies whose shares are traded in the stock market. Insurance companies insure individuals while investors assume aggregate insurance risk via insurance company share prices. This implies that in our context (traded) stock companies are better than mutuals as agents investing in stock companies can adjust their risk exposures by changing their stock holdings even doing this over time, while members of a mutual not only have limited control on what proportion of the mutual they own but they also have limited ability to adjust that proportion over time.

The paper is structured as follows: as we want to introduce a new framework we will spend quite a bit of time after this introduction giving a detailed description of the different aspects that define the model: first we will describe the agents and their risks, then the insurance market and the stock market, and after that we will put everything together to define agents' budget constraints and the corresponding notion of equilibrium. The section following the description of the model contains the analysis of optimal insurance demand and conditions that lead to full insurance purchases. Insurance market efficiency is then proven in section 3, we study the corresponding equilibrium insurance and asset demands in section 4, where we show the conditions from section 2 apply giving us full insurance purchases. Section 5 discusses how the results extend to insurance markets with more complex risks and considers what will happen if agents have HARA preferences. We then provide some concluding remarks.

Only the short and simpler proofs are in the text, the rest we have put in the appendix.

1 The Framework

We will construct a general economy which we will call $\mathcal{E}(B)$ and which represents the general framework we will be working with. Also, both for expository purposes as well as for purposes of comparison with existing results it will be useful to consider a more restricted model which we will call $\mathcal{E}(A)$. Economy $\mathcal{E}(A)$ is a simpler economy that is designed to resemble the standard insurance model (and is referred to as such): in $\mathcal{E}(A)$ all agents have the same preferences and endowments and are subject to independently and identically distributed risks (assumptions A.1 and A.2 below). Economy $\mathcal{E}(B)$ is a more general economy where agents have heterogeneous preferences and endowments and risks may not be independent (assumptions B.1 and B.2 below).

Our framework incorporates a time dimension that is not standard in insurance models. This is done so that we can combine a regular insurance market with a standard model of financial markets. The way this is done may be seem confusing at first so let us start by describing how time enters into the model. We will assume time goes from now, date $t = 0$, to some point in the future, $t = 1$. Consumption and insurance take place at dates zero and one, while investment decisions are made at every t between now and the future ($t \in [0, 1]$).

Agents have endowments of goods at date zero and one (for simplicity we assume that there is only one good for consumption at each date). At date zero agents decide their consumption and their optimal insurance purchases, taking into account that they can also invest now and that their future income (at date one) will depend on their (investment and insurance) decisions and some idiosyncratic risk.

Agents' idiosyncratic risk is the possibility that they suffer 'an accident' between date zero and one. The real effects of an accident is a loss of period one ($t = 1$) endowment. In addition to making consumption and insurance decisions, agents can invest (choose the contents of their financial portfolios). Formally (as in standard models in finance), agents choose trading strategies that involve buying and selling shares and bonds in financial markets. These strategies can involve portfolio changes at any time between dates zero and one, but insurance contracts are not traded in stock markets (and we do not treat them as other financial assets).

At date one, agents income available for consumption is determined by (1) their endowments, (2) whether they suffered accidents or not, (3) the amount of insurance they had chosen at date zero, and (4) their investment strategies.

1.1 The Model: Preferences and Risk

We analyze an economy with a finite number of agents, $n < \infty$. Agents can consume at date 0 and date 1. In the standard insurance model all agents have the same preferences and endowments. Agents are expected utility maximizers with common priors and their preferences are given by:

Assumption A.1 For all $i = 1, \dots, n$, $U_i(x) = u(x(0)) + \beta E(u(x(1)))$, where u is an increasing, strictly concave differentiable function satisfying the standard Inada conditions[1].

In the general framework, preferences can be quite heterogeneous:

Assumption B.1 For all $i = 1, \dots, n$, $U_i(x) = u_i(x(0)) + \beta_i E\left[v_i(x(1))\right]$, where both u_i and v_i are increasing, strictly concave differentiable functions satisfying the standard Inada conditions.

Agents' date one endowments are risky in the sense that if an agent has an accident his date one endowment will be lower. Agent i's endowment is denoted $e_i \equiv (e_i(t))_{t\in\{0,1\}}$, the aggregate endowment is denoted $e \equiv (e(t))_{t\in\{0,1\}}$, $e(t) = \sum_{i=1}^{n} e_i(t)$, and we use the random variable $N_i(1)$ to count the number of accidents suffered by agent i. We will assume that $N_i(1)$ can take values 0 or 1. In the standard insurance model:

Assumption A.2 All agents have the same endowments and they will have a date one endowment loss of magnitude $L < w$ with probability p (independent across agents) so that $e_i(0) = w > 0$ and $e_i(1) = w - N_i(1)L$.

It will be useful to keep track of the total number of accidents in the economy via the random variable $N(1) \equiv \sum_{i=1}^{n} N_i(1)$.

[1] The Inada conditions are outlined in Appendix A.1

1.2 Insurance and the stock market

Insurance and stock markets interact in this model as agents can purchase private insurance and invest. The link between the two is provided by insurance companies who sell insurance and issue shares. We combine a standard model of financial markets with a standard insurance market with the added time dimension. First we define the individual's insurance decision. Then, we describe insurance companies and the stock market. After that we'll provide some details about the information that is revealed and which will affect stock prices. Finally, we will conclude this section with a description of financial asset prices and the formal definitions of the economies $\mathcal{E}(A)$ and $\mathcal{E}(B)$.

The insurance decision: Agents can buy insurance to compensate them for losses from accidents. Agent i faces a price per unit of coverage denoted S_i^I and can choose his optimal level of coverage α_i. The endowment together with the insurance decision leave the following amounts for consumption and investment: at date zero $e_i(0) - \alpha_i S_i^I$, and at date one $e_i(1) + N_i(1)\alpha_i$. In the familiar standard insurance model, $\mathcal{E}(A)$ at date zero agents are left with $w - \alpha_i S_i^I$ and at date one $w - N_i(1)(L - \alpha_i)$.

Insurance companies and stock markets: Stock markets are slightly more complex to model. We first define an insurance company and the stock market. Then describe the interaction between share prices and information.

An insurance company is an institution whose objective is to sell private insurance contracts and raise capital to cover future indemnity payments. It can raise capital by issuing shares and by borrowing from the money market. To simplify the exposition we will assume there is a fixed number of insurance companies, J. Each company issues one perfectly divisible share. Companies are fully equity financed. They invest the insurance premia in a riskless bond and at date one issue dividends equal to the firm's assets (premia plus interest) minus indemnities payable. Because we do not allow default the owners of insurance firms stand to loose money if there are too many accidents and will require compensation for taking this risk[2].

The stock market is an institution in which an auctioneer continuously sets prices to facilitate share trading. We assume there is no private information or agency costs and the auctioneer sets prices such that no arbitrage opportunities exist. Agents can go to the stock market and trade shares at the announced prices at any time without any costs, frictions or constraints. As agents have common priors, trades are purely motivated by the desire to control risk exposures.

Information: The relevant information in this market is agents' accidents (or lack thereof). As explained above, an accident to agent i can happen at any time between dates zero and one. We denote the random time of the accident to i by $\tau_i \in [0, 1]$. Although the consequences of the accident on i's endowment will not be realized until date one, the information will have an immediate effect on share prices as it reveals information on future insurance company liabilities (and hence future dividends).

We assume that for each agent i the arrival time of an accident is a

[2]An alternative and equivalent way to model insurance companies is to require them to raise an amount of equity sufficient to fully reserve against all contingencies. Then, the value of the company's capital would equal the value of the old share plus the value of the necessary reserve capital. The reserves will then be invested in the riskless asset and used to ensure the value of net liabilities never falls below zero.

smooth function of calendar time. More specifically, if τ_i is the time that agent i suffers an accident, τ_i is distributed as an exponential distribution with parameter λ (naturally, $\tau_i > 1$ implies agent i suffers no accident at date one, i.e. $N_i(1) = 0$). Under Assumption A.2 agents' risks are independent so that the parameter λ is related to p via the following equation:

$$p = 1 - \exp(-\lambda) \;\Leftrightarrow\quad \lambda = -\ln(1 - p).$$

In the more general model, agents' risks need not be completely independent. In order to generalize the risk process we need to introduce some formal concepts and notation (the reader just interested in the intuition can jump to Assumption B.2 below and the examples thereafter). Here we provide an introductory explanation and all the technical details are given in Appendix A.

We assume there is a probability space $(\Omega', \mathcal{F}_1, P)$ describing all uncertainty in the economy, where Ω' represents all the states of the world (including the number of accidents and the corresponding arrival times) and $\mathcal{F}_1$ represents all the information that will be revealed up to date one. A set $A \subseteq \Omega'$ is an event and P is the probability measure that tells us how probable any event A is. We use $\mathbf{1}_A$ to denote the indicator function of an event A, i.e. for all $\omega \in \Omega'$, $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and zero otherwise. All agents have common priors so they agree on P. $E_P[x]$ represents the expectation of the random variable x with respect to measure P. Given a stochastic process $(x(t))_{t\in[0,1]}$, define for any $t \in [0, 1]$ the random variable $x(t-) \equiv \lim_{s\uparrow t} x(s)$. In the stock market, as we saw earlier, the information that drives prices is accident information. We keep track of this information with the processes $(N_i(t))_{i=1}^n$ and $N(t)$, where $N_i(t)$ counts the number of accidents that have occurred to agent i up to (and including) date t and $N(t) = \sum_{i=1}^{n} N_i(t)$ counts the total number of accidents for the whole economy. All the information generated by $(N_i(t))_{i=1}^{n}$ is public information and is formally described by the filtration $(\mathcal{F}_t)_{t\in[0,1]}$.

