ENDOGENOUS PARTY PLATFORMS: "STOCHASTIC" MEMBERSHIP

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Abstract. We analyze existence of divergent equilibrium in a model of endogenous party platforms with stochastic membership. The parties proposals depend on their membership, while the membership depends both on the proposals and the unobserved idiosyncratic preferences of citizens over parties. It is shown that when citizens view the parties as similar, apart from their policy proposals (i.e., the party platform is a good predictor of individual membership decision), the divergent equilibria exist. We analyze the relationship between parties policy proposals and the unobserved idiosyncratic characterisitics of parties and we obtain conclusions different from the ones provided in exisiting literarure.

1. Introduction

The issue of party platform formation has been a subject of substantial attention in political economy. The major idea in this literature is that platforms of political parties are formed in response to preferences of their members, whereas the memberships themselves are, at least in part, determined by the platforms. Thus, in equilibrium the party platforms should respond to the preferences of members that they attract. An early paper putting forward a political competition framework to define an equilibrium concept in which party ideology and its membership are endogenously determined was Baron (1993). This equilibrium concept was related to the one used in the "voting with one's feet" models developed in the study of local public goods (see Caplin and Nalebuff 1997 for an abstract framework that covers both the political economy and public finance applications). In related work Aldrich (1983a,b), Gerber and Ortuño-Ortín (1998), and Poutvaara (2003) have considered the interrelationship between partisan policy platforms and political activism.

A major issue in this literature has been the problem of existence of *divergent* equilibria, in which parties, though, possibly, *ex ante* identical propose different policies and attract members with different policy preferences. In a deterministic model of this type (such as Ortuño-Ortín and Roemer1998 or Gomberg, Marhuenda and Ortuño-Ortín 2004) such and equilibrium, if it exists, involves a full sorting of agents in terms of their preferences over the policy space: even minute policy differences between parties induce a unique party choice by almost all citizens (in the

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preliminary and incomplete.

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party activist literature, along the lines of Aldrich 1983a, where there is a third possibility - that of non-participation - it is still normally assumed that those actually actively taking part in partisan activities do it in the ideologically closest party). However, such perfect sorting is not commonly observed in reality: even ideologically identical people may frequently find themselves in different parties based on idiosyncratic non-policy considerations (perhaps, historical esthetical or personal). These non-policy issues might not even be observable by an outsider, making the observed policy preferences only stochastic predictors of individual party choice. This is, of course, not a new idea in political science, where the study of stochastic models of voting have been widespread for a long time (see Coughlin 1992 for a survey). Our "stochastic" model of endogenous membership follows some of the same intuitions. Our focus is, however, somewhat different. In particular, rather than considering the vote-maximizing parties in an electoral context we restrict our attention to parties aggregating members' preferences and try to establish to which extent the results of an older deterministic models (such as our own Gomberg *et al.* 2004) extend to this new setting. In modeling parties as aggregating preferences of their members, while membership is, in turn, determined in part (but not fully) by party policy positions our paper is related to the work by Roemer (2007). Our approach, however, is different in such crucial aspects as, among others, our more explicit modeling of membership decisions, the nature of intraparty decision rules (which in Roemer's case discriminate among members of different ideologies based on belonging to a "partisan core"). Our objective is likewise distinct: we want to establish to which extent the results for the stochastic membership model may be viewed as an extension of those for the older deterministic model.

A seemingly major difficulty in this extension is that the studies of the deterministic model have crucially used the sorting nature of equilibrium to derive the results from the properties of the space of sorting partitions (see Caplin and Nalebuff 1997 and Gomberg *et al.* 2004; in the context of local public goods this approach goes back to Westhoff 1977). In the absence of perfect sorting this approach is, of course, not feasible. However, the crucial feature of the deterministic model is, in fact, not the sorting *per se*, but the instability of pooling: if two, in other respects identical, parties propose the same policy than the entire population is indifferent and can be split to support that as an equilibrium, but even minor policy perturbations would result in full population sorting and sharply divergent policies. Thus, whenever it is possible to show that such an equilibrium may not be unique, existence of sorting equilibrium is, in fact, guaranteed! In this paper we show that this intuition, in part, extends to the stochastic context: if the convergent equilibrium exists but is unstable to small policy perturbations, it may be used to detect existence of divergent equilibria. In fact, as the addition of the stochastic component adds continuity to the model, in a sense the results become, in fact, more transparent in this setting. In particular, in a context of a "generalized example" in our framework, we show that when parties are perceived by voters to be very similar in non-policy terms, so that the observed randomness of individual partisan choice is relatively small, the results of the deterministic model extend to the stochastic case.

For the moment (and for simplicity) we abstract from possible strategic electoral competition by parties (in the terminology of Caplin and Nalebuff 1997 our parties are "membership-based"). The main reason here is methodological: we believe that the issue of endogenizing party membership is distinct from the issue of strategic behavior by party leaders in a democratic election. Our main concern here is the former, and we want to consider it separately. This assumption may be viewed as appropriate for either a model of parties in a setting without commitment (*e.g.,* when voters would not believe a party, once in office, can implement policies not supported by its membership) or in a setting without true electoral competition (*e.g.*, if parties' share of the office is determined through non-electoral means). Of course, we do intend to explore extending our results to cover the case of possible strategic interactions between parties.

Our model generates interesting predicitons on the relation between the policy proposals of parties and their idiosyncratic non-policy characterisitics. It is often claim that when ideological parties strongly differ in non-policy characterisitics (that are exogenously given) they have more incentives to propose divergent policies [references]. This is due to the fact that proposing a more "radical" policy is not that costly to a party since voters decision are very much influenced by the large differences in the non-policy variable. In our model, however, this does not to be the case. If agents preferences over the policy variables are independent of their preferences over the non-policy characterisitc of parties, increasing the differences between those non-policy caharacteristics might yield more convergency of the policy proposals. The intuition is clear: If parties are very different in their non-policy characteristics, their membership is basically determined by such non-policy characteristic. In this case the members of the two parties will be quite similar regarding their preferences on the policy variables and since parties just aggregate the preferences of their members their policy proposals will be very similar.

The rest of this paper is organized as follows. Section 2 presents the model and develops a general existence result, section 3 presents the results for the mean and median voter rules in a single dimension of issue space and section 4 concludes. Appendix 1 provides some stylized empirical facts to support some of our assumptions.

2. Model

There are two parties, that propose policy vectors $x_j \in X$, where X is a non-empty compact and convex subset of $\mathbb{R}^n$ with non-empty interior (in the topological sense). In addition to a policy x_j a party is characterized by a non-policy variable $y \in Y \subset \mathbb{R}$, which may be interpreted as reflecting currently fixed characteristics, which may matter for individual preferences.

There is a continuum of agent types with preferences over both policy and non-policy characteristics of parties. Specifically, each agent of type $(\alpha, \beta) \in A \times E \subset \mathbb{R}^n \times \mathbb{R}$ has Euclidean preferences represented by the utility function

$$u(x, y; \alpha, \beta) = -||(x, y) - (\alpha, \beta)||$$

where $x \in X$ is the policy platform adopted by the party and $y \in Y$ is the intrinsic characteristic of the party. Here, E is a subset of $\mathbb{R}$. We shall take $A = X$, so that (for a fixed y) an agent of type (α, β) may be identified with his/her ideal policy.

There is a measure space of agents (citizens) $(A \times E, \mathcal{B}, \nu)$, where $\mathcal{B}$ is the Borel σ-algebra on $A \times E$ and ν is a measure on $\mathcal{B}$ such that $\nu(A \times E) = 1$. We denote the distribution function of ν as $F(\alpha, \beta)$ and we assume existence of a non-zero continuous density function $f(\alpha, \beta)$ with conditional density functions denoted as $f_1(\alpha|\beta)$ and $f_2(\beta|\alpha)$.

Citizens play a twofold role. Each agent is a voter and a member of the party. Regarding party membership, given the parties policy in intrinsic characteristics (x_1, y_1) and (x_2, y_2), citizens join the party they like the most. Thus, the individual party choice is unambiguous. However, from the point of the party, the second coordinate of individual type β is unobservable. Thus, for the parties the observable individual preferences over policies (given by α) may serve only as an imperfect predictor of individual party choice. Therefore, the agent's decision appears stochastic.

Let $\mathcal{A}$ be the Borel σ-algebra on A. Ignoring zero-measure sets, a party membership is observed as a population measure σ_j on A. We shall restrict ourselves to measures on A which induce a continuous corresponding population density defined on A. A population partition $\sigma = (\sigma_1, \sigma_2)$ shall be considered *admissible* if for every $S \in \mathcal{A}$, we have that $\sigma_1(S) + \sigma_2(S) = \nu(S)$ and the corresponding induced density is continuous. The set of admissible population partitions shall be denoted as Σ endowed with the supremum norm. Thus, $(\Sigma, \| \|)$ is a Banach space. On Σ^2 we consider the product topology.

A political party $j = 1, 2$ chooses its policy by aggregating the observed policy preferences of its members according to some fixed rule P_j, which we shall call its statute. As parties do not observe β the aggregation applies only to α. As an example of such rule we may consider, for instance, the mean (resp. the median voter rules) which assigns to each party an ideal policy of its mean voter (resp. median voter). (See Sections 4 and 5). We shall denote the profile of party statutes as $P = (P_1, P_2)$. The mapping $P : \Sigma^2 \to X^2$ assigns to every population partition a policy profile $x = (x_1, x_2)$.

Assumption 1 (Scale Invariance)**.** For $j = 1, 2$, the function $P_j : (\Sigma, \| \|) \to X$ is scale invariant. That is $P_j(t\sigma) = P_j(\sigma)$ for every $t \in \mathbb{R}_+$ and every $\sigma \in \Sigma$.

Assumption 2 (Regularity)**.** For $j = 1, 2$, The function $P_j : (\Sigma, \| \|) \to X$ is continuous and Fréchet[1] differentiable.

Consider the exogenous idiosyncratic party characteristic parameters y_1 and y_2. Each policy proposal profile $x = (x_1, x_2) \in X^2$ defines a partition of A as follows. Let

$$\begin{aligned} A_1(x) &= = \{(\alpha, \beta) \in A \times E : u(x_1, y_1; \alpha, \beta) \geq u(x_2, y_2; \alpha, \beta)\} \\ A_2(x) &= = \{(\alpha, \beta) \in A \times E : u(x_2, y_2; \alpha, \beta) \geq u(x_1, y_1; \alpha, \beta)\} \end{aligned}$$

Note that $A_1(x) \cap A_2(x)$ has measure zero. Then, $\sigma(x) = (\sigma_1(x), \sigma_2(x))$ is determined by the density functions

$$g_i(\alpha; x) = \int_{A_i(x)} f(\alpha, \beta) d\beta, \quad i = 1, 2$$

the parties that give them highest utility. Thus, for each Lebesgue measurable set $S \subset X$,

$$\sigma_i(x)(S) = \int_S g_i(\alpha; x)\, d\alpha$$

[1]see Luenberger (1969) for definitions of differentiability of functions from and into function spaces.

Without loss of generality, me may assume from now that $y_1 \leq y_2$ and $x_1 \leq x_2$. If parties choose the proposals $x = (x_1, x_2)$ with $x_1 < x_2$ and $y_1 > y_2$, the agents affiliate to one or the other party, depending on which side of the line

$$z(t;x) = \frac{x_1 - x_2}{y_2 - y_1} t + \frac{y_1 + y_2}{2} - \frac{x_1^2 - x_2^2}{2(y_2 - y_1)} \tag{1}$$

they fall. Thus,

$$g_1(\alpha;x) = \int_{-\infty}^{z(\alpha;x)} f(\alpha,\beta)\, d\beta \qquad g_2(\alpha;x) = \int_{z(\alpha;x)}^{+\infty} f(\alpha,\beta)\, d\beta$$

whereas, if $y_1 = y_2$, the induced population densities are

$$g_1(\alpha;x) = \begin{cases} \int_{\mathbb{R}} f(\alpha,\beta)\, d\beta & \text{if } \alpha < \frac{x_1+x_2}{2} \\ \frac{1}{2}\int_{\mathbb{R}} f(\alpha,\beta)\, d\beta & \text{if } \alpha = \frac{x_1+x_2}{2} \\ 0 & \text{if } \alpha > \frac{x_1+x_2}{2} \end{cases} \qquad g_2(\alpha;x) = \begin{cases} \int_{\mathbb{R}} f(\alpha,\beta)\ \, d\beta & \text{if } \alpha > \frac{x_1+x_2}{2} \\ \frac{1}{2}\int_{\mathbb{R}} f(\alpha,\beta)\ \, d\beta & \text{if } \alpha = \frac{x_1+x_2}{2} \\ 0 & \text{if } \alpha < \frac{x_1+x_2}{2} \end{cases}$$

On the other hand, if $x_1 = x_2$ and $y_1 < y_2$, we have that

$$g_1(\alpha;x) = g_1(\alpha) = \int_{-\infty}^{\frac{y_1+y_2}{2}} f(\alpha,\beta) d\beta \qquad g_2(\alpha;x) = g_2(\alpha) = \int_{\frac{y_1+y_2}{2}}^{+\infty} f(\alpha,\beta) d\beta = g_2(\alpha) \tag{2}$$

The mapping $\sigma : X^2 \to \Sigma^2$ is well defined. That is, the functions $g_1(\alpha;x)$ and $g_2(\alpha;x)$ are continuous on $\alpha \in A$. The proof of the following Lemma is provided in the Appendix.

Lemma 3. If $y_1 < y_2$ the mapping $\sigma_i : X^2 \to \Sigma$ is Fréchet differentiable, for each $i = 1, 2$.

As a consequence, we have the following.

Corollary 4. If $y_1 < y_2$ the mapping $\sigma_i : X^2 \to \Sigma$ is continuous, for each $i = 1, 2$.

3. Divergent Equilibria

Throughout this section, unless explicitly stated, we assume $y_1 < y_2$. The multiparty equilibrium under free mobility of population may be defined as follows.