As we are assuming agents are exposed only to one possible accident then in economy $\mathcal{E}(A)$, $\lambda_i(t) = \lambda \mathbf{1}_{\{N_i(t-)=0\}}$, i.e. after an accident, when $N_i(t) = 1$, agent i is no longer at risk and his hazard is zero.

In the general case we assume that accidents are subject to possible contagion effects. We do this by assuming the hazard rate of an accident is described by a function $g(t, N(t-))$ instead of the constant λ (when it is not zero, i.e. prior to the accident), $\lambda_i(t) = g(t, N(t-))\mathbf{1}_{\{N_i(t-)=0\}}$. Substituting λ by $g(t, N(t-))$ allows for two effects: one, a non-time homogenous hazard rate, e.g. an accident can be more (less) likely at the beginning (at the end) of the trading period ($[0, 1]$); and two, the hazard rate can depend on $N(t-)$, i.e. on the current total number of accidents. The latter effect allows us to model contagion by allowing the hazard rate to increase (or fall) as the number of accidents in the economy increases.

Assumption B.2 All agents will have a date one endowment loss of magnitude $L < w_{i,1}$ with probability p so that $e_i(0) = w_{i,0} > 0$ and $e_i(1) = w_{i,1} - N_i(1)L$. The distribution of accident arrivals is described by a hazard rate $\lambda_i(t) = g(t, N(t-))\mathbf{1}_{\{N_i(t-)=0\}}$.

To illustrate this assumption consider the following two examples.

Example 1 (forest fires): consider the risk of fire destroying houses in a wooded neighborhood. Describe the risk of a fire affecting your house using two constants, $\lambda > 0$ and $\gamma > 1$, as follows: at the beginning, if $N(t) = 0$, the hazard rate for every agent is λ. If one agent's house suffers a fire at time s, so that $N(t) \geq 1$ for $t > s$, then there is a chance of a forest fire and everyone else's hazard rate increases to $\lambda\gamma$. The hazard rate will then be: $g(t, 0) = \lambda$ and $g(t, x) = \lambda\gamma$ for $x \geq 1$.

Example 2 (structural collapse): consider the risk of houses falling down because their structure fails. Assume all houses are the same. Then one would think that the risk of a house's structure collapsing is independent from that of others but that the probability that such a collapse occurs between today and tomorrow is increasing with the time since the house was built. This can be easily incorporated by letting $g(t, N(t-)) = \beta t^{\alpha}$ with $\beta > 0$ and $\alpha > 1$.

Asset prices: We have looked at the information that will be revealed. Now we focus on how that information relates to the price of insurance company shares, but first, one small bit of notation. Each of the J insurance companies will sell insurance to a set of agents. For each $j = 1, \ldots, J$ the index set $I_j \subset \{1, \ldots, n\}$ denotes the set containing the indices (the 'names') which identify the agents that buy insurance from firm j.

There are two types of dynamically traded assets: insurance company shares (assets $j = 1, \ldots, J$), and a zero coupon bond (asset $j = 0$). Any dynamically traded asset is formally described by a final dividend d_j (a random variable on $(\Omega', \mathcal{F}_1, P)$) and a stochastic price process $S_j(t)$, with $S_j(1) = d_j$. For the riskless zero coupon bond: $d_0 = 1$, while for insurance company j, its dividend will depend on the value of its assets minus that of its expected liabilities, which depends on $\sum_{i \in I_j} N_i(1)$. As the auctioneer sets prices so that there are no arbitrage opportunities, we know at least since the work of Harrison and Kreps(1979), that there exists a probability measure Q on

$(\Omega', \mathcal{F}_1)$ and an interest rate process $r(t)$ such that for all $j = 0, 1, \ldots, J$, the price of asset j can be described by:

$$\forall t \in [0,1], \quad S_j(t) = E_Q\left[d_j e^{-\int_t^1 r(s)\,\mathrm{d}s} \middle| \mathcal{F}_t\right] \qquad P-\text{a.s.} \tag{1}$$

We will use the notation $\mathbf{r} = \int_0^1 r(t)\,\mathrm{d}t$ for the interest rate, and $\mathcal{D} = (S_j(t))_{j=0}^J$ for the set of assets traded in stock markets.

Our two economies: Having defined agents preferences and endowments, the insurance problem and the stock market, the traded assets and the way asset prices are described, we can formalize the two economies, $\mathcal{E}(A)$ and $\mathcal{E}(B)$, and do some analysis. The two economies are distinguished by the set of assumptions on preference-endowment pairs and risks. In the standard insurance model, economy $\mathcal{E}(A)$, we have homogenous preferences (A.1) and homogenous endowments and independent risks (A.2). In the more general framework, economy $\mathcal{E}(B)$, agents have heterogeneous preferences (assumption A.1 is replaced by assumption B.1) and heterogeneous endowments and non-independent risks (B.2 for A.2). We will also distinguish an economy by the types of markets they have: economies with state-contingent commodities (as in the original Arrow-Debreu setup) and economies with asset markets substituting for state-contingent commodity ones. An economy with state-contingent commodity markets is denoted by $\mathcal{E}(x)$, where x stands for A, the simple economy, or B, the more general one. An economy with asset markets instead of state-contingent commodity ones is denoted by $\mathcal{E}(x, \mathcal{D})$, where x stands for the type of economy (A or B) and $\mathcal{D}$ acts (for now) only as a marker for the presence of insurance and financial markets. When analyzing economy A and economy B with financial markets, we allow everyone to buy private insurance coverage and invest in the bond and insurance company shares.

1.3 The budget constraint

A crucial component in the agent's decision process is his budget constraint, the set of consumption allocations the agent can achieve given his endowment (and trading opportunities). In an economy with financial markets, the agent can only buy insurance and trade assets to alter his consumption pattern. In terms of insurance, agent i can buy as much (non-negative) coverage as he wishes, $\alpha_i \in \mathbf{R}^+$. He can also use financial markets to reallocate consumption and his set of possible changes in consumption patterns is determined by the set of existing asset prices. These asset prices are summarized by the vector[3] $\mathcal{D} \equiv ((S_j(t))_{t\in[0,1]})_{j=0}^{J}$. Given prices $\mathcal{D}$, let agent i's investment strategy in asset j be represented by $\theta_j^i(t)$ where $\theta_j^i(t)$ is the number of units of asset j agent i plans to hold going into date t. The set of allowable investment strategies is denoted by Θ. We make the natural restriction that there are no trading strategies in Θ that anticipate information or that require an agent to hold an infinite amount of any asset. We also assume that allowable trading strategies do not imply adding or reducing wealth between dates 0 and[4] 1.

We assume that each agent is endowed with some (possibly zero) shares of insurance companies and no one starts with any amount of the riskless asset;

[3]The careful reader will have noticed the coincidence of notation with the financial markets marker above. Economy $\mathcal{E}(x, \mathcal{D})$ is economy x with insurance plus financial markets where assets $\mathcal{D}$ are traded. Insurance prices are not reflected in the notation to reduce clutter.

[4]This is a condition arising from the budget constraint but we include it in the definition of allowable trading strategies to simplify the presentation.

for $i = 1, \ldots, n$ and $j = 0, 1, \ldots, J$, let $\theta^i_{j,0}$ denote the number of units of asset j agent i is endowed with at date zero (and in particular $\theta^i_{0,0} = 0$ for all $i = 1, \ldots, n$). Agent i's full initial endowment is $e_i(0) = w + \sum_{j=1}^{J} \theta^i_{j,0} S_j(0)$.

Agent i's budget constraint is then determined by asset prices, the price of insurance he faces, S^I_i, and the set of allowable trading strategies Θ. The budget constraint is denoted $B_i(\mathcal{D}, S^I_i)$ and is given by:

$$B_i(\mathcal{D}, S^I_i) = \left\{ x = (x(0), x(1)) \;\middle|\; \begin{array}{l} \exists \theta^i \in \Theta,\ \alpha_i \in \mathbf{R}^+ \\ x(0) = e_i(0) - \alpha_i S^I_i - \sum_{j=0}^{J} \theta^i_j(0) S_j(0) \\ x(1) = e_i(1) + \alpha_i N_i(1) + \sum_{j=0}^{J} \theta^i_j(1) d_j \end{array} \right\}$$

2 Optimal Insurance and Trading

The first question we consider is how the presence of additional investment opportunities (the stock market) affects insurance decisions in competitive insurance markets. In particular, we will show that the result in Mossin(1968): “if the [insurance] premium is actuarially unfavorable, then it will never be optimal to take full coverage” does not mean that observing actuarially unfavorable prices is incompatible with observing agents purchasing full coverage in equilibrium.