Definition 5. Given the profile of party statutes P and the idiosyncratic party characteristics y_1, $y_2$we say that $(x^*, \sigma^*) \in X^2 \times \Sigma^2$ is a **multi–party equilibrium** if:

(i) $x^* = P(\sigma^*)$
(ii) $\sigma^* = \sigma(x^*)$

Furthermore, the equilibrium is **divergent** if $x_1^* \neq x_2^*$. Otherwise, we say that the equilibrium is **convergent**.

Consider the mapping $\phi : X^2 \to X^2$ defined by $\phi(x) = P(\sigma(x))$. Clearly, an equilibrium is just a fixed point of this mapping.

Proposition 6. Let Assumption 2 hold. Then, there is an equilibrium x^*.

Proof. Consider the mapping $\phi : X^2 \to X^2$ defined by $\phi(x) = P(\sigma(x))$. The fixed points of this mapping correspond to equilibria of the model. The mapping is clearly defined on the entire X^2 and continuity follows from assumption 1. As X^2 is compact and convex, by Brouwer's fixed point theorem there must exist at least one such fixed point (possibly convergent). □

Note that if $y_1 = y_2$, so that individual membership is fully determined by policy positions, this model becomes deterministic and fully falls into the framework posed by Caplin and Nalebuff (1997). Thus, the interesting case for us in this paper is $y_1 \neq y_2$. In fact, as long as what we are interested in, is merely existence of *some* multi-party equilibrium, the present model still fits the same approach: imposing basic continuity and minimal internal support assumptions on the statutes P_j (as in Gomberg *et al. 2004*), together with the exogenously imposed difference between parties, would, indeed, lead into existence of an equilibrium, which, in fact, would involve full sorting of agents in the $A \times E$ space. This sorting, however, would be incomplete once projected onto the observable A space. It may be entirely caused by the difference in y's.

As a matter of fact, there is nothing that prevents the observed policy positions x_j of the parties to coincide (converge). Indeed, whenever such equilibrium exists, the older results, by themselves, are silent on the existence of divergent equilibria. Let $\Delta = \left\{(x, x') \in X^2 : x = x'\right\}$ denote the diagonal of the policy space.

Definition 7. The two parties are *ex ante* identical if they use the same policy rule $P_1 = P_2$ and the distribution of population is such that $g_1(\alpha; x)$ and $g_2(\alpha; x)$, given by equation 2, satisfy that there is some $t \in \mathbb{R}_+$ such that

$$g_1(\alpha; x) = t g_2(\alpha; x)$$

for every $\alpha \in A$, and for every $x \in \Delta$.

Let

$$\lambda_i(x) = \sigma_i(x)(A) = \int_A g_i(\alpha; x)\, d\alpha$$

be the vote share of party $i = 1, 2$ that results from $x \in X \times X$. Clearly $\lambda_1 + \lambda_2 = 1$.

Remark 8. If parties are *ex ante* identical and the proposals of the parties satisfy $x \in \Delta$, then $g_i(\alpha; x) = g_i(\alpha)$ for every $\alpha \in A$ and $i = 1, 2$. Hence, for $x \in \Delta$,

(1) $\lambda_i(x) = \lambda_i$, $i = 1, 2$ does not depend on x.
(2) the induced partition of the population is described by the density functions

$$\frac{1}{\lambda_1} g_1(\alpha) = \frac{1}{\lambda_2} g_2(\alpha) = g(\alpha)$$

Under the conditions assumed in Remark 8, it should be noted that the vote shares are themselves independent of the common ideological position of the parties and are entirely determined by y_1 and y_2. In particular, for $i = 1, 2$, σ_i is constant on the diagonal $\Delta \subset X^2$. It follows that there can be at most a unique pooling equilibrium policy profile x^*.

Proposition 9. Let the parties be *ex ante* identical. Then, there is unique convergent equilibrium $x^* \in \Delta$.

Proof. Let the parties be *ex ante* identical. Let $P_1 = P_2 = P$. By remark 8, we may assume that the population partitions are determined by

$$\frac{1}{\lambda_1} g_1(\alpha) = \frac{1}{\lambda_2} g_2(\alpha) = g(\alpha)$$

with g_1 and g_2 given by (2). By Assumption 1,

$$x_1^* = P(g_1(\cdot, x)) = x_2^* = P(g_2(\cdot, x))$$

for every $x \in X^2$. Hence, $x^* = (x_1^*, x_2^*)$ is the unique convergent equilibrium policy profile. $\square$

Hence, when the two parties are *ex ante* identical, a unique convergent equilibrium exists. Our question in this paper shall be to find out the conditions for existence of other, divergent, equilibria.

In order to do this, we shall, following Caplin and Nalebuff (1997) and Gomberg *et. al* (2004), postulate, in addition to the two previously introduced assumptions, two additional hypotheses on policy rules. The first one ensures that parties choose interior ideological positions. For this purpose, consider the following property.

Assumption 10 (Minimal Internal Support)**.** There is $\delta > 0$ such that the following holds. For each party $j = 1, 2$, any $x = (x_1, x_2) \in X \times X$ and any value $t \in X$

$$\frac{1}{|A_j(X)|} \int_{\{(\alpha,\beta)\in A_j(x):u(P_j(g_j(\cdot,x),y_j;\alpha,\beta)>u(P_j(t,y_j;\alpha,\beta)\}} g_j(\alpha, x) d\alpha > \delta$$

where $|A_j(x)|$ denotes the Lebesgue measure of the set $A_j(x)$.

In a deterministic version of the model Caplin and Nalebuff (1997) have postulated the assumption that the party policy rules would never result in identical policies if party populations have opposing preference, in the sense of being divided by a hyperplane in the ideological space. This assumption, problematic even in that model, unless the policy rules are just aggregating intraparty preferences ("membership-based" in their terminology) would be entirely unapplicable here, as, in general, the sorting is not perfect in this model. Fortunately, it turns out, that what was driving the Caplin and Nalebuff result was not this, but a weaker condition: instability of the convergent equilibrium under adjustment dynamics. We shall assume the following:

Assumption 11 (Instability of Pooling)**.** Let $x^* \in \Delta \subset X^2$ be an equilibrium policy profile. Then, there exists an open neighborhood $O \subset X^2$ containing x^* such that for any boundary point $x \in \partial O \backslash \Delta$, we have that $\phi(x) \notin O$.

Intuitively, this states that once the convergent equilibrium is perturbed, the induced population partition induces a further policy divergence. It is not difficult to check that, in a deterministic model, the requirement of distinct policies from sorting partitions imposed in earlier work implies this (any minute policy difference in a deterministic model induces a full population sorting which, in turn, induces distinct policies, which, by continuity of policy rules, cannot be close to the diagonal). This weaker assumption is, in fact, sufficient for our next result.

Proposition 12. Let the parties be *ex ante* identical. Suppose the unique convergent equilibrium $x^* \in \Delta$ is a Lefschetz fixed point of ϕ^2. If the dimension of the policy space n is odd and Assumptions 1, 2, 10 and 11 hold then, there shall exist a divergent equilibrium.

Proof. The equilibrium $x^* \in \Delta$ is *stable* along the diagonal, Δ, since any policy profile on Δ is mapped directly into x^*. Furthermore, this equilibrium is either isolated or else, there is at least a divergent equilibrium. Thus, we may assume this equilibrium is an isolated fixed point.

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[2]That is, if all the eigenvalues of $d_x\phi$ are all unequal to $+1$.

We will show that x^* cannot be the unique equilibrium. Assume, for contradiction that x^* is the unique equilibrium. We argue that the boundary of X^2 is unstable. Let $x = (x_1, x_2) \in \partial X^2$. By making x'_j arbitrarily close to x_j, and using assumption 2, we can make an arbitrarily large proportion of party members strictly prefer x'_j. Hence, $\phi(x) \in \text{int}(X^2)$ and the boundary of X^2 is unstable.

As ϕ is a mapping from the compact and convex set X^2 to itself, and as x^* is assumed to be Lefschetz, by the Lefschetz Fixed Point Theorem (see Guillemin and Pollack 1974, pp. 119-130) the total sum of the indices of the fixed points x^* must be equal to 1 (the Euler characteristic of X^2). Recall, that the index $\text{ind}(x^*)$ of a Lefschetz fixed point x^*may be calculated as $(-1)^d$, where d is the dimension of the unstable manifold of x^*.

Now, by assumption 3, the equilibrium x^* is unstable off diagonal. As the co-dimension of the diagonal Δ is n, the index of the diagonal fixed point equals $(-1)^n$, which implies it cannot be unique if n is odd. Hence, a divergent equilibrium must exist. □

Of course, this proposition, on its own, is of limited interest: unless we can show that assumption 3 holds for cases where $y_1 \neq y_2$ the result is vacuous. Fortunately, for a restricted case when the parties use common preference aggregation rules, such as the mean and the median voter rule, we can show that for the difference between parties sufficiently small the assumption does hold.

4. The Mean Voter Rule

Consider, first, the mean voter rule. Suppose that for each $x \in X \times X$, the induced population partition $\sigma_j(x)$ is represented by the density $g_j(\alpha; x)$. Then, each party chooses

$$P_j(\sigma_j(x)) = \frac{\int_A \alpha g_j(\alpha; x) d\alpha}{\int_A g_j(\alpha; x) d\alpha} = \frac{\int_A \alpha g_j(\alpha; x) d\alpha}{\sigma_j(A)}$$

Recall that an equilibrium is a fixed point of the mapping $\phi : X^2 \to X^2$ defined by

$$\phi(x) = (P_1(\sigma_1(x)), P_2(\sigma_2(x))).$$

Lemma 13. The map $\phi : X^2 \to X^2$ is continuous and differentiable.

It turns out that as the y's get closer (but remain distinct), existence of equilibrium can be assured. In fact, the following proposition holds. The proof is provided in the Appendix.

Proposition 14. Let $n = 1$ and the parties be using the mean voter rule. Let $\nu \in \Sigma$ be an overall population distribution on $A \times E$ which is described by a continuous density function $f(\alpha, \beta)$ and such that the parties be *ex ante* identical. Let $x \in \Delta$ and $\mu = P_1(\sigma_1(x)) = P_2(\sigma_2(x))$. If

$$|y_2 - y_1| < \frac{1}{\lambda_1 \lambda_2} \left(\int_A (\alpha - \mu) f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha \right)^2 \tag{3}$$

then, there exists a divergent equilibrium.

The boundary established by proposition 14 depends only on two bits of population statistics: the mean of ideal points in the observable ideological space of those citizens who would be indifferent between parties in the absence of ideological differences between them, and the relative size λ_1 of the part of the population that exogenously prefers one party to another when there is no ideological difference between them.

While, strictly speaking, exceeding the boundary does not guarantee the uniqueness of the convergent equilibrium, examples of the latter are not hard to find.

Example 15. Let the population distribution be uniform ($f(\alpha,\beta)=1$ on $[0,1]^2$) and $y_1=\frac{1}{4}<\frac{3}{4}=y_2$. The unique equilibrium in this case has $x_1=x_2=\frac{1}{2}$ *(show).* Conversely, for any (y_1,y_2) such that $|y_2-y_1|<\frac{1}{12\lambda_1(1-\lambda_1)}$ divergent equilibria exist by our proposition.

Note that the number in the left hand side of equation (3) depends only on some statistics of the population.

Since small exogenous differences between the parties implies that the citizens' membership decision is mostly determined by the observed policy differences (in particular, if $y_2=y_1$ we are reduced to the deterministic model), this result shows that the deterministic case is not isolated. Rather, in this case there is continuity: a small amount of uncertainty about individual membership decision does not affect the existence of a divergent equilibrium.

It should be noted that the continuity result of Proposition 14 can be extended to policy rules other than the mean voter rule, though the precise boundary would be different. In particular, suppose that, instead of choosing the ideal point of the mean of its voter distribution, parties propose policies according to a different rule

$$P_j(\sigma_j(x))=\frac{\int_A Q(\alpha;x)g_j(\alpha;x)d\alpha}{\int_A g_j(\alpha;x)d\alpha}$$

where $Q:A\times X\to A$ is a non-constant continuous mapping. Let $x\in\Delta$ (so $\sigma_j(x)$ does not depend on x) and denote the policy society as a whole would adopt as $\chi=\int_A Q(\alpha)f_1(\alpha)d\alpha\in\operatorname{int}(A)$. Following the steps of the proof of proposition 14 we shall easily obtain the following boundary on the exogenous difference between parties that guarantees existence of divergent equilibria:

$$0<|y_2-y_1|<\frac{1}{\lambda_1(1-\lambda_1)}\int_A(Q(\alpha)-\chi)(\alpha-\chi)f(\alpha,\frac{y_1+y_2}{2})d\alpha$$

which implies that, as long as $\int_A(Q(\alpha)-\chi)(\alpha-\chi)f(\alpha,\frac{y_1+y_2}{2})d\alpha>0$ for these rules, likewise, sufficiently small uncertainty about individual membership choices leads to the existence of divergent equilibria.

5. The Median Voter Rule

It is possible to use the same techniques to establish similar bounds for other rules, that do not belong to the class described above. For example, suppose parties instead use the median voter rule, i.e. $P_j(\sigma_j(x))$ is defined by

$$\int_{\{\alpha:\alpha\le P_j(\sigma_j(x))\}}g_j(\alpha;x)d\alpha=\frac{1}{2}$$

Clearly, what we are after is finding the fixed points of the mapping $\phi:X^2\to X^2$ given by an implicit equation

$$\int_{\{\alpha:\alpha\le\phi:Xj(\sigma_j(x))\}}g_j(\alpha;x)d\alpha=\frac{1}{2}$$

Denoting the median of the whole population distribution as m, it is not hard to observe that the point $(m, m) \in \Delta$ is indeed the fixed point of this mapping, and, therefore, corresponds to a convergent equilibrium. As before, we would like to establish the (in)stability of ϕ around the convergent fixed point, that is we would like to establish conditions under which inequality 4 holds.