We know (e.g. Smith Mayers (1983)) that when there are investment as well as insurance opportunities, then in general the optimal amount of coverage will depend on the interaction between indemnity payments and the returns on the rest of the assets in the economy. We will show that this may not occur in competitive insurance markets, in particular, we will show that under some, quite general circumstances, the demand for insurance is independent of the returns of other investments. Furthermore, we will show that the demand for insurance will be a demand for full insurance coverage even if premia are unfair.

The conditions we use are that insurance prices are competitive and in this economy the following two conditions hold: (i) only undiversifiable risk carries a risk premium, and (ii) agents can tailor their exposure to aggregate risk.

Above we discussed how no arbitrage opportunities implies a pricing relation (equation 1) for financial assets. Here we want to extend this notion to insurance pricing and define what we mean by competitive insurance prices: given a set of financial asset prices that can be described by the pair (Q, r) as in (1), we say that insurance prices are *competitive* under (Q, r) if for all $B \subseteq \{1, \ldots, n\}$, one can construct $(S(t), d)$ satisfying relation (1) such that $S(0) = \sum_{i \in B} S_i^I$ and $d = \sum_{i \in B} N_i(1)$.

This condition states, literally, that the price of insurance should be determined as it where a financial asset which pays one unit of consumption when the accident occurs and the prices satisfies the no-arbitrage condition. We do not believe one can treat insurance contracts as standard financial assets priced via no-arbitrage arguments, among other things because they would be a highly illiquid type of asset. But this condition can be defended as a result of competition (e.g. Bertrand-type successive price cutting among insurance firms).

Let us elaborate on this. Private insurance markets are defined by firms that issue private insurance and finance the risks through the stock market. Consider a company selling a single insurance contract at price at $S_i^I + \epsilon$, where S_i^I is what we call its competitive price, $S_i^I = e^{-\mathbf{r}} E_Q[N_i(1)]$ and $\epsilon > 0$. The market value of the liabilities from this insurance contract at date zero is $e^{-\mathbf{r}} E_Q[N_i(1)]$ so that the manager of the firm can compensate investors for the risk from insurance company liabilities by paying them S_i^I and keeping ϵ. This extra gain of ϵ for the manager attracts another manager who can offer insurance to the customer i at a price of $S_i^I + \epsilon/2$. The new manager will drive out the old one at a profit and still compensate his investors for the new liabilities. This undercutting will continue until insurance is offered at its competitive price, S_i^I, and the initial value of every insurance company will be zero. Note that it is not reasonable for a manager to sell insurance at a price below $e^{-\mathbf{r}} E_Q[N_i(1)]$ as he will need to add money from his own pockets to compensate investors in order to raise enough capital.

We now turn to defining formally the conditions (i) and (ii) stated at the beginning of this section (for easy of exposition we have relegated the more technical definitions to Appendix A.1). Consider a filtered probability space $(\Omega', (\mathcal{F}_t)_{t\in[0,1]})$ and two absolutely continuous measures P and Q. Let P_t and Q_t define the restriction to $(\Omega', \mathcal{F}_t)$ of P and Q respectively.

Definition 1 *In economy $\mathcal{E}$,* a measure Q is said to price only aggregate risk *if there exists a real-valued function f such that $\xi(1) = f(e(1))$ where*

$$Q(\omega) = P(\omega)\xi(1,\omega); \qquad \xi(t,\omega) \equiv \frac{\mathrm{d}Q_t}{\mathrm{d}P_t}(\omega) = \frac{\xi(1,\omega)}{E_P[\xi(1)|\mathcal{F}_t]}$$

Definition 2 *In economy $\mathcal{E}$,* a set of assets $\mathcal{D}$ is said to span aggregate uncertainty *if for every consumption allocation x such that there exists g_i :*

$\Omega' \to \mathbf{R}$ *and* $x(1) = g_i(e(1))$ *there exists* $\theta \in \Theta$ *such that*

$$x(1) = \sum_{j=0}^{J} \theta_j(0) S_j(0) + \int_0^1 \sum_{j=0}^{J} \theta_j(t) \, \mathrm{d}S_j(t)$$

If measure Q prices only aggregate risk then the price of an asset depends only on its relationship with aggregate risk. If some arbitrary given set of assets spans aggregate uncertainty then the agent is free to transfer consumption between different realizations of aggregate uncertainty arbitrarily. Under these conditions we can relate insurance and investments markets in an economy with competitive insurance even if it is not in equilibrium.

Theorem 3 *In economy $\mathcal{E}(B)$, with prices $(\mathcal{D}, S_i^I)$, agent i's optimal insurance demand includes buying full insurance if:*
(i) the set of assets $\mathcal{D}$ span aggregate uncertainty;
(ii) there exists (Q, r) such that $\mathcal{D}$ satisfy Relation 1 and Q prices only aggregate risk; and,
(iii) insurance prices, $(S_i^I)_{i=1}^n$, are competitive under (Q, r).

Corollary 1 *Under the conditions of Theorem 3 every agent can make his optimal investment decision independently of his demand for insurance.*

Thus, if an economy has a sufficiently rich set of assets traded in the stock market (in terms of what agents can achieve using those assets) and if the investors in that economy only care for aggregate risk then we should observe some very special behavior in competitive insurance markets: we should observe people buying full insurance coverage. Note that nothing has been said about whether prices are actuarially fair or not but the conditions are sufficiently general that it should be quite possible for (Q, r) to exist such that insurance prices are not actuarially fair and the conditions of Theorem 3 holds. The skeptical reader will find an example of such a (Q, r) in Section 4, but before we get to a particular example we want to construct an insurance market equilibria and check its efficiency properties.

3 Insurance Efficiency and Equilibrium

3.1 Known Equilibrium Results

The approach we will follow is to compare what happens in an economy with insurance and other assets with an economy with state-contingent commodity markets. We use the state-contingent equilibrium as our point of reference because we know from the first welfare theorem that the equilibrium allocations in the state-contingent commodity economy will be Pareto efficient. If those allocations can be attained as an equilibrium with insurance and financial assets then insurance markets will be efficient.

We will also explore the details of financial markets and their relationship with efficiency. Take the simplest economy, $\mathcal{E}(A)$. We know from Arrow's pioneering work (Arrow 1964) that without dynamic trading one needs 2^n financial assets to obtain the state-contingent equilibrium of $\mathcal{E}(A)$ as a financial markets equilibrium. Malinvaud (1973) establishes that the economy $\mathcal{E}(A, \mathcal{D})$ with only insurance contracts and no other financial markets will not be efficient in general (specially if $n < \infty$) because individual risks generate aggregate risk. Efficiency cannot be attained in general because n insurance contracts are insufficient to deal both with individual risk and aggregate risk. If n is very large then, by the law of large numbers, aggregate risk almost disappears so that n insurance contracts suffice to obtain an approximate equilibrium. The results of Arrow(1964) are extended by Duffie and Huang(1985) to markets with dynamic trading and asset market equilibrium (as defined by Radner (1972)). If agents can change their asset positions over time, then the number of assets needed to obtain efficiency can be much smaller than 2^n, namely you need at most $K + 1$ 'appropriate' assets. The value of K is a fixed number determined by prices and allowable trading strategies. In Appendix B we demonstrate that for our economy $K = n$. Within our framework, and in addition to the results on full insurance demand we hope to say something more. We want to be specific about what assets are traded, how are they traded and by whom.

First we will establish how existing results relate to our framework formally. State-contingent equilibrium (Arrow-Debreu equilibrium) is a standard concept so we will not define it here. For the market with financial assets it is standard to use the notion of Radner equilibrium (e.g. Penalva(2001)) but our model includes insurance contracts that are illiquid/not traded dynamically so we define a new equilibrium for insurance markets. The notion of equilibrium we want to use is one in which agents are making their consumption, insurance and trading decisions in an optimal manner, given insurance and asset prices. These prices, set by the auctioneer, have to lead to market clearing in the goods market. Also, these prices cannot allow arbitrage opportunities in asset trading. We also require that private insurance markets are competitive as defined in the previous section.

Definition 4 *A triple* $((x_i)_{i=1}^n, (S_i^I))_{i=1}^n, \mathcal{D})$ *is an insurance market equilibrium if:*

(i) for all $i = 1, \ldots, n$, $x_i \in B(\mathcal{D}, S_i^I)$ and for all $x' \in B(\mathcal{D}, S_i^I)$, $U_i(x_i) \geq U_i(x')$;
(ii) there exists (Q, r) such that every $S_j \in \mathcal{D}$ satisfies Relation (1); and
(iii) insurance prices are competitive.

Remark 1 *For the economy $\mathcal{E}(B)$ there exists an insurance market equilibrium $((x_i^*)_{i=1}^n, (S_i^I))_{i=1}^n, \mathcal{D})$ such that $((x_i^*)_{i=1}^n)$ is Pareto optimal.*

Without any restrictions on the set of dynamically traded assets, $\mathcal{D}$, we can get this result just by ensuring that $\mathcal{D}$ contains enough appropriate securities as was demonstrated in Duffie Huang(1985). But this is not our objective; we want to say something more precise about the interaction of $\mathcal{D}$ with insurance prices and insurance coverage decisions, and about the way agents invest in this economy.