Differentiating ϕ implicitly with respect to x_i we obtain

$$g_j(\phi_j(x); x)\partial_{x_i}\phi_j(x) + \int_{\{\alpha:\alpha\le\phi_j(\sigma_j(x))\}} \partial_{x_i} g_j(\alpha; x)d\alpha = 0$$

from which it follows that

$$\partial_{x_i}\phi_j(x) = -\frac{\int_{\{\alpha:\alpha\le\phi_j(\sigma_j(x))\}} \partial_{x_i} g_j(\alpha; x)d\alpha}{g_j(\phi_j(x); x)\partial_{x_i}\phi_j(x)}$$

Taking the limit of the expression as $x \to (m, m)$ and substituting from equations 9, 9 and 5 the formulas for $\partial_{x_i} g_j(\alpha; x)$, we obtain

$$\lim_{x\to(m,m)} \partial_{x_j}\phi_j(x) = \frac{\int_{\{\alpha:\alpha\le m\}}(m-\alpha)f(\alpha, \frac{y_1+y_2}{2})d\alpha}{\lambda_j(y_2-y_1)f_1(m)} = -\lim_{x\to(m,m)} \partial_{x_i}\phi_j(x), i \neq j$$

Thus, computing the determinant and rearranging terms we obtain the following proposition:

Proposition 16. Let $n = 1$ and the parties be using the median voter rule. Let $\nu \in \Sigma$ such that the parties be *ex ante* identical and a continuous population density function $f(\alpha, \beta)$ exists. If

$$\frac{1}{\lambda_1(1-\lambda_1)}\frac{1}{f_1(m)}\int_{\{\alpha:\alpha\le m\}}(m-\alpha)f(\alpha, \frac{y_1+y_2}{2})d\alpha > |y_2 - y_1| > 0$$

then, there exists a divergent equilibrium.

It should be noted that when the density of citizens's ideological viewpoints at the median point of the whole distribution is $f_1(m) = 0$, then the bound on $|y_2 - y_1|$ explodes, as the minor changes of policies cause the intraparty medians to move at an infinite rate.[3] Another interesting observation is, that, at least for the uniform distribution of citizens over $A \times E = [0, 1]^2$, the boundary for the $|y_2 - y_1|$ implied by the median voter rule (which, in this case is easily computed to be equal to $\frac{1}{2}$ for *ex ante* identical parties with $\lambda_1 = \lambda_2 = \frac{1}{2}$) is weaker than that for the mean voter rule (for which it is $\frac{1}{3}$): as the policy difference between parties induces ideologically skewed memberships within each party, the medians move towards the edges faster than the means.

Finally, it should be noted that the divergent equilibria exist even when parties use distinct rules. Of course, the result is trivially true if there is no convergent equilibrium. Thus, if party 1 uses the mean voter rule, while party 2 uses the median voter rule a convergent equilibrium only exists if the mean equals the median for the overall population distribution $f_1(\alpha)$, i.e. if $m = \mu$. If $m \neq \mu$ whatever equilibria exist (and some equilibria involving sorting over $A \times E$ do exist from the results, for instance, of Caplin and Nalebuff 1997) would be divergent here. Still, even for the case when $m = \mu$ we can guarantee existence of a divergent equilibria for $|y_2 - y_1|$

[3] though, strictly speaking, the function ϕ is not differentiable (not even necessarily continuous) in this case, the instability of the convergent equilibrium and the consequent existence of a divergent equilibrium can be easily shown using standard approximation techniques.

small enough. In fact, using the inequality 4 we can establish that such equilibria exist in this case whenever

$$0 < |y_2 - y_1| < \frac{1}{\lambda_1} \int_A (\alpha - \mu)^2 f(\alpha, \frac{y_1 + y_2}{2}) d\alpha + \frac{1}{1-\lambda_1} \frac{1}{f_1(\mu)} \int_{\{\alpha : \alpha \leq \mu\}} (\mu - \alpha) f(\alpha, \frac{y_1 + y_2}{2}) d\alpha$$

6. Conclusions and Further Research

In this paper we introduce a model in which the citizens' party choice is determined both by the ideological difference between the parties and the unobserved non-ideological attitudes. As the membership choice is only incompletely determined by observed policy proposals, it may be interpreted as stochastic in this model. The party membership, in turn, determines party policy stances by means of intraparty preference aggregation rules.

In this context with two parties we show that, at least when the parties aggregate preferences by choosing ideal points of their mean (or median) voters, than, if parties are perceived by citizens as "similar" in the sense that the non-policy difference is small compared to the mean of the agents' preferences in the ideological space, we are guaranteed existence of divergent equilibria even in an *ex ante* symmetric model. In this sense, the present stochastic model shows continuity with the deterministic endogenous platform model we studied earlier (Gomberg *et al.* 2004).

It remains to consider how the results extend to increasing the number of parties and of policy dimensions, as well as considering different party decision-making rules (including, possible strategic interaction in a democratic context).

7. Appendix 1

The model assumes that party $j = 1, 2$ chooses its policy by aggregating the preferences of its members according to some fixed rule P_j. This aggregation applies only to the observed variable α. Here we provide some empirical evidence suggesting that this might be a realistic assumption. For a selection of countries we analyze the political platforms of the main parties and the average ideal policy of their supporters. We find that in countries with only two major parties (US, UK) the political platform of the party and the average ideal policy of its supporters are strikingly similar. This is not the case in countries with more than two major parties or in a clear unstable political period.

We assume that the policy space X is one-dimensional and can be identified with the Left-Right ideological position. The information about the ideological position of voters is obtained from the World Values Survey (WVS) while the information on parties' platforms comes from the Comparative Manifesto Project (CMP) [reference]. The WVS provides information about the respondants self-reported ideological position on the (1-10) Left-Right scale and which party the respondant would vote for. Thus, we can compute the average ideological position of the supporter of each party. The CMP measures the policy position of the parties in many issues in the electoral period for a series of democratic countries. In particular, it provides the Left-Right position of parties (see Laver and Budge 1992 for an explanation of the methodology used to obtain such positions). Since the information in the WVS and in the CMP often cover different years we select for each year reported in the CMP the closest earlier year in the WVS if the difference between the two is less than three years. Table 1 reports for three electoral periods in Great Britain the positions of the major parties and the average position of their supporter. For example, according to the WVS in year 1990 the average ideological position of the individuals supporting the Labour Party was 4.28 whereas the average position for the supporters of the Conservative Party was 6.74. On the other hand, according to the CMP the position of the Labour Party in the 1992 electoral period was 4.13, and the position of the Conservative party 6.76. Table 2 shows for US the same type of information for the five electoral periods for which the required data is available. The closeness between the two numbers is even clearer than in the case of Grat Britain. However, it would be wrong to expect that those values are so similar in all countries. Our model tries to capture the situation in two-party systems during "stable" periods. Table 3 shows the results for the case of Portugal during the 90's. Even tough the two parties reported in the table (Socialdemocrat and Socialist) obtained together close to 80% of the vote–so that the two-party system can be a reasonable assumption here– the position of the Socialdemocrat Party is quite different from the average position of its supporters suggesting that an "equilibrium" has not been reached yet. Other countries like, for example, Belgium, Finland, France and Spain also present a big difference between the ideological position of the main parties and the average (or median) position of their supporters (the tables with this information for a series fo 20 countries is avaliable from the authors upon request).

Even tough we do not carry out a rigourous statistical empirical analysis, the data provided here suggest that, in certain countries, the proposals of parties might be very close to the average, or median, positions of their respectives supporters.

Table 1. Left-Right Position. Great Britain

Party	WVS	Manifesto	WVS average	Manifesto	Sample Size WVS
Conservative	1981	1983	7.69	6.81	99
	1990	1992	6.74	6,76	490
	1999	2001	6.37	6.17	163
average			6.93	6.58	
Labour	1981	1983	4	3.74	91
	1990	1992	4.28	4.13	475
	1999	2001	4.53	5.75	282
average			4.27	4.54	

Source: World Values Survey and Comparative Manifesto Project. The original values from the CMP have been transformed to the 1-10 scale.

Table 2. Left-Right Position. US

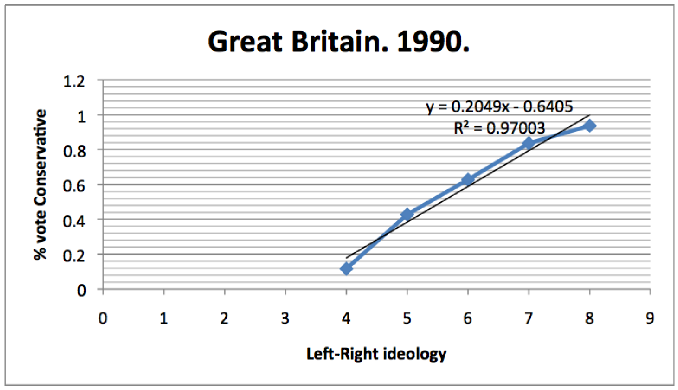
Party	WVS	Manifesto	WVS average	Manifesto	Sample Size WVS
Republican	1982	1984	6.7	7.01	451
	1990	1992	6.23	6.87	561
	1995	1996	6.57	6.59	548
	1999	2000	6.73	7	350
	2006	2008	6.9	6.63	352
average			6.62	6.82	
Democrat	1982	1984	5.68	4.87	934
	1990	1992	5.51	6.05	740
	1995	1996	5.25	5.9	588
	1999	2000	5.42	5.34	555
	2006	2008	4.92	6	498
average			5.35	5.63	

Table 3. Left-Right Position. Portugal

Party	WVS	Manifesto	WVS average	Manifesto	Sample Size WVS
Socialdemocrats	1990	1991	7.24	5.11	285
	1999	1999	7.04	5.53	132
average			7.14	5.32	
Socialist	1990	1991	5.07	4.97	269
	1999	1999	5.01	4.7	299
average			5.04	4.835	

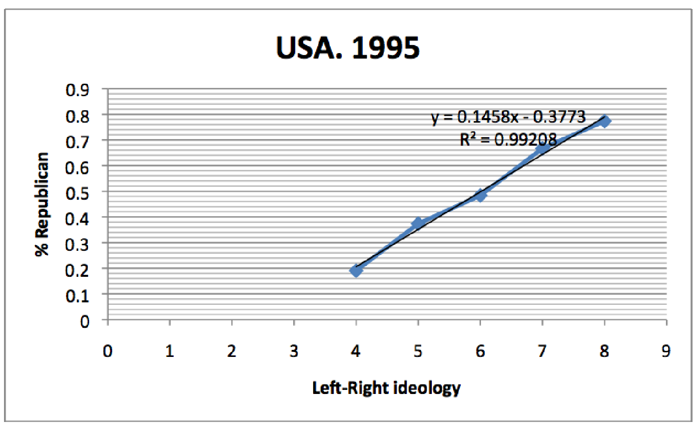
8. Appendix 2

Here we continue with the data provided in the World Values Survey about the self-reported ideological position and which party respondants would vote for. We mainly focus on US and Great Britain but similar results can be obtained for other democracies with two major political parties. Consider the case of Great Britain in year 1990. We take all the respondants that would vote either for the Conservative party or for the Labour party. Then we compute for each left-rigth ideological position the percentage of those individuals who would vote for the Conservative party. Since the number of individuals reporting a given ideological positions outside the interval $[4, 8]$ is always very small (less than 50 people for each position) we only consider people in such interval (734 people representing 77% of all the individuals). Figure 1 plots such information and its best linear fit.



Only 11% of the individuals at ideological position of four reported to be willing to vote Conservative. Such percentage increases in an apparently linear manner with the ideological position, reaching a value of 93% for individuals at position of eight, i.e. 93% of the individuals that reported an ideological position of eigth would vote Conservative.

Figure 2 shows the same type of information for the case of the Republican party and the Democratic party in US in 1995, 1996[4].

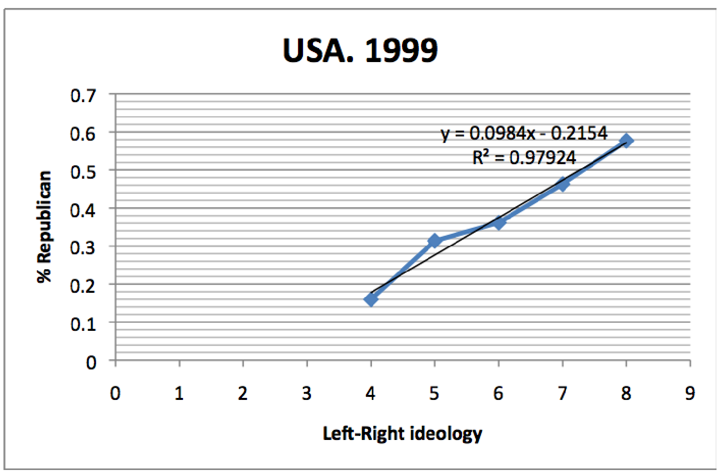


We also obtain similar type of plots with such a good linear fit to the data for countries like, for example, Australia, Germany and New Zeland.

Take the Left-Right dimension as our policy space X . Then, these plots seem consistent with the existence of a non-policy variable y such that the conditional density function of ideal points, $f_2(\beta|\alpha)$, is independent of α. To see that assume that voters have Euclidean preferences on the space $A \times E$ and all their ideal points, (α, β), are in the rectangle $[1, 10] \times [0, 1] \subseteq A \times E$. Figure 3 shows the case of Great Britain in 1990, as in Figure 1, but now the y-axis represents the non-policy variable. We next assume that the policy proposal of the party, $P_j(\sigma_j(x))$, is the mean rule. Thus, these parties' proposals coincide with the average ideal point of the citizens

[4]These plots are very similar to the ones we have obtained using data from the National Election Study instead of the World Values Survey.

supporting each party, and they are basically the same as the propossals reported in the Comparative Manifesto Project (see Appendix 1). In our case those policies are $x_1 = 4.13$ for the Labour party and $x_2 = 6.76$ for ther Conservative party (see Table 1 in Appendix 1). The straight line $y = a + bx$ represents the best fit to the data showed in Figure 1, where $a = -0.6405$ and $b = 0.2049$. Given such "separating line" $y = a + bx$ we can find the non-policy positions y_1 and y_2. To do that we first need to find the line $y = a' - (1/b)x$ (perpendicular to $y = a + bx$), such that the intervals d_1 and d_2 shown in the figure have the same lenght. The values of the two non-policy variables are given by the intersections of such line $y = a' - (1/b)x$ and the vertical lines given by policy positions x_1 and x_2. In our particular example we obtain $y_1 = 6.5$ and $y_2 = -5.52$. Thus, we can suppose that for Labour $(x_1, y_1) = (4.13, 6.5)$ and for Conservative $(x_2, y_2) = (6.76, -5, 52)$ and–given that preferences are Euclidan–all the agents with ideal positions on the Southeast side of $y = a + bx$ vote Conservative, and the agents with ideal positions on the other side of the line vote Labour. Thus, the two sides of the separating line $y = a + bx$ give the population partition $\sigma = (\sigma_1, \sigma_2)$.