3.2 Efficiency and Insurance Demand

From the discussion above and Remark 1, we know that markets will be complete with dynamic trading of $n + 1$ assets and agents will be able to achieve efficient allocations in an insurance market equilibrium. We have shown in the previous section (Theorem 3) that under certain conditions agents can attain their optimal allocations by buying insurance and trading assets that span aggregate uncertainty. Putting these results together we obtain the following new results: (a) the number of assets in $\mathcal{D}$ can be just two (not n); and, (b) we can be very specific about what these two assets are – they turn out to be very intuitive ones: a zero coupon bond and an equally-weighted portfolio of insurance company shares.

Theorem 5 *For economy $\mathcal{E}(B)$, there exists an insurance market equilibrium $((x_i^*)_{i=1}^n, (S_i^I))_{i=1}^n, \mathcal{D})$ such that $((x_i^*)_{i=1}^n)$ is Pareto optimal for $\mathcal{E}(B)$ with $\mathcal{D}$ containing only two assets: a zero coupon bond and an equally weighted portfolio of insurance company shares.*

Corollary 2 *The standard insurance market economy, $\mathcal{E}(A)$, with aggregate risk only requires dynamic trading in an equally weighted portfolio of insurance company shares and a riskless bond to be efficient.*

Insurance markets can be efficient, even if there is aggregate risk, when the economy has a sophisticated stock market. And, the stock market does not have to allow trading in very many different assets. In fact, it only needs to have an equally weighted share index and a riskless bond. As was suggested by Marshall (1974a), the presence of a stock market eliminates not only the need for the law of large numbers to apply but also independence between risks. An economy can have efficient insurance markets even if it is exposed to large risks such as natural disasters (earthquakes and hurricanes). Insurance markets for risks such as satellite launches can function efficiently even though the number of risks is very small and not independent. The aggregate risk is assumed by the investors in insurance company shares and these investors are appropriately compensated with a risk premium on their investment. In the next section we will show that this risk premium is financed by a loading on private insurance over and above the actuarially fair price of insurance.

4 Insurance and Trading in Equilibrium

Now we turn to the effect of stock markets on insurance decisions and agents' investment in insurance companies. If Theorem 3 (and its Corollary) hold in equilibrium then it is reasonable to observe agents who buy full insurance coverage. We show that not only is Theorem 3 true in equilibrium but also that we can be very specific about how agents will invest in insurance company shares and bonds, we can write down very explicitly what their investment strategies will be.

Theorem 6 *In the insurance market equilibrium of Theorem 5 the trades of every agent i are characterized by:*
a. buying full insurance coverage: $\alpha_i = L$;
b. dynamically trading an equally-weighted portfolio in insurance company shares and a bond according to strategies $\theta^i_M(t)$ and $\theta^i_0(t)$ respectively. These strategies are constructed using $n+1$ deterministic functions: $\hat{\theta}^i_M(t,j)$ and $\hat{\theta}^i_0(t,j)$, $j = 1, \ldots, n+1$, as follows:

$$\theta^i_M(t) = \sum_{j=0}^{n} \mathbf{1}_{\{N(t-)=j\}} \hat{\theta}_M(t,j)$$
$$\theta^i_0(t) = \sum_{j=0}^{n} \mathbf{1}_{\{N(t-)=j\}} \hat{\theta}_0(t,j)$$

The $\hat{\theta}$ functions are constructed in Appendix D.

There are two important new results here. One relates to insurance demand and the other to investment decisions. In terms of insurance demand, we identify the relationship between efficiency and insurance decisions in equilibrium:

Corollary 3 *Efficiency implies full insurance coverage.*

Theorem 6 states that the insurance market equilibrium of Theorem 5 will lead to full insurance purchases. Going over the proof, it becomes clear that this is because that insurance market equilibrium is decentralizing a Pareto efficient state-contingent commodity equilibrium, and condition (a.) and (b.) in Theorem 6 are the way the agent structures his insurance and investments so as to attain his Pareto efficient allocation. Nevertheless, Theorem 3 shows that it is not really the efficiency of equilibrium per se that generates the purchase of full insurance coverage, i.e. it's not a consequence of markets equilibrating, but the combination of rich financial markets, asset prices that do not price idiosyncratic risks and competitive insurance pricing. These combination of factors occurs in an insurance equilibrium that decentralizes an efficient state-contingent commodity one but it can also occur out of equilibrium.

As for investment decisions, we would like to emphasize that there are no results in the literature that we know of that characterize trades as we do (both in terms of the exact assets (an equally weighted portfolio of shares and the bond) and the related trading strategies). There are known explicit solutions for continuous trading behavior for given asset price processes but these are limited to models where agents have very specific preferences (of the Hyperbolic Absolute Risk Aversion, HARA, type; we discuss these preferences in the Extensions Section). Solutions for the optimal trading rule with general preferences are for the most part stated either in terms of the value function or as a stochastic differential equation (for approaches based on martingale methods). Essentially what we do is provide an explicit solution to the stochastic differential equations obtained using martingale methods, which in our model turn out to have particularly simple structure and which can be easily computed.

There is still a final question that we want to address and that is: what is the relationship between the equilibrium full insurance decision and the price of insurance. Insurance prices in equilibrium are fair if the price per unit of coverage equals the (discounted) probability of the loss. The difference between the price and the fair value if described using a loading factor. If the price per unit of coverage for agent i is S_i and p is the probability of loss, then the loading γ_i is defined from the following relation: $S_i = p(1+\gamma_i)e^{-\mathbf{r}}$. Note that the time element implies that the actuarially fair price per unit of coverage is not p but $pe^{-\mathbf{r}}$ as indemnities are paid in period one while premia are collected in period zero. We can show that the loading will be strictly positive and independent of i.

Theorem 7 *For the equilibrium of economy $\mathcal{E}(B)$ in Theorem 5, the price per unit of coverage for all $i \in I$ is the same, $S_i^I = S$, and it has a strictly positive loading, i.e. $\gamma > 0$ where*

$$S = p(1+\gamma)e^{-\mathbf{r}} \tag{2}$$

Thus, we cannot take Mossin's result to the data directly. If we observe people buying full insurance we cannot conclude that prices are fair. Furthermore, if people purchase full coverage and prices are known to be unfair it does not mean that people are irrational or have strange preferences. The so-called 'full coverage puzzle' is not really a puzzle: full insurance and unfair prices are a *characteristic* of equilibrium behavior in competitive insurance markets when agents have access to sufficiently flexible stock markets.

Theorem 7 also has an important implication for how insurance prices are determined, namely

Corollary 4 *The competitive price of private insurance does not depend on the insured's willingness to pay for coverage.*

The price of insurance for agent i, S_i^I, depends only on its actuarially fair value, p, the loading γ determined from the market price for risk (see Appendix E), and the market interest rate, $\mathbf{r}$.

5 Extensions

We have considered only a single risk and agents having only one accident. What happens if you consider more than one type of risk? This question has been answered in a similar setting by Penalva(2001): each additional risk an agent is exposed to will require an additional insurance contract. But, our framework is different in a number of ways: our model includes actual insurance companies, insurance is not treated as a (dynamically) tradeable financial asset and in our model the number of insurance contracts needed even with multiple types of risk is just one. The one insurance contract takes a very special form: it covers every risk (a bit like a full coverage multi-peril contract).

To see why one contract is enough consider car insurance (without a deductible). A car can have very different types of accidents ranging from total damage to a scratch. If the conditions of Theorem 3 hold, agents' insurance demand is to have full coverage and thus they will only need one contract that provides full coverage for all different types of damage. This does not happen in Penalva(2001), where each agent would require a different contract for each different loss magnitude.

On the asset market side however, agents have access to a more restricted set of securities, namely shares of insurance companies. In such a case and with multiple perils we may find that condition (i) may not be satisfied with only one portfolio of insurance shares. Penalva (2001) suggests that one may at worst need as many as one distinct portfolio per type of risk so that the question of whether the investment opportunities provided by insurance company shares is sufficient remains open.

An alternative extension to consider is to allow more than one accident per person. The main results stated before will hold true but with some caveats. The full insurance coverage result will continue to hold but again, condition (i) may require additional investment opportunities. One of the reasons condition (i) may not be satisfied with a single portfolio has to do with the informational content of a second accident. For example, compare the informational content of one agent having two accidents versus two different agents having one accident each. If (from the point of view of aggregate risk) these two events are equivalent, then condition (i) will continue to hold, but if they are not then after one agent has had an accident investors need to consider at least three different contingencies (one accident to a different agent, a second accident to this agent, and no more accidents) and this uncertainty may require additional investment portfolios. A minor issue that arises with multiple accidents is that we have required agents to always have some endowment left so one needs to be careful about not allowing agents to loose more than they have, but this issue can be dealt quite easily. A final remark regarding multiple accidents has to do with the insurance loading. The method we have used to prove Theorem 7 is valid only for a single accident. Whether the same result can be proven with multiple accidents requires further research.

A final extension we want to consider is what happens if one puts restrictions on agents' preferences. In particular, consider restricting preferences to those in the class of hyperbolic risk aversion (HARA) which are extensively used in financial modelling. If agents have preferences of this type then we can show that stock market trading behavior is very simple:

Proposition 1 *If agents' preferences in $\mathcal{E}(B)$ are of the form*

$$U_i(x) = v_i(x(0)) + \beta_i E(u_i(x(1))),$$

where $-u_i'(x)/u_i''(x) = a_i + bx$, and if agents have access to a bond, full insurance and an equally-weighted portfolio of insurance company shares, then agents' optimal investment strategies are to buy-and-hold the bond and the portfolio and purchase full coverage.

This Proposition can be easily shown using a well-known result, namely that with HARA preferences the optimal risk sharing rule is linear (see Huang and Litzenberger(1988) p.135 for a very nice proof of this). The linearity of the risk sharing rule means that there exists constants $\kappa_i \in \mathbf{R}$, $i = 1, \ldots, n$, such that $x_i^*(1) = \kappa_i e(1)$. We know that the equally weighted portfolio and the aggregate endowment are linear functions of $N(1)$ (see the proof of Theorem 5 in Appendix D), i.e. there exists $\alpha, \beta \in \mathbf{R}$ such that $d_M = \alpha + \beta N(1)$, and (assumption B.2) there exist $\alpha_e, \beta_e \in \mathbf{R}$ such that $e(1) = \alpha_e + \beta_e N(1)$.

Putting these together the agent can replicate the optimal risk sharing rule by buying full coverage and investing θ_0^i in the bond and θ_M^i in the portfolio, where θ_0^i and θ_M^i solve:

$$\begin{cases} x_i^*(0) &= w_{i,0} - S_i^I L - \theta_0^i S_0(0) - \theta_M^i S_M(0) \\ \kappa_i(\alpha_e + \beta_e N(1)) &= w_{i,1} + \theta_0^i + \theta_M^i(\alpha + \beta N(1)) \\ x_i^*(0) &= w_{i,0} + E_Q[w_{i,1}]e^{-\mathbf{r}} - \kappa_i E_Q[e(1)]e^{-\mathbf{r}} \end{cases}$$

After some simple algebraic manipulation we obtain

$$\theta_M^i = \frac{\kappa_i \beta_e}{\beta}, \qquad \theta_0^i = \alpha_e \kappa_i - w_{i,1} - \alpha \frac{\kappa_i \beta_e}{\beta}$$

■

Note that Proposition 1 is proven without reference to risks and that it implies no dynamic trading. Hence, Proposition 1 holds even if there are many different types of risks and multiple accidents can occur.

6 Concluding Remarks

We have constructed a framework in which to study the interaction between insurance and financial markets. Our framework explicitly models both the insurance and the financial market, assuming insurance companies are pure financial intermediaries whose value is determined by the market value of their net assets. Within this framework we have addressed the relationship between efficiency and full insurance. We have studied the conditions on insurance and asset prices and on investment opportunities that will lead agents to purchase full insurance. We have shown that these conditions are satisfied in an efficient insurance market equilibrium but they could also hold outside of equilibrium. Hence, in equilibrium agents will purchase full insurance coverage and the risks will be assumed by all agents acting as investors who diversify across all insurance companies.

We have shown that by allowing trading of insurance company stocks in financial markets, insurance markets have a great deal of scope for optimally distributing risks. These investors will be compensated by the risks they assume with a risk premium and this risk premium is financed by a loading on private insurance. We have shown that the efficiency of insurance markets does not depend on having agents with the same tastes for risk or same wealth levels. Even risks do not have to be independent as long as agents can trade in financial assets. Furthermore, we have shown that the financial assets needed for efficient risk sharing are neither many nor complex, so that an economy can achieve optimal risk sharing with relative ease.

The way we model the risks underlying share prices enables us to be very specific about how agents trade, and to construct exact trading rules for economies with very general preferences and continuous trading, something that is rare in the literature. Also, in the last section, we showed that the standard finance assumption of HARA preferences leads to very simple investment strategies: pure buy-and-hold strategies on a portfolio of insurance companies' shares.

Our model allows us to look at the insurance market in detail and the presence of the stock market greatly simplifies what we observe. Agents separate insurance from investment decisions and purchase full coverage even if the presence of aggregate risk pushes insurance prices above actuarially fair levels. Thus, there is no ‘full coverage puzzle’ in the sense that it is not inconsistent to observe actuarially unfair prices together with purchases of complete coverage, at least in equilibrium with competitive insurance markets.

Overall, our results suggest that the development of financial markets and financial innovation is highly beneficial for insurance markets. Increasing the investor base and providing cheap and flexible investment opportunities enhances the possibilities for insurance companies to increase the number and types of risks insured so that it can develop with the rest of the economy and provide coverage for such risks as satellite launches and earthquake damage to microchip manufacturing plants.

A Formal Definitions

The Inada conditions on an increasing function $u : \mathbf{R}_+ \to \mathbf{R}$ are: $\inf_x u'(x) = 0$ and $\sup_s u'(x) = +\infty$. Note that this condition is sufficient though not necessary for the results in the paper (they are used to guarantee existence and representative agent characterization of prices), and can be extended to economies with more general preferences in the standard way.

Let τ_i be the random time $\tau_i \in [0, 1]$ agent i has an accident and if agent i has no accidents between dates zero and one let $\tau_i = \infty$. Let $\mathcal{T} = [0, 1] \cup \{\infty\}$. The set of states of the world, $\Omega' = \mathcal{T}^n$ (a standard measurable space). The hazard rates $(\lambda_i(t))_{i=1}^n$ define a probability measure P on Ω'. Clearly, Ω' is uncountable. Equivalently, we could have defined Ω' using the space of counting functions $(N_i(t))_{i \in I}$ on $[0, 1]$ - for a more detailed discussion on these issues see Bremaud(1981). Throughout the paper we use the definition of Ω' (in terms of counting functions or stopping times) that is more convenient or intuitive for the context at hand.

The revelation of information is described by $(\mathcal{F}_t)_{t \in [0,1]}$, the filtration generated by the random vector process $\tilde{N}(t) \equiv (N_i(t))_{i=1}^n$, i.e. let $\sigma(x)$ denote the sigma-algebra generated by the random variable x, then for each $t \in [0, 1]$, let $\mathcal{F}_t$ be the σ-algebra representing all the information embodied in the vector process $\tilde{N}$ up to and including date t, i.e. $\mathcal{F}_t = \cap_{s \leq t} \sigma(\tilde{N}(s))$. Note that $\{\tau_i = \infty\} = \{N_i(1) = 0\} \in \mathcal{F}_1$. A process $x(t)$ is said to be adapted to $(\mathcal{F}_t)_{t \in [0,1]}$ if for all t, $x(t)$ is $\mathcal{F}_t$-measurable. A process $x(t)$ is said to be $(\mathcal{F}_t)_t$-predictable if $x(t)$ is measurable with respect to $\mathcal{F}_{t-} \equiv \cap_{s<t} \sigma(\tilde{N}(s))$. Let $x(t-)$ denote $\lim_{s \uparrow t} x(s)$. A process $x(t)$ is said to be $(P, (\mathcal{F}_t)_{t \in [0,1]})$-integrable if $x(t)$ is measurable with respect to $\mathcal{F}_{t-}$ and for all t $\int |x(t)| P(\,d\omega) < \infty$.

Future claims, d are $\mathcal{F}_1$-measurable random variables. The price process $S(t)$ $t \in [0,1]$ is a stochastic price process adapted to $(\mathcal{F}_t)_{t\in[0,1]}$ such that $S(1) = d$.

In an economy with J assets, let $\mathcal{D} = ((S_j(t))_{t\in[0,1]})_{j=1}^J$ and Θ denote the vector of allowable trading strategies given $\mathcal{D}$. In the usual way, an allowable trading strategy on asset j is an $\mathcal{F}_t$-predictable and $(P, \mathcal{F}_t)$-integrable stochastic process θ_j. To simplify notation in the text we have included the self-financing condition from the budget constraint into the definition of allowable strategies, i.e. $\theta \in \Theta$ implies

$$\sum_{j=0}^{J} \theta_j(t) S_j(t) = \sum_{j=0}^{J} \theta_j(0) S_j(0) + \sum_{j=0}^{J} \int_0^t \theta_j(s)\, \mathrm{d}S_j(s), \quad \forall t \in [0,1].$$

B Martingale Dimension

Duffie and Huang(1985) show how to decentralize a state-contingent (AD) equilibrium as a Radner equilibrium if you have one riskless asset plus K appropriate risky assets (see the original for the exact definition of 'appropriate'). The number K is equal to the dimension of the space of $(Q, (\mathcal{F}_t)_{t\in[0,1]})$-martingales, where a martingale is an $\mathcal{F}_t$-adapted and integrable process $X(t)$ such that $X(t) = E_Q[X(s)|\mathcal{F}_t]$ for all $0 \leq t \leq s \leq 1$, and $E_Q[y|\mathcal{F}_t]$ represents the expectation of random variable y conditional on the sigma algebra $\mathcal{F}_t$ using probability measure Q. This measure Q is derived from the equilibrium price of the AD equilibrium one is trying to decentralize (in Appendix D we construct one such measure).