Notice that the example described in Figure 3 represents a **multi–party equilibrium** since: i) $x_1 = 4.13$ is the average left-right position of agents in $\sigma_1(x)$, i.e. of the agents on the Northwest side of $y = a + bx$ and $x_2 = 6.76$ is the average of agents on $\sigma_2(x)$, the Southeast side of $y = a + bx$; ii) Given $(x_1, y_1) = (4.13, 6.5)$ and $(x_2, y_2) = (6.76, -5, 52)$ the separating line that determines the population partition $\sigma = (\sigma_1(x), \sigma_2(x))$ is $y = a + bx$.

Since the ideal points on the non-policy variable y are all on the interval $[0, 1]$ and, by construction, for any given ideological position α those points coincide with the percentage of agents on favor of the Conservative party, the conditional distribution $f_2(\beta|\alpha)$ must be independent of α.Thus, the data seems consistent with the existence of a non-policy variable y such that agents' preferences over it are independent of their preferences on the Left-Right ideological space.

Of course we do not mean this example to be conclusive proof that our model is correct. The only objective of this empirical exercise is to show that in principle our model could be consistent with some stilized facts. In particular, the example does not inquestionably prove the existence of a non-policy variable y. In principle the varibale y could be also chosen by parties (using some specific rule, perhpas different from the mean or median rule). However, the fact that in Figure 1 the true "separating line" $y = a + bx$ is a straight line strongly suggests the existence of a **non-policy** variable y. In the contrary case, i.e. in the event that this variable y was chosen by policy parties, it would be hard to believe that agents' preferences over the left-right variable x and their preferences over the variable y were independent, as the data suggest.

9. Appendix 3: Proofs

Proof Lemma 3. We do the proof for $i = 1$. Let $x = (x_1, x_2) \in X^2$. Note that

$$\partial_1 g_1(\alpha; x) = f(\alpha, z(\alpha; x)) \frac{\alpha - x_1}{y_2 - y_1}$$

and

$$\partial_2 g_1(\alpha; x) = f(\alpha, z(\alpha; x)) \frac{x_2 - \alpha}{y_2 - y_1}$$

From the above expressions, the Fréchet derivative can be easily computed. First of all, a simple computation shows that

$$z(\alpha; x + e) - z(\alpha; x) = \frac{h(\alpha - x_1) + k(x_2 - \alpha) + (k^2 - h^2)/2}{y_2 - y_1}$$

We use the notation $e = (h, k)$ and $\|e\| = \sqrt{h^2 + k^2}$. Fix $x \in X^2$. There is $M > 0$ such that

$$\frac{|z(\alpha; x + e) - z(\alpha; x)| \leq 1}{\|e\|} \leq M$$

for any $\alpha \in A$ and e such that $\|e\| \leq 1$.

Let $\varepsilon > 0$. By the Mean Value Theorem, there is $\beta(\alpha) \in [z(\alpha; x), z(\alpha; x + e)]$ such that

$$\begin{aligned} g_1(\alpha; x + e) - g_1(\alpha; (x)) &= \int_{z(\alpha;x)}^{z(\alpha;x+e)} f(\alpha, \beta)\, d\beta = (z(\alpha; x + e) - z(\alpha; x))(f(\alpha, \beta(\alpha)) = \\ &= \frac{f(\alpha, \beta(\alpha))}{y_2 - y_1} \left(h(\alpha - x_1) + k(x_2 - \alpha) + \frac{k^2 - h^2}{2} \right) \end{aligned}$$

Let $D_x : \mathbb{R}^2 \to C(A)$ be defined as $Dx(e) \in C(A)$ is the function

$$D(\alpha; e) = \frac{f(\alpha, z(\alpha; x)}{y_2 - y_1} (h(\alpha - x_1) + k(x_2 - \alpha)) = \int_{z(\alpha;x)}^{z(\alpha;x+e)} f(\alpha, z(\alpha; x))\, d\beta - \frac{k^2 - h^2}{2}$$

Note that D_x is linear in e and continuous in all the variables. We have that,

$$g_1(\alpha; x + e) - g_1(\alpha; (x)) - D(\alpha; e) = \int_{z(\alpha;x)}^{z(\alpha;x+e)} (f(\alpha, \beta) - f(\alpha, z(\alpha; x)))\, d\beta - \frac{k^2 - h^2}{2}$$

Since, z is continuous and $f(\alpha, \beta)$ is uniformly continuous in $A \times [z(\alpha; x), z(\alpha; x+e)]$, given $\varepsilon > 0$, there is a $0 < \delta < 1$ such that if $\|e\| \leq \delta$, then

$$|f(\alpha, \beta) - f(\alpha, z(\alpha; x))| \leq \frac{\varepsilon}{M}$$

Thus, as long as $\|e\| \leq \delta$ we have that

$$|g_1(\alpha; x + e) - g_1(\alpha; (x)) - Dx(\alpha; e)| \leq \frac{\varepsilon}{M} |z(\alpha; x + e) - z(\alpha; x)| + \frac{k^2 + h^2}{2}$$

So, for any $\alpha \in A$ we have that

$$\frac{|g_1(\alpha; x+e) - g_1(\alpha; (x)) - Dx(\alpha; e)|}{\|e\|} \leq \varepsilon + \frac{k^2+h^2}{2\|e\|} = \varepsilon + \frac{\|e\|}{2}$$

Hence, have that

$$\sup_{\alpha \in A} \frac{|g_1(\alpha; x+e) - g_1(\alpha; (x)) - Dx(\alpha; e)|}{\|e\|} \leq \varepsilon + \frac{\|e\|}{2}$$

Hence,

$$\frac{\|g_1(\alpha; x+e) - g_1(\alpha; (x)) - Dx(\alpha; e)\|}{\|e\|} \leq \varepsilon + \frac{\|e\|}{2}$$