Take economy $\mathcal{E}(B)$. Equation (1) plus the conditions on preferences and endowments imply that $Q(\omega) = \xi(\omega)P(\omega)$ where $\xi(\omega) > 0$ P-a.s. so that Q

is absolutely continuous relative to P. Fortunately, the martingale dimension of the space is invariant to an absolutely continuous change of measure so that it suffices to show:

Lemma 1 *The space of martingales on* $(\Omega', \mathcal{F}_1, (\mathcal{F}_t)_t, P)$ *has martingale dimension of* n*.*

Proof The martingale dimension is given by the minimal number of martingales, $M_1, \ldots, M_K$, that have the property that for any $(\mathcal{F}_t, P)-$martingale there exists $(\mathcal{F}_t, P)$ integrable and predictable vector process y such that given $M = (M_1, \ldots, M_K)$,

$$X(t) - X(0) = \int_0^t y(s)\,\mathrm{d}M(s) \tag{3}$$

Let $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where for $j = 1, \ldots, n$, $\lambda_j = \lambda$ for $N_j(t-) = 0$ and zero otherwise. For $j = 1, \ldots, n$, let $M_j(t) = \int_0^t \tilde{\lambda}_j \,\mathrm{d}s - N_j(t)$ and $M(t) = (M_j(t))_{j=1}^n$.

From the martingale representation theorem for marked point processes (see Last and Brandt(1991, pp. 342-346) for a general version and proof of this theorem), for every $(P, (\mathcal{F}_t)_t)$-martingale, $X(t)$, there exists a $(P, (\mathcal{F}_t)_{t\in[0,1]})$-integrable predictable process $y(t)$ such that equation (3) holds. It is quite straightforward to show that $M_j(t)$ is a $(P, \mathcal{F}_t)$-martingale. Furthermore, they are pairwise orthogonal, so that the vector M is minimal. Hence, the martingale dimension is equal to n. ∎

C Proof of Theorem 2.1 and Corollary 2.1

Proof. The agent's problem is

[Problem A]

$$\max_x U_i(x) \quad \text{s.t.} \quad x \in B(\mathcal{D}, S_i^I)$$

Let x^* be maximal in the alternative problem

[Problem B]

$$\max_x U_i(x) \qquad \text{s.t.} \quad x^*(0) + E_Q[x^*(1)]e^{-\mathbf{r}} = e_i(0) + E_Q[e_i(1)]e^{-\mathbf{r}},$$

where Q and $r(t)$ are defined in the statement of Theorem 3. We proceed by proving the following three statements: (1) $x^*(1) = f(e(1))$, (2) $x \in B(\mathcal{D}, S_i^I) \Rightarrow x$ satisfies the constraint in problem B, and (c) $x^* \in B(\mathcal{D}, s_i^I)$ and is optimal. We drop the i subscript for clarity.

(a). The problem B has Lagrangian

$$L = U(x) - \lambda \left(x(0) + E_Q[x(1)]e^{-\mathbf{r}} - e_i(0) + S^I L - we^{-\mathbf{r}} \right)$$

The first order condition is:

$$v'(x(0)) = \lambda$$

$$\forall \omega \in \Omega',\ \beta u'(x(1,\omega)) = \lambda\, \xi(e(1,\omega))e^{-\mathbf{r}}$$

where λ is the Lagrange multiplier in the constrained maximization problem. The properties of u ensure that u'^{-1} is a well-defined function so that we can define $x^*(1) = f(e(1))$ as:

$$f(e(1)) \equiv u'^{-1} \left(\frac{v'(x^*(0))\xi(e(1))e^{-\mathbf{r}}}{\beta} \right)$$

where $x^*(0)$ is the constant that solves

$$x^*(0) + E_Q[f(e(1))]e^{-\mathbf{r}} = e_i(0) - S_i^I L + e^{-\mathbf{r}} e_i(1)$$

The properties of the problem (u_i, v_i increasing concave differentiable functions satisfying Inada conditions plus the linearity of the constraint) imply that such a $x^*(0)$ exists.

(b). By conditions (*ii*) and (*iii*) of Theorem 3, for $x \in B(\mathcal{D}, S_i^I)$ there exist α and θ such that

$$\begin{cases} x(0) &= e_i(0) - \alpha S^I - \sum_{j=0}^{J} \theta_j(0) S_j(0) \\ x(1) &= e_i(1) + \alpha N_i(1) + \sum_{j=0}^{J} \left(\theta_j(0) S_j(0) + \int_0^1 \theta_j(s)\, \mathrm{d}S_j(s) \right) \end{cases}$$

$$\Rightarrow \quad x(0) = e_i(0) - \alpha E_Q[N_i(1)]e^{-\mathbf{r}} - \sum_{j=0}^{J} \theta_j(0) S_j(0)$$

$$x(0) = e_i(0) + E_Q[e_i(1) - x(1)]e^{-\mathbf{r}}$$

(c). We can write x^* as

$$x^*(0) = e_i(0) - \alpha_i S_i^I - c$$

$$x^*(1) = w_1 - (L - \alpha_i) N_i(1) + \tilde{x}(1)$$

If we let $\alpha_i = L$, then $x^*(1) = w_1 + \tilde{x}(1)$. By the premise of Theorem 3 there exists θ such that $\tilde{x}(1) = \sum_{j=0}^{J} \theta_j(0) S_j(0) + \int_0^1 \sum_{j=0}^{J} \theta(s)\, \mathrm{d}S_j(s)$. Hence, if we let $c = \sum_j \theta_j(0) S_j(0)$, then $x^* \in B(\mathcal{D}, S_i^I)$. As x^* is optimal in problem B and every other $x \in B(\mathcal{D}, S^I)$ is feasible in problem B then x^* is optimal in the more restricted problem, A.

Theorem 3: Step (c) shows that full insurance ($\alpha_i = L$) is used in constructing the strategy used to attain the (Problem A)-optimal x^*.

Corollary 1: As the insurance decision is always the same ($\alpha_i = L$), it is independent of θ used to attain $\tilde{x}(1)$. ∎

D Proof of Theorems 3.1 and 4.1

Theorem 5 can be proven using abstract martingale representation arguments (see Duffie Huang(1985) and Penalva(2001)). We complement their arguments by explicitly constructing the strategies for the proof of Theorem 5 in the proof of Theorem 6 (thereby embedding the proof of Theorem 6 in the proof of Theorem 5).

Proof of Theorem 5: we proceed to construct the equilibrium in stages: (1) We show that a stage-contingent equilibrium exists and define asset and insurance prices $(\mathcal{D}, (S_i^I)_{i=1}^n)$ using state-contingent equilibrium prices; (2) we show that the state-contingent equilibrium consumption allocations are in the agent's budget constraint, $B(\mathcal{D}, S_i^I)$, by constructing the appropriate trading strategies; (3) we show optimality of the allocations.

Stage 1. Let $\mathcal{E}$ denote an economy with n agents indexed by i, with common priors (given by the measure P), preferences of the form

$$U_i(x) = u_i(x(0)) + \beta_i E_P(v_i(x(1))),$$

with u_i and v_i are concave, differentiable and satisfying the standard Inada conditions, $\beta_i \in (0, \infty)$, and endowments $e_i = (e_i(0), e_i(1))$ such that $\sum_i e_i(t) > 0$ for $t \in \{0, 1\}$. We use the following well-known result (for example, Constantinides(1982)).

Proposition 2 *For any AD equilibrium of $\mathcal{E}$, $((x_i^*)_{i=1}^n, \tilde{\pi})$, there exists a representative agent representation of prices with strictly concave vonNeumann-Morgenstern preferences*

$$v_0(x(0)) + \beta_0 E_P[v_1(x(1))]$$

such that

$$\forall \omega \in \Omega,\ \tilde{\pi}(\omega) = \frac{\beta_0 v_1'(e(1))}{v_0'(e(0))} \tag{4}$$

where $v_t'(x)$ is the first derivative of the representative agent's utility function at date $t = 0, 1$. For all i, there exists $f_i : \mathbf{R} \to \mathbf{R}$ such that $x_i^*(1) = f_i(e(1))$.

From this equilibrium, define Q and r for the insurance market equilibrium as:

$$Q(\omega) = \frac{\tilde{\pi}(\omega)}{\int_{\omega \in \Omega} \tilde{\pi}(\omega)\, \mathrm{d}\omega}$$

Let P_t and Q_t define the restriction to $(\Omega, \mathcal{F}_t)$ of P and Q respectively. Define the Radon-Nikodym derivative process as

$$Q(\omega) = P(\omega)\xi(1, \omega); \qquad \xi(t, \omega) = \frac{\xi(1, \omega)}{E_P[\xi(1)|\mathcal{F}_t](\omega)} \equiv \frac{\mathrm{d}Q_t}{\mathrm{d}P_t}(\omega)$$

$$e^{-\mathbf{r}} \equiv \int_{\omega\in\Omega} \tilde{\pi}(\omega); \qquad \exp\left(-\int_t^1 r(s)\,\mathrm{d}s\right) \equiv e^{-\mathbf{r}(1-t)}$$

Define $\mathcal{D} = (S_0(t), S_M(t))$ as follows: for all $t \in [0,1)$

$$S_0(t) = \exp\left(-\int_t^1 r(s)\,\mathrm{d}s\right)$$

$$S_M(t) = \frac{1}{J} S_j(t)$$

where the price of a share in firm j is the market value of the insurance company's assets (the premia collected plus interest) minus the market value of the liabilities (future indemnities):

$$S_j(t) = \sum_{i\in I_j} \alpha_i S_i^I e^{\mathbf{r}t} - E_Q\left[\sum_{i\in I_j} \alpha_i N_i(1) e^{-\mathbf{r}(1-t)} \middle| \mathcal{F}_t\right]$$

Let $S_i^I = E_Q[N_i(1)]e^{-\mathbf{r}}$ – this implies that for any insurance company $S_j(0) = 0$.

Stage 2. We want to check whether $x_i^* \in B(\mathcal{D}, S_i^I)$. From Proposition 2, for all $i = 1, \ldots, n$, $x_i^*(1) = f_i(e(1))$. As $e(1) = nw - N(1)L$, and abusing the f notation $x_i^*(1) = f_i(N(1))$. We need to show that $x_i^*(1)$ can be achieved using a dynamic trading strategy with only the bond and the portfolio. We could appeal to martingale representation theorems, as in Penalva (2001) and Duffie Huang (1985), to show that such dynamic trading strategies exists. Instead we prove existence by constructing those strategies.

Stage 2.a We first look at the aggregate process $N(t)$. Consider the general case, $\lambda_i(t) = g(t, N(t-))\mathbf{1}_{N_i(t-)=0}$, where $g : \mathbf{R} \times \mathbf{N} \longrightarrow \mathbf{R}$ with $g(t, N(t-)) = \lambda$ as a special case. Let $\lambda^N(t) = \sum_{i=1}^n \lambda_i(t)$. Then

$$\lambda^N(t) \quad = \quad \sum_{i=1}^n g(t, N(t-))\mathbf{1}_{N_i(t-)=0} = g(t, N(t-)) \sum_{i=1}^n \mathbf{1}_{N_i(t-)=0}$$

$$= \quad g(t, N(t-))N(t-),$$

and we can say that $N(t)$ admits the $\mathcal{F}_t$-intensity λ^N (see Brémaud for a detailed exposition of these concepts and related results).

Stage 2.b We now turn to the price process for the portfolio. Define the discounted price of the portfolio as S_M^*:

$$\begin{aligned} S_M^*(t) &\equiv \sum_{j=1}^{J} \frac{1}{J} S_j^*(t) = \sum_{j=1}^{J} \frac{1}{J} e^{-\mathbf{r}t} S_j(t) \\ &= e^{-\mathbf{r}t} \left(\sum_{j=1}^{J} \frac{1}{J} \left(\sum_{i \in I_j} \alpha_i S_i^I e^{\mathbf{r}t} - E_Q \left[\sum_{i \in I_j} \alpha_i N_i(1) e^{-\mathbf{r}(1-t)} \middle| \mathcal{F}_t \right] \right) \right) \end{aligned}$$

Let $C_j = \sum_{i \in I_j} \alpha_i S_i^I / J$. If $\alpha_i = L$ for all i, then

$$\begin{aligned} S_M^*(t) &= e^{-\mathbf{r}t} \left(\sum_{j=1}^{J} C_j e^{\mathbf{r}t} - \frac{L}{J} E_Q \left[N(1) e^{-\mathbf{r}(1-t)} \middle| \mathcal{F}_t \right] \right) \\ &= \sum_{j=1}^{J} C_j - \frac{L}{J} E_Q \left[N(1) \middle| \mathcal{F}_t \right] e^{-\mathbf{r}} \end{aligned}$$

Let $C = L/J$. Similarly, for $x(1)$ define

$$X^*(t) \equiv e^{-\mathbf{r}t} E_Q \left[x_i^*(1) e^{-\mathbf{r}(1-t)} \middle| \mathcal{F}_t \right] = E_Q[x_i^*(1) | \mathcal{F}_t] e^{-\mathbf{r}}.$$

Stage 2.c We want to identify how we will replicate $X^*(t)$ using the portfolio, S_M^*. Standard martingale representation arguments such as in Penalva (2001) prove that such a portfolio exists. Instead we will construct a portfolio, $\theta_M^i(t) \in \Theta$, that replicates $X^*(t)$. Such a portfolio will need to satisfy the following conditions (we drop the i superscripts for the rest of the proof):

$$\begin{cases} \theta_M(0)S_M^*(0) &= X^*(0) \\ \theta_M(t)S_M^*(t) &= X^*(t) \\ \theta_M(t)S_M^*(t) &= \theta_M(0)S_M^*(0) + \int_0^t \theta_M(s)\,\mathrm{d}S_M^*(s) \end{cases}$$

So that to construct θ_M we need to solve

$$\mathrm{d}X^*(t) = \theta_M(t)\,\mathrm{d}S_M^*(t) \tag{5}$$

From $x_i^*(1) = f_i(N(1))$ and the definitions of X^* and S_M^*:

$$\begin{aligned} \mathrm{d}X^*(t) &= e^{-\mathbf{r}}\,\mathrm{d}(E_Q[x_i^*(1)|\mathcal{F}_t]) \\ &= e^{-\mathbf{r}}\,\mathrm{d}(E_Q[f_i(N(1))|\mathcal{F}_t]) \\ \&\ \mathrm{d}S_M^*(t) &= -e^{-\mathbf{r}}C\,\mathrm{d}E_Q[N(1)|\mathcal{F}_t] \end{aligned}$$

So that to replicate X^* we need to solve

$$\mathrm{d}E_Q[f_i(N(1))|\mathcal{F}_t] \;=\; -\theta_M(t)C\,\mathrm{d}E_Q[N(1)|\mathcal{F}_t] \tag{6}$$

Stage 2.d We want to use the properties of $E_Q[y|\mathcal{F}_t]$ where y is an arbitrary deterministic function of $N(1)$, $y = y(N(1))$.

$$\begin{aligned} E_Q[y|\mathcal{F}_t] &= \frac{E_P[\xi(t)y|\mathcal{F}_t]}{E_P[\xi(1)|\mathcal{F}_t]} \\ &= \sum_{k=N(t)}^{n} \frac{P\{N(1)=k|\mathcal{F}_t\}\xi(1)y(k)}{E_P[\xi(1)|\mathcal{F}_t]} \end{aligned}$$

As λ^N is a function only of t and $N(t-)$, the probability $P\{N(1) = k|\mathcal{F}_t\}$ is only a function of $N(t)$, k and t so that for $N(t) = N$, $P\{N(1) = k|\mathcal{F}_t\} =$

$\mathcal{P}(t,k,N)$. This function can be easily constructed recursively for time homogenous hazards, i.e. $\lambda^N(t,N(t-)) = N(t-)g(N(t-))$, (using the shorthand $\lambda_k = \lambda^N(t,k)$) as:

$$\begin{aligned}
\mathcal{P}(t,k,k) &= \exp(-\lambda_k(1-t)) \\
\mathcal{P}(t,j+1,k) &= \int_t^1 \mathcal{P}(t+s,j,k)\lambda_j \exp(-\lambda_{j+1}(1-s))\,\mathrm{d}s,\ j=k,\ldots,n-1
\end{aligned}$$

This definition is extended to the non-time homogenous case by using $\lambda_j(s) = \lambda^N(s,j)$ and substituting $\lambda_j(1-t)$ (and $\lambda_{j+1}(1-s)$) with $\int_t^1 \lambda_j(u)\,\mathrm{d}u$ (and $\int_s^1 \lambda_{j+1}(u)\,\mathrm{d}u$) in the above equations. Note that for the independence case, $\lambda^N = N(t-)\lambda$, $\mathcal{P}(t,k,N)$ is given by the binomial probability of $(k-N)$ successes out of $(n-N)$ trials with probability of success $p(t) = 1-\exp(-\lambda(1-t))$.