Since, $\varepsilon > 0$ is arbitrary, we see that

$$\lim_{\|e\| \to 0} \frac{\|g_1(\alpha; x+e) - g_1(\alpha; (x)) - Dx(\alpha; e)\|}{\|e\|} = 0$$

and the Lemma is proved. □

Proof Lemma 13. For α and $j = 1, 2$ fixed, the function $g_j(\alpha; x)$ depends continuously on x. Thus, $P_j(\sigma_j(x))$ depends continuously on x. In addition, by Lemma 3 the mapping $g_j(\alpha; x)$ is Fréchet differentiable in x. It is easy to see that the maps

$$\begin{array}{rcl} T : \Sigma & \to & X \\ f(\alpha) & \mapsto & \mu_f \end{array}$$

where

$$\mu_f = \int_A \alpha f(\alpha)\, d\alpha$$

and

$$\begin{array}{rcl} S : \Sigma & \to & X \\ f(\alpha) & \mapsto & \int_A f(\alpha)\, d\alpha \end{array}$$

are continuous. For example, given $\varepsilon > 0$, Let

$$\delta = \frac{\varepsilon}{\int_A \alpha\, d\alpha}$$

If $\sup\{|f(\alpha) - g(\alpha)| : \alpha \in A\} \leq \delta$ then

$$|\mu_f - \mu_g| \leq \int_A \alpha |f(\alpha) - g(\alpha)|\, d\alpha \leq \delta \int_A \alpha\, d\alpha = \varepsilon$$

So T is continuous. The proof that S is continuous is similar. Since, S and T are continuous, they are Fréchet differentiable. Therefore $P_j(\sigma_j)$ is also differentiable and, hence continuous. It follows that $\phi(x$ is also continuous and differentiable. □

Lemma 17.

$$\lim_{x \to (\mu,\mu)} \partial_i \phi_j(x) = \frac{1}{\lambda_j} \int_A (\alpha - \mu)(\partial_i g_j(\alpha; (\mu, \mu)) d\alpha$$

Proof. Note that

$$\begin{aligned} \partial_i \phi_j(x) &= \frac{1}{(\int_A g_j(\alpha; x)\, d\alpha)^2} \left(\int_A \alpha (\partial_i g_j(\alpha; x)\, d\alpha \int_A g_j(\alpha; x)\, d\alpha - \int_A \alpha g_j(\alpha; x)\, d\alpha \int_A (\partial_i g_j(\alpha; x)\, d\alpha \right) \\ &= \frac{\int_A (\alpha - \phi_j(x))\, \partial_i g_j(\alpha; x)\, d\alpha}{(\int_A g_j\, (\alpha; x) d\alpha)} \end{aligned}$$

Thus,

$$\lim_{x\to(\mu,\mu)} \partial_i\phi_j(x) = \frac{1}{\lambda_j}\int_A (\alpha-\mu)\partial_i g_j(\alpha;(\mu,\mu))d\alpha$$

and the Lemma follows. □

Proof of Proposition 14. Recall that, given the proposals $x=(x_1,x_2)$ of the parties, we use the notation

$$g_j(\alpha;x) = \int_{\{(\alpha,\beta):||(x_j,y_j)-(\alpha,\beta)||\geq||(x_i,y_i)-(\alpha,\beta)||,i\neq j\}} f(\alpha,\beta)d\beta$$

when we want to make explicit the dependence of the density functions that describe the induced population partitions on the policies proposed by the parties.

Note that, for the mean voter rule, if the proposal of the parties satisfy $x\in\Delta$, then $P_1(\sigma_1(x)) = P_2(\sigma_2(x)) = \mu$, the observed mean of the overall population on A. Therefore, (μ,μ) is a fixed point of $\phi(x) = (\phi_1(x),\phi_2(x)) = P(g_1(\alpha;x),g_2(\alpha;x))$. We want to determine conditions under which the fixed point (μ,μ) is unstable off diagonal. As the stability along the diagonal is immediate from the definition of the rule (from anywhere on Δ the function ϕ immediately maps to (μ,μ)) we should simply need that the eigenvalues of the matrix

$$B(x) = \begin{pmatrix} \partial_1\phi_1(x)-1 & \partial_2\phi_1(x) \\ \partial_1\phi_2(x) & \partial_2\phi_2(x)-1 \end{pmatrix}$$

have different signs around (μ,μ). In other words, the necessary condition for assumption 3 to hold is that

$$\lim_{x\to(\mu,\mu)} \det(B(x)) < 0 \tag{4}$$

When parties choose the proposals $x=(x_1,x_2)$, the agents affiliate to one or the other party, depending on which side of the line

$$z(t;x) = \frac{x_1-x_2}{y_2-y_1}t + \frac{y_1+y_2}{2} - \frac{x_1^2-x_2^2}{2(y_2-y_1)}$$

they fall. For example,

$$g_1(\alpha;x) = \int_{-\infty}^{z(\alpha;x)} f(\alpha,\beta)\,d\beta$$

holds for x_1 close enough to μ. Hence,

$$\partial_1 g_1(\alpha;x) = f(\alpha,z(\alpha;x))\partial_1 z(t;x)|_{t=\alpha} = f(\alpha,z(\alpha;x))\frac{\alpha-x_1}{y_2-y_1}$$

Furthermore,

$$\partial_2 g_1(\alpha;x) = f(\alpha,z(\alpha;x))\partial_2 z(t;x)|_{t=\alpha} = f(\alpha,z(\alpha;x))\frac{x_2-\alpha}{y_2-y_1}$$

which implies that

$$\partial_1 g_1(\alpha;(\mu,\mu)) = -\partial_2 g_1(\alpha;(\mu,\mu)) = f\left(\alpha,\frac{y_1+y_2}{2}\right)\frac{\alpha-\mu}{y_2-y_1}$$

Since, for party 2 the relevant population density is

$$g_2(\alpha;x) = \int_{z(\alpha;x)}^{\infty} f(\alpha,\beta)\,d\beta$$

we get that

$$\partial_2 g_2(\alpha;(\mu,\mu)) \quad = \quad -\partial_1 g_2(\alpha;(\mu,\mu)) = \partial_1 g_1(\alpha;x)$$

To ease the notation we will write $g_i = g_i(\alpha;(\mu,\mu))$. Applying Lemma 17 and using the above formulae for $\partial_i g_j$ we have to establish conditions under which

$$\begin{aligned}
\lim_{x\to(\mu,\mu)} |B(x)| &= \begin{vmatrix} \frac{1}{\lambda_1}\int_A(\alpha-\mu)\partial_1 g_1 d\alpha - 1 & \frac{1}{\lambda_1}\int_A(\alpha-\mu)\partial_2 g_1 d\alpha \\ \frac{1}{\lambda_2}\int_A(\alpha-\mu)\partial_1 g_2 d\alpha & \frac{1}{\lambda_2}\int_A(\alpha-\mu)\partial_2 g_2 d\alpha - 1 \end{vmatrix} \\
&= 1 - \frac{1}{\lambda_1\lambda_2\,(y_2-y_1)}\left(\int_A(\alpha-\mu)f\left(\alpha,\frac{y_1+y_2}{2}\right)d\alpha\right)^2 < 0
\end{aligned}$$

which clearly holds if

$$|y_2-y_1| < \frac{1}{\lambda_1\lambda_2}\left(\int_A(\alpha-\mu)f\left(\alpha,\frac{y_1+y_2}{2}\right)d\alpha\right)^2$$

□

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