Using these definitions

$$\begin{aligned}
E_Q[y|\mathcal{F}_t] &= \sum_{k=N(t)}^{n} \frac{\mathcal{P}(t,k,N(t))\xi(1,k)y(k)}{E_P[\xi(1)|\mathcal{F}_t]} \\
\text{and } \xi(t) &= \frac{\xi(1)}{E_P[\xi(1)|\mathcal{F}_t]}
\end{aligned}$$

Given the properties of $\xi(1)$ and $\mathcal{P}$ (and abusing notation) we can define deterministic functions $\xi(1,N)$ and $\xi(t,j,N)$:

$$\begin{aligned}
&\forall j,k \in \{0,1,\ldots,n\} \\
\xi(1,k) &= \xi(1,\omega)\mathbf{1}_{N(1,\omega)=k}, \\
\forall t \in [0,1),\ \xi(t,j,k) &= \begin{cases} \dfrac{\xi(1,j)}{\sum_{s=k}^{n} \mathcal{P}(t,s,k)\xi(1,s)} & n \geq j \geq N \\ 0 & \text{otherwise} \end{cases} \\
\Rightarrow E_Q[y|\mathcal{F}_t] &= \sum_{j=0}^{n} \mathbf{1}_{\{N(t)=j\}} \left(\sum_{k=j}^{n} \mathcal{P}(t,k,j)\xi(t,k,j)y(k) \right)
\end{aligned}$$

Equation (6) must hold at all accident times. For arbitrary adapted process $x(t)$ let $\Delta x(t) = (x(t-) - x(t))\mathbf{1}_{\{N(t)-N(t-)=1\}}$. As $\theta_M(t)$ has to be predictable, then the following equation characterizes $\theta_M(t)$

$$\Delta E_Q[f_i(N(1))|\mathcal{F}_t] \quad = \quad -\theta_M(t) C \Delta E_Q[N(1)|\mathcal{F}_t] \tag{7}$$

We use equation (7) to define $n+1$ deterministic functions $\hat{\theta}_M(t,k)$, $k = 0, \ldots, n$. Let $\hat{\theta}_M(t,n) = 0$ and for $k \in \{0, \ldots, n-1\}$ define

$$\hat{\theta}_M(t,k) = \frac{1}{C} \frac{\sum_{j=k}^{n} \mathcal{P}(t,j,k)\xi(t,j,k) f_i(j) - \sum_{j=k+1}^{n} \mathcal{P}(t,j,k+1)\xi(t,j,k+1) f_i(j)}{\sum_{j=k}^{n} \mathcal{P}(t,j,k)\xi(t,j,k) j - \sum_{j=k+1}^{n} \mathcal{P}(t,j,k+1)\xi(t,j,k+1) j}$$

As θ_M has to be predictable and equation (7) has to hold then

$$\theta_M(t) = \sum_{j=0}^{n} \mathbf{1}_{\{N(t-)=j\}} \hat{\theta}_M(t,j)$$

Stage 2.e We now define the strategies on the bond, $\theta_0(t)$. Define $S_0^*(t) = e^{-\mathbf{r}t} S_0(t) = e^{-\mathbf{r}}$ and let

$$\theta_0(t) = \frac{X^*(t) - \theta_M(t) S_M^*(t)}{S_0^*(t)}.$$

Note that $\theta_0(t)$ is predictable and positive because $S_0^*(t)$ is predictable by construction plus the definition of $\theta_M(t)$ makes $X^*(t) - \theta_M(t) S_M^*(t)$ predictable. From the previous stage it is clear that the above equation can be expressed using deterministic functions $\hat{\theta}_0(t,k)$ such that

$$\theta_0(t) = \sum_{k=0}^{n} \mathbf{1}_{\{N(t-)=k\}} \hat{\theta}_0(t,k)$$

Stage 2.f Finally, we need to show that $x_i^* \in B(\mathcal{D}, S_i^I)$. To obtain x_i^*, the agent can buy full insurance ($\alpha_i = L$) and trade dynamically the mutual fund using the strategy $\theta_M(t)$ defined above, and trade dynamically the bond but using $\tilde{X}(t) = X^*(t) - w_{1,i}e^{-\mathbf{r}}$ instead of X^*. Following this strategy leaves the agent with the following consumptions:

$$\begin{aligned} c(0) &= w_{i,0} - S_i^I L - \theta_M(0)S_M(0) - \theta_0(0)S_0(0) \\ &= w_{i,0} - S_i^I L - \theta_M(0)S_M(0) - \frac{\tilde{X}(0) - \theta_M(0)S_M^*(0)}{S_0^*(0)} S_0(0) \end{aligned}$$

Recall $S_0^*(0) = e^{-\mathbf{r}} = S_0(0)$, $S_M^*(0) = E_Q[d_M]e^{-\mathbf{r}} = S_M(0)$, and $\tilde{X}(0) = E_Q[x_i^*(1) - w_{i,1}]e^{-\mathbf{r}}$. Also, recall the definitions of Q and r from Stage 1. As x_i^* satisfies the state-contingent commodity budget constraint

$$\begin{aligned} c(0) &= w_{i,0} - S_i^I L - E_Q[x_i^*(1)]e^{-\mathbf{r}} + w_{i,1}e^{-\mathbf{r}} \\ &= x_i^*(0) \end{aligned}$$

As for the consumption obtained from this strategy at date one:

$$\begin{aligned} c(1) &= w_{i,1} - N_i(1)(L - \alpha_i) + \theta_M(1)S_M(1) + \theta_0(1)S_0(1) \\ &= w_{i,1} + \theta_M(1)S_M(1) + \frac{\tilde{X}(1) - \theta_M(1)S_M^*(1)}{S_0^*(1)} S_0(1) \\ &= w_{i,1} + \theta_M(1)d_M(1) + \frac{(x_i^*(1) - w_{i,1})e^{-\mathbf{r}} - \theta_M(1)d_M(1)e^{-\mathbf{r}}}{e^{-\mathbf{r}}} \times 1 \\ &= x_i^*(1) \end{aligned}$$

Stage 3. As shown in the proof of Theorem 3, all consumption allocations in $B(S_i^I, \mathcal{D})$ are also feasible in the state-contingent budget constraint so that x_i^* is also optimal in the insurance market equilibrium. ∎

Theorem 6: The reader will find the strategies characterized and constructed in Stages 2.d and 2.e of the previous proof. In Stage 2.f it is shown that those strategies do indeed attain the desired equilibrium allocations. ∎

E Proof of Theorem 4.2

We prove the result in a more general way using the concept of exchangeability. Let B_i represent an event of the form $\{N_i(1) = 0\}$ or $\{N_i(1) = 1\}$ and $A_i = \{N_i(1) = 1\}$. Also, let $\mathbf{1}_A$ denote the indicator function of an arbitrary event A.

Remark 2 *If $\lambda_i(t) = g(t, N(t))\mathbf{1}_{\{N_i(t-)=0\}}$, where $g : \mathbf{R} \times \mathbf{N} \longrightarrow \mathbf{R}$, then the events $B_1, B_2, \ldots, B_n$ are exchangeable events, i.e. for all permutations of the indexes, $\iota(n) : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, ι a one-to-one function,*

$$P(B_1, B_2, \ldots, B_n) = P(B_{\iota(1)}, B_{\iota(2)}, \ldots, B_{\iota(n)})$$

That this is true can be seen from the way the function $\mathcal{P}$ was constructed in Appendix D

Exchangeability implies that for $i, j, k \in \{1, \ldots, n\}$, $P(A_i) = P(A_k) = p$ and

$$\begin{aligned} P(N = j, N_i = 1) &= P(N = j, N_k = 1) \\ &= P(N = j)\binom{n-1}{j-1}/\binom{n}{j} = P(N = j)\frac{j}{n} \end{aligned}$$

Let $q_i = E_Q[A_i]$, $\xi = \mathrm{d}Q/\,\mathrm{d}P$ and recall n is finite, then

$$\begin{aligned} q_i &= \sum_{\omega \in A_i} P(\omega)\xi(\omega) \\ q_i &= \sum_{j=0}^{n} P(N = j|A_i)P(A_i)\xi(N = j) \\ q_i &= pE_P[\xi(N(1))|A_i] \end{aligned}$$

and by exchangeability, $q_i = pE_P[\xi|A_k]$, $\forall k \in \{1, \ldots, n\}$, so that for all $i = 1, \ldots, n$, $q_i = q$. Let $p_j \equiv P(N = j)$, then

$$P(N = j|A_i) - P(N = j) = \frac{p_j}{p}\frac{j}{n} - p_j = \frac{p_j}{np}(j - np)$$

Using $p = \sum_{k=0}^{n} P(N = k, A_i)$ and $P(N = k, A_i) = p_k k/n$

$$\begin{aligned} P(N = j|A_i) - P(N = j) &= \frac{p_j}{np}\left(j - n\sum_{k=0}^{n} p_k \frac{k}{n}\right) \\ &= \frac{p_j}{np}(j - E[N]) \end{aligned}$$

As $E[(j - E[N])] = 0$ and N is increasing, then for all $k \leq n$

$$F_{N|A_i}(k) \equiv \sum_{j=0}^{k} P(N = j|A_i) \leq F_N(k) \equiv \sum_{j=0}^{k} P(N = j)$$

and the inequality is strict at least for $N = 0$. That is, $N(1)|A_i$ first-order stochastically dominates[5] $N(1)$. The economy has a representative agent representation with strictly increasing and concave utility (see Lemma 2) so that the equilibrium $\xi(N)$ will be strictly increasing. By definition $E[\xi] = 1$. Stochastic dominance of $F_{N|A_i}$, ξ increasing, and $E_P[\xi] = 1$ imply $E_P[\xi|A_i] > 1$ and $q > p$.

$$S = E_Q[N_i(1)]e^{-\mathbf{r}} = qe^{-\mathbf{r}} = p(1+\gamma)e^{-\mathbf{r}}$$

with $\gamma > 0$.

∎

[5] The notion of first-order stochastic dominance is quite standard, see Huang and Litzenberger (1988) for a definition and more details.

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